On the Tate and Langlands–Rapoport conjectures for special fibres of integral canonical models of Shimura varieties of abelian type

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ABSTRACT. We prove the isogeny property for special fibres of integral canonical models of Shimura varieties of A_n, B_n, C_n, and $D_n^{\mathbb{R}}$ type that have compact factors.[1] The approach used shows that large classes of crystalline cycles on abelian varieties over finite fields are algebraic. These two results have many applications. First, we prove the conditional Langlands–Rapoport conjecture for all special fibres of integral canonical models of Shimura varieties of A_n, B_n, C_n, and $D_n^{\mathbb{R}}$ type that have compact factors. Second, for certain isogeny sets we prove the unconditional Langlands–Rapoport conjecture (like for many basic loci). Third, we prove that all integral canonical models of Shimura varieties of Hodge type that are of A_n, B_n, C_n, and $D_n^{\mathbb{R}}$ type and that have compact factors, are closed subschemes of integral canonical models of Siegel modular varieties.

Key words: finite fields, Shimura varieties, reductive group schemes, integral models, abelian varieties, p-divisible groups, isogenies, complex multiplication, F-crystals, deformation theories, stratifications, and Newton polygons.

MSC 2000: 11G10, 14G15, 11G18, 11G35, 14F30, 14F55, 14K22, 14L05, and 14L15.

Contents

1 The assumption "have compact factors" can be removed throughout the paper; however, in this first version of the paper, it has been incorporated in order to keep the arguments short.

1. Introduction

Let $p \in \mathbb{N}$ be a prime. Let $\mathbb{F}$ be an algebraic closure of the field $\mathbb{F}_p$ with p elements. Let $W(\mathbb{F})$ be the ring of Witt vectors with coefficients in $\mathbb{F}$ and let $B(\mathbb{F})$ be the field of fractions of $W(\mathbb{F})$. In this paper we study abelian varieties over $\mathbb{F}$. In all that follows, by an *algebraic cycle* on an abelian variety $\triangle$ we mean a $\mathbb{Q}$–linear combination of irreducible subvarieties of $\triangle$. The main two goals of the paper are:

(a) to show that many crystalline cycles on abelian varieties over $\mathbb{F}$ are in fact algebraic (i.e., are crystalline realizations of algebraic cycles);

(b) to get combinatorial descriptions of certain isogeny classes of principally polarized abelian varieties over $\mathbb{F}$ endowed with families of crystalline (presumed algebraic) cycles.

Behind (a) and (b) are the *Tate* and *Langlands–Rapoport conjectures* on special fibres of *integral canonical models* of *Shimura varieties of Hodge type*. In order to describe these two conjectures and our results pertaining to them, we will first review basic things on integral canonical models of Shimura varieties.

1.1. A review of Shimura varieties of Hodge type. Let (W,ψ) be a symplectic space over $\mathbb{Q}$ and let $(G,X) \hookrightarrow (\boldsymbol{GSp}(W,\psi),S)$ be an injective map of Shimura pairs. Let $d \in \mathbb{N}$ be such that $\dim(W) = 2d$. Let $(v_\alpha)_{\alpha\in\mathcal{J}}$ be a family of tensors of the tensor algebra $\mathcal{T}(\mathrm{End}(W))$ of $\mathrm{End}(W) = W \otimes_{\mathbb{Q}} W^*$ such that G is the subgroup of $\boldsymbol{GL}_W$ that fixes each v_α with $\alpha \in \mathcal{J}$. Let L be a $\mathbb{Z}$-lattice of W such that we have a perfect alternating form $\psi : L \times L \to \mathbb{Z}$. We assume that:

(*) The Zariski closure $G_{\mathbb{Z}_{(p)}}$ of G in $\boldsymbol{GL}_{L\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$.

Let $E(G,X)$ be the reflex field of (G,X). Let $\mathrm{Sh}(G,X)$ be the canonical model over $E(G,X)$ of the complex Shimura variety defined by (G,X), cf. [De1]. Let $\mathbb{A}_f := \widehat{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{Q}$ be the ring of finite adèles of $\mathbb{Q}$. Let $\mathbb{A}_f^{(p)}$ be the ring of finite adèles of $\mathbb{Q}$ with the p-component omitted; we have $\mathbb{A}_f = \mathbb{Q}_p \times \mathbb{A}_f^{(p)}$. The group $G(\mathbb{A}_f)$ acts naturally on $\mathrm{Sh}(G,X)$. For a compact subgroup $\dagger$ of $G(\mathbb{A}_f)$, let $\mathrm{Sh}_\dagger(G,X)$ be the quotient of $\mathrm{Sh}(G,X)$ by $\dagger$.

Let $K_p := \boldsymbol{GSp}(L,\psi)(\mathbb{Z}_p)$ and $H_p := K_p \cap G(\mathbb{Q}_p) = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$; they are hyperspecial subgroups of $\boldsymbol{GSp}(L,\psi)(\mathbb{Q}_p)$ and $G(\mathbb{Q}_p)$ (respectively). We have $\mathrm{Sh}(G,X)(\mathbb{C}) = G(\mathbb{Q})\backslash[X \times G(\mathbb{A}_f)]$ and $\mathrm{Sh}(\boldsymbol{GSp}(W,\psi),S)(\mathbb{C}) = \boldsymbol{GSp}(W,\psi)(\mathbb{Q})\backslash[S \times \boldsymbol{GSp}(W,\psi)(\mathbb{A}_f)]$, cf. [De2]. We also have a functorial closed embedding $\mathrm{Sh}(G,X) \to \mathrm{Sh}(\boldsymbol{GSp}(W,\psi),S)_{E(G,X)}$, cf. [De1, Cor. 5.4]. It is easy to see that the last two sentences imply that we have as well a functorial closed embedding

$$\mathrm{Sh}_{H_p}(G,X) \hookrightarrow \mathrm{Sh}_{K_p}(\boldsymbol{GSp}(W,\psi),S)_{E(G,X)}.$$

1.1.1. Hodge cycles. We will use the terminology of [De3] on *Hodge cycles* on an abelian scheme B over a reduced $\mathbb{Q}$–scheme Y. Thus we write each Hodge cycle v on B as a pair $(v_{\mathrm{dR}}, v_{\acute{e}t})$, where v_{dR} and $v_{\acute{e}t}$ are the de Rham and the étale component of v (respectively). The étale component $v_{\acute{e}t}$ as its turn has an l-component $v_{\acute{e}t}^l$, for each rational prime l.

In what follows we will be interested only in Hodge cycles on B that involve no Tate twists and that are tensors of different tensor algebras $\mathcal{T}(\ddagger)$ (with $\ddagger$ a finite dimensional vector space). Accordingly, if Y is the spectrum of a field E, then in applications $v_{ét}^p$ will be a suitable $\mathrm{Gal}(\overline{E}/E)$-invariant tensor of $\mathcal{T}(\mathrm{End}(H^1_{ét}(B_{\overline{Y}}, \mathbb{Q}_p)))$, where $\overline{Y} := \mathrm{Spec}(\overline{E})$. If moreover $\overline{E}$ is a subfield of $\mathbb{C}$, then we will also use the Betti realization v_B of v: it is a tensor of $\mathcal{T}(\mathrm{End}(H^1((B \times \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}}, \mathbb{Q})))$ that corresponds to v_{dR} (resp. to $v_{ét}^l$) via the canonical isomorphism that relates the Betti cohomology of $(B \times_Y \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}}$ with $\mathbb{Q}$–coefficients with the de Rham (resp. the $\mathbb{Q}_l$ étale) cohomology of $B_{\overline{Y}}$ (see [De3, Sect. 2]). We recall that v_B is also a tensor of the F^0-filtration of the Hodge filtration of $\mathcal{T}(\mathrm{End}(H^1((B \times_Y \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}}, \mathbb{C})))$.

1.1.2. Integral canonical models. Let v be a prime of $E(G, X)$ that divides p. Let $O_{(v)}$ be the localization of the ring of integers of $E(G, X)$ at v. It is known that property 1.1 (*) implies that $O_{(v)}$ is an étale $\mathbb{Z}_{(p)}$-algebra, cf. [Mi4, Cor. 4.7 (a)].

Let $\mathcal{M}$ be Mumford's moduli scheme over $\mathbb{Z}_{(p)}$ that parametrizes isomorphism classes of principally polarized abelian schemes of relative dimension d over $\mathbb{Z}_{(p)}$-schemes that have compatibly level-N symplectic similitude structures for all $N \in \mathbb{N}$ prime to p, cf. [MFK, Thms. 7.9 and 7.10]. One can identify naturally $\mathrm{Sh}_{K_p}(\boldsymbol{GSp}(W, \psi), S) = \mathcal{M}_{\mathbb{Q}}$ (cf. [De1, Prop. 4.17]) and thus one can speak about the Zariski closure $\mathcal{N}^{\mathrm{cl}}$ of $\mathrm{Sh}_{H_p}(G, X)$ in $\mathcal{M}_{O_{(v)}}$.

Let $\mathcal{N}$ be the normalization of $\mathcal{N}^{\mathrm{cl}}$. It is known that $\mathcal{N}$ is a regular, formally smooth $O_{(v)}$-scheme and that it is the integral canonical model of $\mathrm{Sh}_{H_p}(G, X)$ over $O_{(v)}$ or of the Shimura quadruple (G, X, H_p, v) in the strongest sense of [Va1, Def. 3.2.3 6)] (for $p \geq 5$, see [Va1, Thm. 6.4.1 and Subsubsect. 3.2.12] and [Va5]; for p arbitrary see [Va11, Parts I and II]). We emphasize that the morphism $\mathcal{N} \to \mathcal{M}_{O_{(v)}}$ (equivalently, $\mathcal{N} \to \mathcal{N}^{\mathrm{cl}}$) is in fact finite (see [Va1, proof of Prop. 3.4.1], [Va6, Prop. 3 (b)], [Va11, Part I, Prop. 2.2.1 (b)]) and not only pro-finite.

1.1.3. Moduli interpretation. Let $(\mathfrak{A}, \lambda_{\mathfrak{A}})$ be the principally polarized abelian scheme over $\mathcal{N}$ which is the natural pull back of the universal principally polarized abelian scheme over $\mathcal{M}$. For each $N \in \mathbb{N}$ prime to p, one has a natural level-N symplectic similitude structure on $(\mathfrak{A}, \lambda_{\mathfrak{A}})$ that is defined by an isomorphism $\eta_N : (L/NL)_{\mathcal{N}} \xrightarrow{\sim} \mathfrak{A}[N]$ of constant étale group schemes over $\mathcal{N}$. If $N_1 \in N\mathbb{N}$, then η_N is naturally induced by η_{N_1}.

The choice of the $\mathbb{Z}$-lattice L of W and of the family of tensors $(v_\alpha)_{\alpha \in \mathcal{J}}$ allows a moduli interpretation of $\mathrm{Sh}(G, X)$, cf. [De1], [De2], [Mi4], and [Va1, Subsect. 4.1 and Lemma 4.1.3]. For instance, $\mathrm{Sh}(G, X)(\mathbb{C}) = G(\mathbb{Q}) \backslash [X \times G(\mathbb{A}_f)]$ is the set of isomorphism classes of principally polarized abelian varieties over $\mathbb{C}$ of dimension d, that carry a family of Hodge cycles indexed by the set $\mathcal{J}$, that have compatible level-N symplectic similitude structures for all $N \in \mathbb{N}$, and that satisfy some additional axioms. This interpretation endows the abelian scheme $\mathfrak{A}_{E(G,X)}$ with a family $(w_\alpha^{\mathfrak{A}})_{\alpha \in \mathcal{J}}$ of Hodge cycles; all realizations of pulls back of $w_\alpha^{\mathfrak{A}}$ via $\mathbb{C}$-valued points of $\mathcal{N}_{E(G,X)}$ correspond naturally to v_α.

1.1.4. On $\mathbb{F}$-valued points. We consider a point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$. Let $(A, \lambda_A) := y^*(\mathfrak{A}, \lambda_{\mathfrak{A}})$. Let (M, ϕ, ψ_M) be the principally quasi-polarized (contravariant) Dieudonné module of the principally quasi-polarized p-divisible group of (A, λ_A). For each $N \in \mathbb{N}$ prime to p, let $\eta_{N,y} := y^*(\eta_N)$; it is an isomorphism $(L/NL)_{\mathbb{F}} \xrightarrow{\sim} A[N]$ of constant étale group schemes over $\mathbb{F}$ that defines a level-N symplectic similitude structure on (A, λ_A).

For each $\alpha \in \mathcal{J}$, let t_α be the tensor of the tensor algebra $\mathcal{T}(\mathrm{End}(M[\frac{1}{p}]))$ of $\mathrm{End}(M[\frac{1}{p}])$ which is the crystalline realization of the Hodge cycle $\tilde{z}^*(w_\alpha^{\mathfrak{A}})$ on the lift $\tilde{z}^*(\mathfrak{A})$ of A defined by a (any) lift $\tilde{z} : \mathrm{Spec}(V) \to \mathcal{N}$ of y, with V as a finite, discrete valuation ring extension of $W(\mathbb{F})$. We conclude that to the point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ one associates naturally a septuple

$$(A, \lambda_A, M, \phi, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M, (\eta_{N,y})_{N \in \mathbb{N} \setminus p\mathbb{N}}).$$

If $l \in \mathbb{N}$ is a prime different from p, then the sequence of isomorphisms $(\eta_{l^n,y})_{n \in \mathbb{N}}$ defines naturally a $\mathbb{Q}_l$-linear isomorphism $W \otimes_{\mathbb{Q}} \mathbb{Q}_l \xrightarrow{\sim} (\mathrm{proj.lim.}_{n \in \mathbb{N}} A[l^n]) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ which maps v_α to the l-adic component of the étale component of the Hodge cycle $\tilde{z}^*(w_\alpha^{\mathfrak{A}})$ on $\tilde{z}^*(\mathfrak{A})$. If a is a $\mathbb{Q}$–endomorphism of A, then the mentioned $\mathbb{Q}_l$-linear isomorphism allows us to view naturally the l-adic realization of a as an element of $\boldsymbol{GSp}(W, \psi)(\mathbb{Q}_l)$. In particular, it makes sense to say that a is or is not an element of $G(\mathbb{Q}_l)$ (or of $G(\mathbb{A}_f^{(p)})$).

Let $L^* := \mathrm{Hom}(L, \mathbb{Z})$. If $p > 2$, then there exists a $W(\mathbb{F})$-linear isomorphism (cf. [Va9, Thm. 1.2 and Rm. 4.4 (a)])

$$(M, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M) \xrightarrow{\sim} (L^* \otimes_{\mathbb{Z}} W(\mathbb{F}), (v_\alpha)_{\alpha \in \mathcal{J}}, \psi). \tag{1}$$

In what follows, we will assume that:

(**) If $p = 2$, then such an isomorphism (1) exists for each point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$.[2]

Let $\mathcal{G}$ be the Zariski closure in $\boldsymbol{GL}_M$ of the subgroup of $\boldsymbol{GL}_{M[\frac{1}{p}]}$ that fixes each t_α with $\alpha \in \mathcal{J}$. From (1) and (*) we get that $\mathcal{G}$ is a reductive group scheme over $W(\mathbb{F})$ isomorphic to $G_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} W(\mathbb{F})$. It is well known that under the natural action of ϕ on $\mathcal{T}(\mathrm{End}(M[\frac{1}{p}]))$, each t_α is fixed by ϕ (see [Va1, Rm. 5.6.5], [Va6, Subsect. 8.1], or [Va8, Cor. 5.1.6]). Therefore the Lie algebra $\mathrm{Lie}(\mathcal{G})[\frac{1}{p}]$ is left invariant by ϕ (i.e., $(\mathrm{Lie}(\mathcal{G})[\frac{1}{p}], \phi)$ is a Lie sub-F-isocrystal of $(\mathrm{End}(M)[\frac{1}{p}], \phi)$). Each lift $z : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}$ of y defines naturally a direct summand F^1 of M such that we have $\phi(\frac{1}{p}F^1 + M) = M$ and $\psi(F^1, F^1) = 0$ i.e., such that the quadruple (M, F^1, ϕ, ψ_M) is a principally quasi-polarized F-crystal over $\mathbb{F}$. More precisely, F^1 is the Hodge filtration of (M, ϕ) defined by $z^*(\mathfrak{A})$. The inverse of the canonical split cocharacter of (M, ϕ) defined in [Wi, p. 512] fixes each t_α (cf. the functorial aspects of [Wi, p. 513]) and thus it factors through a cocharacter $\mu : \mathbb{G}_m \to \mathcal{G}$ which is a *Hodge cocharacter* of $(M, \phi, \mathcal{G})$ in the sense of [Va8, Def. 1.2.1]. One gets that the triple $(M, \phi, \mathcal{G})$ is a *Shimura F-crystal* over $\mathbb{F}$ in the sense of loc. cit. One says that $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ is a *basic point* if all Newton polygon slopes of $(\mathrm{Lie}(\mathcal{G})[\frac{1}{p}], \phi)$ are 0.

The following conjecture is only an adèlic (and weaker) variation of the classical Tate conjecture for abelian varieties over $\mathbb{F}$.

1.2. Conjecture (Tate conjecture for $\mathbb{F}$-valued points of integral canonical models of Shimura varieties of Hodge type). *Let $y : Spec(\mathbb{F}) \to \mathcal{N}$ and we use the above*

2 We emphasize that (**) follows from [Va11]. In this first version, we will check this (see Lemma 2.3.5) for the case when (G, X) has compact factors in the sense recalled below before Theorem 1.5. Due to this, (in this first version) in all that follows we will not come back to (**) i.e., we will use (**) without any extra comment.

notations. We assume that $(v_\alpha)_{\alpha\in\mathcal{J}}$ is the family of all tensors of $\mathcal{T}(End(W))$ fixed by G. Then each t_α is the crystalline realization of an algebraic cycle $c_{\alpha,A}$ on A.

One philosophy of [Va10] (see [Va10, Sects. 2.6 and 9.5]) can be captured as follows.

1.3. Philosophy. *To prove Conjecture 1.2 for all $\mathbb{F}$-valued points of integral canonical models of Shimura varieties of Hodge type, it suffices to prove Conjecture 1.2 for all $\mathbb{F}$-valued basic points of integral canonical models of Shimura varieties of Hodge type.*

1.4. The isogeny set of y. Let $\mathfrak{E}(y)$ be the set of those elements $h\in\mathcal{G}(B(\mathbb{F}))$ such that the pair $(h(M),\phi)$ is a Dieudonné module over k and there exists an element $h_1\in\mathcal{G}(W(\mathbb{F}))$ for which we have $\phi^{-1}(ph(M)) = hh_1(\phi^{-1}(pM))$. Let

$$\mathfrak{E}_0(y) := \{h\in\mathfrak{E}(y)|h \text{ fixes } \psi_M\}.$$

For each element $h\in\mathfrak{E}_0(y)$, let $(A(h),\lambda_{A(h)})$ be the principally quasi-polarized abelian variety over $\mathbb{F}$ which is $\mathbb{Z}[\frac{1}{p}]$-isogenous to (A,λ_A) and whose principally quasi-polarized Dieudonné module is canonically identified with $(h(M),\phi,\psi_M)$. Let $y(h):\mathrm{Spec}(\mathbb{F})\to\mathcal{M}$ be the $\mathbb{F}$-valued point defined by $(A(h),\lambda_{A(h)})$ and the compatible level-N symplectic similitude structures on it obtained naturally from those on (A,λ_A) associated to the point $y:\mathrm{Spec}(\mathbb{F})\to\mathcal{N}\to\mathcal{M}$ (here $N\in\mathbb{N}$ is prime to p). The following isogeny property was defined and announced to hold in [Va10] (recently Kisin also announced a proof of it).

1.4.1. Definition. We say that the *isogeny property* holds for a point $y:\mathrm{Spec}(\mathbb{F})\to\mathcal{N}$, if for each element $h\in\mathfrak{E}_0(y)$ the point $y(h):\mathrm{Spec}(\mathbb{F})\to\mathcal{M}$ factors naturally through a morphism $y(h):\mathrm{Spec}(\mathbb{F})\to\mathcal{N}$ in such a way that for a (any) lift $z(h):\mathrm{Spec}(W(\mathbb{F}))\to\mathcal{N}$ of it, every tensor $t_\alpha\in\mathcal{T}(\mathrm{End}(M[\frac{1}{p}]))$ is the crystalline realization of the Hodge cycle $z(h)^*(w_\alpha^{\mathfrak{A}})$ on $z(h)^*(\mathfrak{A})$.

If such a factorization $y(h):\mathrm{Spec}(\mathbb{F})\to\mathcal{N}$ exists, then it is unique (cf. [Va1, Rm. 5.6.4]; see Lemma 2.3.3).

1.4.2. Definition. We assume that the isogeny property holds for $y:\mathrm{Spec}(\mathbb{F})\to\mathcal{N}$. Then by the *isogeny set* of y we mean the set $o(y)$ of $\mathbb{F}$-valued points of $\mathcal{N}$ which are $G(\mathbb{A}_f^{(p)})$-translate of points in the set $\{y(h)|h\in\mathfrak{E}_0(y)\}$. Let $o^{\mathrm{big}}(y)$ be the isogeny set of the $\mathbb{F}$-valued of $\mathcal{M}$ defined by y.

Obviously, for each point $\tilde{y}\in o(y)$ we have $o(\tilde{y})=o(y)$. Therefore, if one knows that the isogeny property holds for all $\mathbb{F}$-valued points of $\mathcal{N}$, then one gets a natural disjoint decomposition into $G(\mathbb{A}_f^{(p)})$-equivariant sets

$$\mathcal{N}(\mathbb{F}) = \coprod_{o(y)\in\mathfrak{I}} o(y), \tag{2}$$

where $\mathfrak{I} := \{o(y)|y\in\mathcal{N}(\mathbb{F})\}$. Let Θ be the Frobenius automorphism of $\mathbb{F}$ whose fixed field is the residue field $k(v)$ of v. It acts naturally on the set $\mathcal{N}(\mathbb{F})$ in such a way that: (i) each isogeny set $o(y)$ is left invariant by Θ and (ii) it commutes with the right action of $G(\mathbb{A}_f^{(p)})$ on the set $\mathcal{N}(\mathbb{F})$. Thus, (2) is in fact a decomposition into $\mathbb{Z}\Theta \times G(\mathbb{A}_f^{(p)})$-equivariant sets.

1.4.3. Definition. Let $\mathcal{E}$ be the Zariski closure in the group of invertible elements of $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ (i.e., in the reductive group $\boldsymbol{Aut}(A)_{\mathbb{Q}}$ over $\mathbb{Q}$) of the set of those $\mathbb{Q}$–automorphisms a of A that satisfy the following two properties:

(i) The crystalline realization of a is a $B(\mathbb{F})$-valued point of $\mathcal{G}$.

(ii) For each prime $l \neq p$, the l-adic realization of a is a $\mathbb{Q}_l$-valued point of G (this makes sense, cf. second paragraph of Subsubsection 1.1.4).

Due to the property (ii), the l-adic realizations allow us to view naturally $\mathcal{E}(\mathbb{Q})$ as a subgroup of $G(\mathbb{A}_f^{(p)})$. Let $\widetilde{\mathcal{E}(\mathbb{Q})}$ be the group of those $\mathbb{Q}$–valued points of $\mathcal{E}$ which act on the $\mathbb{Q}$–span of λ (equivalently of ψ_M) via $\mathbb{G}_m(\mathbb{Z}_{(p)})$-multiples.

1.4.4. Definition. We say that the *endomorphism property* holds for the point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$, if $\mathcal{E}$ is a reductive group over $\mathbb{Q}$ of the maximum possible dimension i.e., the crystalline realizations of elements of $\mathcal{E}(\mathbb{Q})$ allow us to view $\mathcal{E}_{B(\mathbb{F})}$ as a reductive subgroup of $\mathcal{G}_{B(\mathbb{F})}$ with the property that we have an identity

$$\mathrm{Lie}(\mathcal{E}) \otimes_{\mathbb{Q}} \mathbb{Q}_p = \{x \in \mathrm{Lie}(\mathcal{G})[\frac{1}{p}] | \phi(x) = x\}.$$

Here and in all that follows, the action of $\mathcal{E}$ on $M[\frac{1}{p}]$ is the composite of the standard isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{E}^{\mathrm{opp}}$ with the natural functorial action of $\mathcal{E}^{\mathrm{opp}}$ on $M[\frac{1}{p}]$, where $\mathcal{E}^{\mathrm{opp}}$ is the opposite group of $\mathcal{E}$. If $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ is a basic point, then the endomorphism property holds for it if and only if the group $\mathcal{E}$ is a $\mathbb{Q}$–form of $\mathcal{G}_{B(\mathbb{F})}$.

1.4.5. Definition. We say the *CM lift property* holds for the point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$, if there exists an element $h \in \mathfrak{E}_0(y)$ for which the morphism $y(h) : \mathrm{Spec}(\mathbb{F}) \to \mathcal{M}$ factors through $\mathcal{N}$ as in Definition 1.4.1 and for which there exist a finite, discrete valuation ring extension V of $W(\mathbb{F})$ and a lift $\tilde{z}(h) : \mathrm{Spec}(V) \to \mathcal{N}$ of $y(h) : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ whose generic fibre is a special point (i.e., the abelian scheme $\tilde{z}(h)^*(\mathfrak{A})$ over V has complex multiplication). If moreover:

(a) the Tate and Hodge conjectures hold for the generic fibre of $\tilde{z}(h)^*(\mathfrak{A})$, then we say that the *Tate–Hodge property* holds for the point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$;

(b) we can take $V = W(\mathbb{F})$, then we say that the *unramified CM lift property* holds for the point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$.

1.4.6. The Langlands–Rapoport conjecture. We define

$$\mathfrak{X}_{0,p}(y) := \{h(M) | h \in \mathfrak{E}_0(y)\} = \mathrm{Im}(\mathfrak{E}_0(y) \to \mathcal{G}(B(\mathbb{F}))/\mathcal{G}(W(\mathbb{F}))).$$

Due to the property 1.4.3 (i), the group $\widetilde{\mathcal{E}(\mathbb{Q})}$ acts via crystalline realizations on $\mathfrak{X}_{0,p}(y)$. Due to the isomorphism (1), one can naturally identify $\mathfrak{X}_{0,p}(y)$ with a certain subset of $G_{\mathbb{Z}_{(p)}}(B(\mathbb{F}))/G_{\mathbb{Z}_{(p)}}(W(\mathbb{F}))$ which is uniquely determined up to $G_{\mathbb{Z}_{(p)}}(W(\mathbb{F}))$-conjugation.

Using such an identification, one gets a natural action of Θ on the set $\mathfrak{X}_{0,p}(y)$. Thus Θ also acts on the product set $\mathfrak{X}_{0,p}(y) \times G(\mathbb{A}_f^{(p)})$ and on the quotient set $\widetilde{\mathcal{E}(\mathbb{Q})}\backslash[\mathfrak{X}_{0,p}(y) \times G(\mathbb{A}_f^{(p)})]$.

If the isogeny property holds for $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$, then Faltings deformation theory implies that we have a natural bijection of $\mathbb{Z}\Theta \times G(\mathbb{A}_f^{(p)})$-equivariant sets

$$o(y) \xrightarrow{\sim} \widetilde{\mathcal{E}(\mathbb{Q})}\backslash[\mathfrak{X}_{0,p}(y) \times G(\mathbb{A}_f^{(p)})] \tag{3}$$

with the property that y is mapped to the equivalence class $[(1_M, 1_W)]$ defined by $(1_M, 1_W) \in \mathfrak{X}_{0,p}(y) \times G(\mathbb{A}_f^{(p)})$ (see Corollary 2.3.4). The $\mathbb{Z}$-equivariant part of the isomorphism (3) follows from its well known equivalent for the set $o^{\mathrm{big}}(y)$ (see [Mi2]) and from the fact that the restriction of the natural map $o(y) \to o^{\mathrm{big}}(y)$ to the set of equivalence classes of the form $[(h, 1_W)]$ with $h \in \mathfrak{X}_{0,p}(y)$, is injective. If the isogeny property, the endomorphism property, and (in the case when y is not basic) the CM lift property hold for y, then the natural map $o(y) \to o^{\mathrm{big}}(y)$ is injective (cf. Corollary 2.3.5). The Langlands–Rapoport conjecture (see [LR], [Mi3], and [Pf]) predicts that the following three things hold.[3]

(a) The isogeny property holds for every point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ (thus formulas (2) and (3) also hold).

(b) The endomorphism property and the CM property hold for every point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$.

(c) The set $\mathfrak{I}$ is in natural bijection to the set of $G(\overline{\mathbb{Q}})$-conjugates (i.e., of isomorphism classes) $[\kappa]$ of *admissible homomorphisms* $\kappa : \mathfrak{P} \to \mathfrak{G}$ of $\overline{\mathbb{Q}}/\mathbb{Q}$–groupoids, where $\mathfrak{P}$ is the pseudo-motivic groupoid of $\mathbb{F}$ as in [Mi1-5] and $\mathfrak{G}$ is the $\overline{\mathbb{Q}}/\mathbb{Q}$–groupoid defined by G. If $[\kappa]$ maps to $o(y) \in \mathfrak{I}$, then moreover $\widetilde{\mathcal{E}(\mathbb{Q})}$ is naturally identified with the group of those elements of $G(\overline{\mathbb{Q}})$ which act on ψ via a $\mathbb{G}_m(\mathbb{Z}_{(p)})$-multiple and which fix κ under inner conjugation.

More recently, the manuscript [Mi5] speaks about the unconditional form of the Langlands–Rapoport conjecture: briefly speaking, this is the same as above but in (c) one requires the admissibility to be in the strongest possible way (that determines the isomorphism classes uniquely). In this first version of the paper, we will not be very precise in differentiating between the conditional and the unconditional form of the Langlands–Rapoport conjecture. We will only say that the conditional form of the Langlands–Rapoport conjecture and suitable forms of the Tate–Hodge property imply the unconditional form of the Langlands–Rapoport conjecture.

We denote by $\star^{\mathrm{ad}}$ and $\star^{\mathrm{der}}$ the adjoint and the derived group schemes (respectively) of a reductive group scheme $\star$. For Shimura types we refer to [Mi4] and [Va3]. Let $(\tilde{G}^{\mathrm{ad}}, \tilde{X}^{\mathrm{ad}})$ be the adjoint Shimura pair of a Shimura pair $(\tilde{G}, \tilde{X})$, cf. [Va1, Subsubsect. 2.4.1]. Following [Va4, Def. 1.1 and Subsect. 2.2], we say that a simple, adjoint Shimura pair (G_0, H_0) has *compact factors*, if there exists a simple factor of $G_{0,\mathbb{R}}$ which is compact. Moreover, we say that the Shimura pair $(\tilde{G}, \tilde{X})$ has compact factors, if each simple, non-trivial factor of the adjoint Shimura pair $(\tilde{G}^{\mathrm{ad}}, \tilde{X}^{\mathrm{ad}})$ has compact factors.

3 The original Langlands–Rapoport conjecture worked with a slightly different but equivalent form of (3) (see Subsection 5.6 for details in a very concrete case).

We recall that the abelian variety A is called *supersingular* if and only if all Newton polygon slopes of (M, ϕ) are $\frac{1}{2}$ (equivalently, if and only if all Newton polygon slopes of $(\mathrm{End}(M)[\frac{1}{p}], \phi)$ are 0).

We have the following basic result which is the very essence of all our main results.

1.5. Basic Theorem. *Let (G_0, H_0) be an arbitrary simple, adjoint Shimura pair of A_n, B_n, C_n, or $D_n^{\mathbb{R}}$ Shimura type. We assume that the group $G_{0,\mathbb{Q}_p}$ is unramified (i.e., it extends to a reductive group scheme over $\mathbb{Z}_p$). Then there exists an injective map $(G, X) \hookrightarrow (\boldsymbol{GSp}(W, \psi), S)$ of Shimura pairs such that the following eight properties hold (for each prime v of $E(G, X)$ that divides p):*

(a) *We have $(G_0, X_0) = (G^{\mathrm{ad}}, X^{\mathrm{ad}})$ and the derived group G^{der} of G is simply connected.*

(b) *There exists a $\mathbb{Z}$-lattice L of W such that the property (*) holds (thus we can use the notations introduced above, with v an arbitrary prime of $E(G, X)$ that divides p).*

(c) *Let $y : Spec(\mathbb{F}) \to \mathcal{N}$ be a basic point. If (G_0, X_0) is of B_n, C_n, or $D_n^{\mathbb{R}}$ type, then the abelian variety A is supersingular and thus the Tate conjecture holds for A.*

(d) *If (G_0, X_0) is of B_n, C_n, or $D_n^{\mathbb{R}}$ type, then the Conjecture 1.2 holds for each basic point $y : Spec(\mathbb{F}) \to \mathcal{N}$. If (G_0, X_0) if of A_n type and if a certain technical condition (***) (defined in Subsection 5.3 (f)) holds (resp. and the technical condition (***) does not hold), then the Conjecture 1.2 holds for each basic point $y : Spec(\mathbb{F}) \to \mathcal{N}$ (resp. we can choose $(v_\alpha)_{\alpha \in \mathcal{J}}$ such that each v_α is an endomorphism of $End(W)$).*

(e) *The isogeny property, the endomorphism property, and the unramified CM lift property hold for all basic points $y : Spec(\mathbb{F}) \to \mathcal{N}$ and in fact all $\mathbb{F}$-valued basic points of $\mathcal{N}$ form one isogeny set. If moreover (G_0, X_0) has compact factors, then the isogeny property holds for all $\mathbb{F}$-valued points of $\mathcal{N}$.*

(f) *There exists a $G(\mathbb{A}_f^{(p)})$-invariant, open subscheme $\mathcal{L}$ of $\mathcal{N}$ which contains all $\mathbb{F}$-valued basic points and which is as well an open subscheme of $\mathcal{N}^{\mathrm{cl}}$. If moreover (G_0, X_0) has compact factors, then in fact we have $\mathcal{L} = \mathcal{N} = \mathcal{N}^{\mathrm{cl}}$.*

(g) (due in essence to Zink and Milne) *We assume that (G_0, X_0) has compact factors. If the endomorphism property and the CM lift property hold for all points $y : Spec(\mathbb{F}) \to \mathcal{N}$, then the (conditional) Langlands–Rapoport conjecture holds for $\mathcal{N}$.*

(h) *If (G_0, X_0) if of A_n type, then we assume that the technical condition (***) holds. Then the unconditional form of the Langlands–Rapoport conjecture holds for the isogeny set of basic points of $\mathcal{N}$.*

Part (d) represents progress towards the proof of Conjecture 1.2. The guiding idea behind (d) is to check that, modulo "relative PEL tricks" as in [Va1, Subsubsect. 4.3.16], the Tate–Hodge property holds for one basic point of $\mathcal{N}$ (cf. Lemma 2.5 and Subsection 5.3 (b) and (c)). Part (f) is a completely new result (before it was known only when (G_0, X_0) is of either C_n type without compact factors or A_n type; see [Zi], [Va3, Cor. 4.10], and [Va5, Prop. 3.2]). Part (e) brings to fruition the program outlined in [Va10, Rm. 9.2.1 (b)]. Part (g) follows from (e), (2), and (3), cf. [Zi] and [Mi3]. In Section 6 we check that if (G_0, X_0) has compact factors, then the endomorphism property and the unramified

CM lift property hold as well for all points $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ (see Corollary 6.4 (a) whose essence is contained in the proof of Corollary 6.2). The proof of Corollary 6.2 involves all Shimura pairs (G_1, X_1) of Hodge type that have compact factors and goes by *induction* on the dimension of the hermitian symmetric domain X_1 (this explains why (g) is stated as it is).

For the sake of completeness, in this paragraph we assume that (G_0, X_0) does not have compact factors (i.e., we assume that each simple factor of $G_{0,\mathbb{R}}$ is non-compact) and we recall what is known in this case. If (G_0, X_0) is of A_n or C_n type, then one can assume that the injective map $(G, X) \hookrightarrow (\boldsymbol{GSp}(W, \psi), S)$ is a PEL type embedding. More precisely, one can assume that $G_{\mathbb{Z}_{(p)}}$ is the closed subgroup scheme of $\boldsymbol{GSp}(L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, \psi)$ that fixes each element of a semisimple $\mathbb{Z}_{(p)}$-subalgebra of $\mathrm{End}(L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$ (see Subsection 5.1 (d) and (f)). Thus if (G_0, X_0) is of A_n or C_n type, then it is well known that the endomorphism property holds for all points $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ and in [Zi, Thm. 4.4] it is proved that a slightly weaker version of the isogeny and the CM lift properties hold for all points $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ (if the Hasse principle holds for G –like if (G_0, X_0) is of either A_n type with n odd or C_n type–, then loc. cit. implies that the isogeny and the CM lift properties hold for all points $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$).

1.6. On contents. Section 2 contains preliminary material on tori, Faltings deformation theory, and on the Tate–Hodge property. In Section 3 we recall basic things on three stratifications of $\mathcal{N}_{k(v)}$. In Section 4 we present reduction steps towards the proof of the isogeny property. In Section 5 we prove Theorem 1.5. Section 6 contains our main applications. More precisely, in Section 6 we combine Theorem 1.5 with results of Milne and Pfau (see [Mi3] and [Pf]) to get general forms of Theorem 1.5 (e), (f), and (g) that do not depend on a particular choice of the injective map $(G, X) \hookrightarrow (\boldsymbol{GSp}(W, \psi), S)$ and that do not require any extra assumption besides "compact factors". In particular, we prove the conditional Langlands–Rapoport conjecture for all special fibres of integral canonical models of Shimura varieties of A_n, B_n, C_n, and $D_n^{\mathbb{R}}$ type that have compact factors (see Corollary 6.2). Several proofs (like the proofs of Theorem 1.5 (c), (d), and (f)) apply as well for large classes of Shimura varieties of $D_n^{\mathbb{H}}$ type (the so called unmixed $D_n^{\mathbb{H}}$ type).[4]

1.7. Leitfaden. For reader's convenience, we include the following diagram of implications between the different properties we have introduced in Subsection 1.4. We will abbreviate "property" by "P" and "for basic points" by "b". Moreover the relevant Subsections are listed on top of arrows. First, we have:

$$\begin{array}{ccc} \textit{Conjecture 1.2 or Tate-Hodge P} & \xrightarrow{4.2} & \textit{Endomorphism P} \\ \downarrow & & \downarrow \\ \textit{Conjecture 1.2 or Tate-Hodge P b} & \xrightarrow{4.2} & \textit{Endomorphism P b} \end{array}$$

4 In fact, due to the relative PEL trick of [Va3, Rm. 4.8.2 (b)] (to be compared with [Va10, Thm. 9.6]), all our results extend to the $D_n^{\mathbb{H}}$ type. The mentioned trick allows us to pass directly the endomorphism property from the C_{2n} type to the $D_n^{\mathbb{H}}$ type (regardless of the fact that we are or we are not in a context with compact factors).

Second, if the derived group G^{der} of G is simply connected, then we have:

$$\textit{Endomorphism P b} \xrightarrow{4.3,4.4,C_1} \textit{Isogeny P b} \xrightarrow{4.1,C_2} \textit{Isogeny P} \xrightarrow{5.5,\ proof\ 6.2} \textit{CM lift P}.$$

Here C_1 refers to some group theoretical assumptions on the injective map $(G,X) \hookrightarrow (\boldsymbol{GSp}(W,\psi),S)$ of Shimura pairs and on the existence of a basic point of $\mathcal{N}$ for which Conjecture 1.2 holds, while C_2 refers to the assumption that (G,X) has compact factors. Third, the starting point for applying these implications is to show that:

Under good arrangements on the injective map $(G,X) \hookrightarrow (\boldsymbol{GSp}(W,\psi),S)$, we can check that the Conjecture 1.2 (and even the Tate–Hodge property in many cases) holds for one basic point of $\mathcal{N}$ (cf. Theorem 1.5 (d) proved in Subsection 5.3).

See [Va10] for an overview of previous works on the Langlands–Rapoport conjecture. We would like to thank Binghamton, Northwestern, and Bielefeld Universities for good conditions with which to write this paper. We would also like to thank Thomas Zink for some comments.

2. Preliminaries

In this Section we list our basic notations and recall basic material on tori, Faltings deformation theory, and on the Tate–Hodge property.

2.1. Notations and conventions. In all that follows, we will use the following notations p, $\mathbb{F}$, $W(\mathbb{F})$, $B(\mathbb{F})$, $(G,X) \hookrightarrow (\boldsymbol{GSp}(W,\psi),S)$, d, $(v_\alpha)_{\alpha\in\mathcal{J}}$, L, $G_{\mathbb{Z}_{(p)}}$, $E(G,X)$, $\mathrm{Sh}_\dagger(G,X)$, $K_p = \boldsymbol{GSp}(L,\psi)(\mathbb{Z}_p)$, $H_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$, v, $O_{(v)}$, $\mathcal{N}^{\mathrm{cl}}$, $\mathcal{N}$, $(\mathfrak{A},\lambda_{\mathfrak{A}})$, η_N with $N \in \mathbb{N}\setminus p\mathbb{N}$, $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$, $(A,\lambda_A,M,\phi,(t_\alpha)_{\alpha\in\mathcal{J}},\psi_M,(\eta_{N,y})_{N\in\mathbb{N}\setminus p\mathbb{N}})$, $\mathcal{G}$, $\mathfrak{E}$, $\mathfrak{E}_0(y)$, $y(h) : \mathrm{Spec}(\mathbb{F}) \to \mathcal{M}$ with $h \in \mathfrak{E}_0(y)$, $o(y)$, Θ, $o^{\mathrm{big}}(y)$, $k(v)$, $\mathcal{E}$, $\widetilde{\mathcal{E}(\mathbb{Q})}$, $\mathfrak{P}$, and $\mathfrak{G}$ we have introduced before Theorem 1.5. Always $l \in \mathbb{N}$ will be a prime different from p and $\mathcal{T}(\ddagger)$ will be the tensor algebra of some vector space $\ddagger$. Let $L_{(p)} := L\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$. If k is a perfect field that contains $\mathbb{F}_p$, let σ_k be the Frobenius automorphism of k, $W(k)$, and $B(k) := W(k)[\frac{1}{p}]$. We recall that ϕ acts naturally on $\mathrm{End}(M)[\frac{1}{p}]$ via the rule: $e \in \mathrm{End}(M)[\frac{1}{p}]$ is mapped to $\phi(e) := \phi\circ e\circ\phi^{-1}$. Thus ϕ acts in the natural tensorial way on $\mathcal{T}(\mathrm{End}(M)[\frac{1}{p}])$.

For a point $y_\star : \mathrm{Spec}(k) \to \mathcal{N}$, let $(A_\star,\lambda_{A_\star},M_\star,\phi_\star,(t_{\star,\alpha})_{\alpha\in\mathcal{J}},\psi_{M_\star},(\eta_{N,y_\star})_{N\in\mathbb{N}\setminus p\mathbb{N}})$ be the septuple that is analogous to $(A,\lambda_A,M,\phi,(t_\alpha)_{\alpha\in\mathcal{J}},\psi_M,(\eta_{N,y})_{N\in\mathbb{N}\setminus p\mathbb{N}})$ but obtained working with $y_\star$ instead of with y. Let $\mathcal{G}_\star$ be the reductive subgroup scheme of $\boldsymbol{GL}_{M_\star}$ whose generic fibre is the subgroup of $\boldsymbol{GL}_{M_\star[\frac{1}{p}]}$ that fixes $t_{\star,\alpha}$ for all $\alpha \in \mathcal{J}$.

As the elements of $Z^0(G)(\mathbb{Q})$ define $\mathbb{Q}$–endomorphisms of the universal abelian scheme $\mathfrak{A}$ over $\mathcal{N}$, we can identify naturally $Z^0(G)_{B(\mathbb{F})} = Z^0(\mathcal{G}_{B(\mathbb{F})})$.

Let (A_0,λ_{A_0}) be a principally polarized abelian variety model of A over a finite field $\mathbb{F}_{p^q}$. Let π_0 be the Frobenius endomorphism of A_0. Let (M_0,ϕ_0,ψ_{M_0}) be the principally quasi-polarized Dieudonné module of (A_0,λ_{A_0}). By taking $q >> 0$, we can assume that the following two properties hold:

(i) For each $\alpha \in \mathcal{J}$, we have $t_\alpha \in \mathcal{T}(M_0[\frac{1}{p}])$.

(ii) The Frobenius torus over $\mathbb{Q}$ generated by sufficiently high powers of π_0, has π_0 as a $\mathbb{Q}$–valued point.

The crystalline realization of ϕ_0^q is an element $\pi_0 \in \boldsymbol{GL}_{M_0}(B(\mathbb{F}_{p^q}))$ that fixes each t_α with $\alpha \in \mathcal{J}$. Thus we have $\pi_0 \in \mathcal{G}(B(\mathbb{F}))$. The point y is basic if and only if $\pi_0 \in Z^0(\mathcal{G}_{B(\mathbb{F})})$ and thus if and only if $\pi_0 \in Z^0(\mathbb{Q})$.

Let $Z(\boxdot)$ be the center of a reductive group scheme $\boxdot$. Let $Z^0(\boxdot)$ be the largest subtorus of $Z(\boxdot)$; the finite group scheme $Z(\boxdot)/Z^0(\boxdot)$ is of multiplicative type.

Let H be a group that acts from the left on two sets B and C. We assume that there exists a subgroup $\tilde{H}$ of H and a subset $\tilde{B}$ of B such that the following three properties hold: (i) we have $B = H\tilde{B}$, (ii) $\tilde{H}\tilde{B} = \tilde{B}$, and (ii) if $h \in H$ is such that $\tilde{B} \cap h\tilde{B}$ is a non-empty set, then we have $h \in \tilde{H}$. Then the natural map of quotient sets

$$\tilde{H}\backslash[\tilde{B} \times C] \to H\backslash[B \times C]. \tag{4}$$

is a bijection.

2.1.1. On $o^{\mathrm{big}}(y)$. Let $\mathfrak{E}_0^{\mathrm{big}}(y)$, $\mathcal{E}^{\mathrm{big}}$, and $\widetilde{\mathcal{E}^{\mathrm{big}}(\mathbb{Q})}$ be the analogues of $\mathfrak{E}_0(y)$, $\mathcal{E}$, and $\widetilde{\mathcal{E}(\mathbb{Q})}$ but obtained working with the composite point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N} \to \mathcal{M}$ instead of with $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$. The Langlands–Rapoport conjecture is known to hold for Siegel modular varieties, cf. [Mi2]. Therefore we have a natural $\mathbb{Z}\Theta \times \boldsymbol{GSp}(W,\psi)(\mathbb{A}_f^{(p)})$-equivariant bijection

$$o^{\mathrm{big}}(y) = \widetilde{\mathcal{E}^{\mathrm{big}}(\mathbb{Q})}\backslash[\mathfrak{E}_0^{\mathrm{big}}(y) \times \boldsymbol{GSp}(W,\psi)(\mathbb{A}_f^{(p)})].$$

2.2. Complements on tori. We recall that a torus over a field is called *anisotropic*, if it has no subtorus isomorphic to $\mathbb{G}_m$. Let T be a torus over $\mathbb{Q}$ such that we have a short exact sequence $0 \to \mathbb{G}_m \to T \to T_c \to 0$, where the torus $T_{c,\mathbb{R}}$ is compact. Thus T has a unique subtorus isomorphic to $\mathbb{G}_m$. We assume that we have a homomorphism $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to T_\mathbb{R}$ such that the subtorus $\mathbb{G}_m$ of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ maps onto the subtorus $\mathbb{G}_m$ of T. In this Subsection we are interested in the smallest subtorus $\tilde{T}$ of T such that h factors through $\tilde{T}_\mathbb{R}$. Let T_1 be a subtorus of T that contains the $\mathbb{G}_m$ subtorus of T. Let T_2 be a subtorus of T which is naturally isogenous to T/T_1 (the below assumptions will imply that such a subtorus is unique). We assume that there exists a number field F_0 and a torus $T_2^{F_0}$ over F_0 such that we have an isogeny

$$T_2 \to \mathrm{Res}_{F_0/\mathbb{Q}} T_2^{F_0}.$$

If l is a prime that splits in F_0, then we have $(\mathrm{Res}_{F_0/\mathbb{Q}} T_2^{F_0})_{\mathbb{Q}_l} = \prod_{i \in \mathrm{Hom}(F_0,\mathbb{Q}_l)} T_2^{F_0} \times_{F_0,i} \mathbb{Q}_l$.

2.2.1. Lemma. *We assume that the following three conditions hold:*

(i) *the torus $\tilde{T}$ surjects onto T/T_2 and its image in T/T_1 is non-trivial;*

(ii) *there exists a prime l which splits in F_0, for which the torus $T_{1,\mathbb{Q}_l}$ is split, and which has the property that for each $i \in Hom(F_0,\mathbb{Q}_l)$ the torus $T_2^{F_0} \times_{F_0,i} \mathbb{Q}_l$ is anisotropic and has no proper subtorus;*

(iii) *either $F_0 = \mathbb{Q}$ or $F_0 \neq \mathbb{Q}$ and there exists an element $i_0 \in Hom(F_0, \mathbb{Q}_l)$ such that the torus $T_2^{F_0} \times_{F_0,i_0} \mathbb{Q}_l$ is not isogenous to any other torus of the form $T_2^{F_0} \times_{F_0,i} \mathbb{Q}_l$ with $i \in Hom(F_0, \mathbb{Q}_l) \setminus \{i_0\}$.*

Then we have $\tilde{T} = T$.

Proof: We can assume that $T = T_1 \times_{\mathbb{Q}} T_2$ and that the isogeny $T_2 \to \mathrm{Res}_{F_0/\mathbb{Q}} T_2^{F_0}$ is an isomorphism. As $T = T_1 \times_{\mathbb{Q}} T_2$, we have natural projection homomorphisms $T \to T_1$, $T \to T_2$, $\tilde{T} \to T_1$, and $\tilde{T} \to T_2$. Let T_0 be the identity component of $\mathrm{Im}(\mathrm{Ker}(\tilde{T} \to T_1) \to T_2)$.

We first assume that $F_0 = \mathbb{Q}$. From conditions (i) and (ii) we get that $\tilde{T}$ surjects onto both T_1 and T_2. From the condition (ii) we also get that T_0 is non-trivial and in fact that it is T_2 itself. Thus $\tilde{T} = T_1 \times_{\mathbb{Q}} T_2 = T$.

We assume now that $F_0 \neq \mathbb{Q}$. From conditions (ii) and (iii) we get that T_0 is non-trivial and moreover $T_{0,\mathbb{Q}_l}$ contains the torus $T_2^{F_0} \times_{F_0,i_0} \mathbb{Q}_l$ (viewed as a direct factor of $T_{2,\mathbb{Q}_l} = (\mathrm{Res}_{F_0/\mathbb{Q}} T_2^{F_0})_{\mathbb{Q}_l} = \prod_{i \in \mathrm{Hom}(F_0,\mathbb{Q}_l)} T_2^{F_0} \times_{F_0,i} \mathbb{Q}_l$ and thus of $T_{\mathbb{Q}_l}$). As T_0 is a torus over $\mathbb{Q}$ and as $T_2 = \mathrm{Res}_{F_0/\mathbb{Q}} T_2^{F_0}$, we get that $T_0 = T_2$. Therefore $\tilde{T}$ contains T_2. From this and the condition (i) we get that $\tilde{T} = T$. $\square$

2.3. Faltings deformation theory. Let $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$. Let $z : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}$ be a lift of y. Let F^1 be the Hodge filtration of M defined by the abelian scheme $z^*(\mathfrak{A})$ over $W(\mathbb{F})$ which lifts A. Each tensor t_α belongs to the F^0-filtration of $\mathcal{T}(\mathrm{End}(M)[\frac{1}{p}])$ defined by $F^1[\frac{1}{p}]$ (in the natural sense; see [Va9, Subsect. 1.1] or [Va11, Part I, Subsect. 2.1]).

Let $\mu : \mathbb{G}_m \to \mathcal{G}$ be a cocharacter such that we have a direct sum decomposition $M = F^1 \oplus F^0$ with the properties that for each $i \in \{0, 1\}$ the torus $\mathbb{G}_m$ acts on F^i through μ via the $-i$-th power of the identitical character of $\mathbb{G}_m$ and that F^1/pF^1 is the kernel of ϕ modulo p. For instance, as μ we can take the inverse of the canonical split cocharacter of (M, F^1, ϕ) defined in [Wi, p. 512] (it factors through $\mathcal{G}$, as due to the functorial aspects of [Wi, p. 513] it fixes each t_α with $\alpha \in \mathcal{J}$).

Let $b := \dim(X)$ be the dimension of X viewed as a complex manifold. For $c \in \mathbb{N} \cup \{0\}$, let $\mathcal{R}_c := W(\mathbb{F})[[x_1, \ldots, x_c]]$. Let Φ_c be the Frobenius endomorphism of $\mathcal{R}_c$ which is compatible with $\sigma_{\mathbb{F}}$ and which takes x_i to x_i^p for all $i \in \{1, \ldots, c\}$.

Let $\mathcal{V}$ be a smooth, closed subscheme of $\mathcal{G}$ which is isomorphic to $\mathrm{Spec}(W(\mathbb{F}))[x_1, \ldots, x_b]$, which contains the identity section of $\mathcal{G}$, and for which we have an identity $\mathrm{Lie}(\mathcal{G}_{\mathbb{F}}) = \mathrm{Lie}(\mathcal{V}_{\mathbb{F}}) \oplus \mathrm{Lie}(\mathcal{P}_{\mathbb{F}})$ of $\mathbb{F}$-vector spaces, where $\mathcal{P}_{\mathbb{F}}$ is the the normalizer of F^1/pF^1 in $\mathcal{G}_{\mathbb{F}}$ and where by $\mathrm{Lie}(\mathcal{V}_{\mathbb{F}})$ we mean the tangent space of $\mathcal{F}_{\mathbb{F}}$ at the identity element (in practice, this tangent space will be an abelian Lie algebra and this is why we denoted it as a Lie algebra). For instance, we can take $\mathcal{V}$ to be the $\mathbb{G}_a^b$ subgroup scheme of $\mathcal{G}$ whose Lie algebra is the largest direct summand of $\mathrm{Lie}(\mathcal{G})$ on which $\mathbb{G}_m$ acts through μ via the identical character of $\mathbb{G}_m$. Let $\mathrm{Spf}(\mathcal{R})$ be the completion of $\mathcal{V}$ along its identity section. Let $u_{\mathrm{univ}} : \mathrm{Spec}(\mathcal{R}) \to \mathcal{V}$ be the universal (natural) morphism. We fix an identification $\mathcal{R} = \mathcal{R}_b = W(\mathbb{F})[[x_1, \ldots, x_b]]$. Let $\Phi := \Phi_b$; it is a Frobenius endomorphism of $\mathcal{R}$. The completion of the local ring of y in $\mathcal{N}$ can be identified with $\mathcal{R}$. Let $\mathcal{Y} : \mathrm{Spec}(\mathcal{R}) \to \mathcal{N}$ be the natural morphism. Faltings deformation theory asserts three basic things (cf. [Fa, Thm. 10 and rmks. after], [Va1, Subsect. 5.4], [Va11, Part I, Subsects. 3.3 and 3.4]).

(a) First it assets that the principally quasi-polarized filtered F-crystal with tensors over $\mathrm{Spec}(\mathcal{R}/p\mathcal{R})$ associated naturally to $\mathcal{Y} : \mathrm{Spec}(\mathcal{R}) \to \mathcal{N}$ is (up to $W(\mathbb{F})$-automorphisms of $\mathcal{R}$ that leave invariant the ideal $(x_1, \ldots, x_b)$) isomorphic to

$$\mathcal{C}_z := (M \otimes_{W(k)} \mathcal{R}, F^1 \otimes_{W(k)} \mathcal{R}, u_{\mathrm{univ}}(\phi \otimes \Phi), \nabla, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M),$$

where ∇ is an integrable, topologically nilpotent, versal connection on $M \otimes_{W(k)} \mathcal{R}$ which is uniquely determined by the requirement that $u_{\mathrm{univ}}(\phi \otimes \Phi)$ is ∇-parallel and which annihilates each t_α with $\alpha \in \mathcal{J}$ (see [Va1, Subsect. 5.4] and [Va11, Part I, Subsects. 3.3 and 3.4]).

(b) Second it asserts that each quintuple

$$\tilde{\mathcal{C}} := (\tilde{M}, \tilde{F}^1, \tilde{\phi}, \tilde{\nabla}, (\tilde{t}_\alpha)_{\alpha \in \mathcal{J}}, \psi_{\tilde{M}})$$

over $\mathcal{R}_c/p\mathcal{R}_c$ which modulo $(x_1, \ldots, x_c)$ is $(M, F^1, \phi, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M)$ and which satisfies the same properties as $\mathcal{C}_z$, is the pull back of $\mathcal{C}_z$ via a unique morphism $\mathrm{Spec}(\mathcal{R}_c) \to \mathrm{Spec}(\mathcal{R})$ of $W(\mathbb{F})$-schemes which at the level of ring homomorphisms takes the ideal $(x_1, \ldots, x_b)$ of $\mathcal{R}$ to the ideal $(x_1, \ldots, x_c)$ of $\mathcal{R}_c$. Here by $\tilde{\mathcal{C}}$ being over $\mathcal{R}_c$ and by satisfying the same properties as $\mathcal{C}_z$, we mean that $\tilde{M}$ is a free $\mathcal{R}_c$-module of rank $2d$, that $\psi_{\tilde{M}}$ is a perfect alternating form on $\tilde{M}$, that $\tilde{F}^1$ is a direct summand of $\tilde{M}$ which has rank d and which is anisotropic with respect to $\psi_{\tilde{M}}$, that $\tilde{\phi} : \tilde{M} \to \tilde{M}$ is a Φ_c-linear endomorphism such that $\tilde{M}$ is $\mathcal{R}_c$-generated by $\tilde{\phi}(\frac{1}{p}\tilde{F}^1 + \tilde{M})$ and for which we have $\psi_{\tilde{M}}(\tilde{\phi}(u), \tilde{\phi}(v)) = p\Phi(\psi_{\tilde{M}}(u, v))$ for all $u, v \in \tilde{M}$, that $\tilde{\nabla}$ is an integrable, topologically nilpotent connection on $\tilde{M}$ which is uniquely determined by the requirement that $\tilde{\phi}$ is $\tilde{\nabla}$-parallel, and that for each $\alpha \in \mathcal{J}$ the tensor $\tilde{t}_\alpha \in \mathcal{T}(\mathrm{End}(\tilde{M}[\frac{1}{p}]))$ is fixed by $\tilde{\phi}$ and belongs to the F^0-filtration of $\mathcal{T}(\mathrm{End}(\tilde{M}[\frac{1}{p}]))$ defined naturally by $\tilde{F}^1$. This implies that each $\tilde{t}_\alpha$ is annihilated by $\tilde{\nabla}$.

(c) Let k be a perfect field of characteristic p. Third it asserts that both (a) and (b) continue to hold for an arbitrary k-valued point of $\mathcal{N}$.

In what follows we will need the following six applications of Faltings deformation theory. From the versality of ∇ we get the following result.

2.3.1. Corollary. *The natural functorial morphism $\mathcal{N} \to \mathcal{M}$ induces formally closed embeddings at the level of complete local rings of residue $\mathbb{F}$.*

2.3.2. Simple properties. The morphism $\bar{\mathcal{Y}} : \mathrm{Spec}(\mathcal{R}/p\mathcal{R}) \to \mathcal{N} \to \mathcal{M}$ induced by $\mathcal{Y}$, depends only on y and not on z. This holds even if $p = 2$, due to Serre–Tate deformation theory and the fact that the quadruple $(M \otimes_{W(k)} \mathcal{R}, u_{\mathrm{univ}}(\phi \otimes \Phi), \nabla, \psi_M)$ is the principally quasi-polarized F-crystal of a unique (up to isomorphism) principally quasi-polarized p-divisible group over $\mathrm{Spec}(\mathcal{R}/p\mathcal{R})$ (cf. the fully faithfulness part of [BM, Thm. 4.1]). Similarly, regardless of what p is, there exists a unique morphism $\mathrm{Spec}(\mathcal{R}_c/p\mathcal{R}_c) \to \mathcal{N} \to \mathcal{M}$ which lifts $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N} \to \mathcal{M}$ and which has the property that the principally quasi-polarized F-crystal over $\mathcal{R}_c/p\mathcal{R}_c$ of the pull back to $\mathrm{Spec}(\mathcal{R}_c/p\mathcal{R}_c)$ of $(\mathfrak{A}, \lambda_{\mathfrak{A}})$ is $(\tilde{M}, \tilde{\phi}, \tilde{\nabla}, \psi_{\tilde{M}})$. The same properties hold for k-valued points of $\mathcal{N}$.

2.3.3. Lemma. *Let $y, y_1 : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ be two points that define the same $\mathbb{F}$-valued point of $\mathcal{N}^{\mathrm{cl}}$. The quadruple which is associated to y_1 and which is the analogue of the quadruple*

$(M,\phi,(t_\alpha)_{\alpha\in\mathcal{J}},\psi_M)$, *is of the form* $(M_1,\phi_1,(t_{1,\alpha})_{\alpha\in\mathcal{J}},\psi_{M_1})=(M,\phi,(t_{1,\alpha})_{\alpha\in\mathcal{J}},\psi_M)$. *We assume that* $\mathcal{G}_1=\mathcal{G}$ *i.e., the subgroup of* $\boldsymbol{GL}_{M[\frac{1}{p}]}$ *that fixes* t_α *for all* $\alpha\in\mathcal{J}$ *is the same as the subgroup of* $\boldsymbol{GL}_{M[\frac{1}{p}]}$ *that fixes* $t_{1,\alpha}$ *for all* $\alpha\in\mathcal{J}$ *(for instance, this hold if for all* $\alpha\in\mathcal{J}$ *we have* $t_\alpha=t_{1,\alpha}$*). Then we have* $y=y_1$.

Proof: Let $z_1:\mathrm{Spec}(W(\mathbb{F}))\to\mathcal{N}$ be a point that lifts y_1 and such that the Hodge filtration of M defined by the abelian scheme $z_1^*(\mathfrak{A})$ over $W(\mathbb{F})$ which lifts A, is F^1 itself (cf. [Va11, Part I, Lemma 3.5.2]). Let $\mathcal{Y}_1:\mathrm{Spec}(\mathcal{R})\to\mathcal{N}$ be the morphism that is the analogue of $\mathcal{Y}$ but for the point z_1 instead of z.

We first assume that either $p>2$ or $p=2$ and the 2-rank of A is 0. This assumption implies that the points z and z_1 are uniquely determined by F^1 and therefore they induce the same $W(\mathbb{F})$-valued point of $\mathcal{N}^{\mathrm{cl}}$. From this and Faltings deformation theory (see Subsection 2.3 (a)) we get that the two morphisms $\mathcal{Y},\mathcal{Y}_1:\mathrm{Spec}(\mathcal{R})\to\mathcal{N}^{\mathrm{cl}}$ coincide. As $\mathcal{N}$ is the normalization of $\mathcal{N}^{\mathrm{cl}}$ and as $\mathcal{R}$ is a normal ring, we get that the two $\mathcal{R}$-valued points $\mathcal{Y}$ and $\mathcal{Y}_1$ of $\mathcal{N}$ coincide. This implies that $y=y_1$.

We assume that $p=2$ and that the 2-rank of A is 0. Due to the assumption 1.1.4 (**), it is easy to see that the two $W(\mathbb{F})$-valued points of $\mathcal{N}^{\mathrm{cl}}$ induced by z and z_1 belong to the same orbit of the action of $G(\mathbb{A}_f^{(p)})$ on $\mathcal{N}^{\mathrm{cl}}(W(\mathbb{F}))$. Therefore, we can choose z_1 such that in fact these two $W(\mathbb{F})$-valued points of $\mathcal{N}^{\mathrm{cl}}$ coincide. As in the previous paragraph we argue that $y=y_1$.

2.3.4. Corollary. *We assume that the isogeny property holds for* $y:\mathit{Spec}(\mathbb{F})\to\mathcal{N}$. *Then the formula (3) holds.*

Proof: Let $y_1,y_2\in o(y)$. We can identify $M[\frac{1}{p}]=M_1[\frac{1}{p}]=M_2[\frac{1}{p}]$ and $t_\alpha=t_{1,\alpha}=t_{2,\alpha}$ for all $\alpha\in\mathcal{J}$. Let $h_1,h_2\in\mathfrak{E}_0(y)$ and $t_1,t_2\in G(\mathbb{A}_f^{(p)})$ be such that $y_1=y(h_1)t_1$ and $y_2 y(h_2)t_2$. The images of y_1 and y_2 in $\mathcal{M}(\mathbb{F})$ (equivalently in $o^{\mathrm{big}}(y)$) coincide if and only if there exists an element $a\in\widetilde{\mathcal{E}^{\mathrm{big}}(\mathbb{Q})}$ such that we have $ay(h_1)=ay(h_2)$ and $at_1=t_2$, cf. Subsubsection 2.1.1. Based on this and Lemma 2.3.3, we get that we have $y_1=y_2$ if and only if we can choose $a\in\widetilde{\mathcal{E}^{\mathrm{big}}(\mathbb{Q})}$ such that moreover the crystalline realization of a fixes each t_α with $\alpha\in\mathcal{J}$.

The identity $at_1=t_2$ implies that the l-adic realizations of a allow us to view a as an element of $G(\mathbb{A}_f^{(p)})$. Thus if $a\in\widetilde{\mathcal{E}^{\mathrm{big}}(\mathbb{Q})}$ is such that $at_1=t_2$, then we have $a\in\widetilde{\mathcal{E}(\mathbb{Q})}$ if and only if the crystalline realization of a is an element of $\mathcal{G}(B(\mathbb{F}))$.

Based on the last two paragraphs we get that the identity $y_1=y_2$ holds if and only if there exist an element $a\in\widetilde{\mathcal{E}(\mathbb{Q})}$ such that we have $ay(h_1)=ay(h_2)$ and $at_1=t_2$. From this the formula (3) follows. $\square$

2.3.5. Corollary. *We assume that the isogeny property and the endomorphism property hold for* $y:\mathit{Spec}(\mathbb{F})\to\mathcal{N}$. *We also assume that either* y *is a basic point or the CM lift property holds for* y *as well. Then the natural map* $o(y)\to o^{\mathrm{big}}(y)$ *is injective.*

Proof: Based on the proof of Corollary 2.3.4, it suffices to show that if $a\in\widetilde{\mathcal{E}^{\mathrm{big}}(\mathbb{Q})}$ is such that $a\in G(\mathbb{A}_f^{(p)})$, then we have $a\in\widetilde{\mathcal{E}(\mathbb{Q})}$. Let A_0, $\mathbb{F}_{p^q}$, and π_0 be as in Subsection 2.1.

If the CM lift property holds for $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ (resp. if y is a basic point), then we can view naturally π_0 as an element of $G(\mathbb{Q})$ (resp. of $Z^0(G)(\mathbb{Q})$) in such way that its centralizers in either $G_{\mathbb{Q}_l}$ or $\mathcal{G}_{B(\mathbb{F})}$ are reductive groups of the same dimension. But as the endomorphism property holds for y, this dimension is precisely $\dim(\mathcal{E})$. From this we get for each prime $l \neq p$, the group of those $\mathbb{Q}$–automorphisms of A whose l-adic realizations belongs to $G(\mathbb{Q}_l)$, is precisely $\mathcal{E}(\mathbb{Q})$. Thus $a \in \mathcal{E}(\mathbb{Q}) \cap \widetilde{\mathcal{E}^{\mathrm{big}}(\mathbb{Q})} = \widetilde{\mathcal{E}(\mathbb{Q})}$. $\square$

2.3.6. Lemma. *We assume that $p = 2$ and that (G, X) has compact factors. Then the smoothness of $\mathcal{N}$ implies that the condition 1.1.4 (**) holds.*

Proof: It is known that there exists a largest open, $G(\mathbb{A}_f^{(p)})$-invariant subscheme $\mathcal{X}$ of $\mathcal{N}_{k(v)}$ such that the condition (**) holds for all points $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{X}$, cf. [Va11, Part I, Subsubsect. 3.5.1]. The scheme $\mathcal{X}$ is non-empty, cf. [Va11, Part I, Lemma 4.1] (this follows from [Va8, Thm. 1.2] using special points of $\mathcal{N}$ which have values in $W(\mathbb{F})$ and which are associated to a maximal torus $T_{\mathbb{Z}_{(p)}}$ of $G_{\mathbb{Z}_{(p)}}$ whose extension to $\mathbb{R}$ has $\mathbb{R}$-rank 1). Faltings deformation theory of Subsection 2.3 (a) holds under no assumption on $\mathcal{G}$ (cf. [Va11, Part I, Subsects. 3.3 and 3.4]). Based on this one easily gets that $\mathcal{X}$ is a closed subscheme of $\mathcal{N}_{k(v)}$. Thus $\mathcal{X}$ is a non-empty, disjoint union of connected components of $\mathcal{N}_{k(v)}$. But $\mathcal{N}$ is a pro-étale cover of a projective $O_{(v)}$-scheme (cf. [Va4, Cor. 4.3]) and the connected components of $\mathcal{N}_{W(\mathbb{F})}$ are permuted transitively by $G(\mathbb{A}_f^{(p)})$, cf. [Va1, Lemma 3.3.2]. This implies that $G(\mathbb{A}_f^{(p)})$-acts transitively on the connected components of $\mathcal{N}_{k(v)}$. We conclude that $\mathcal{X} = \mathcal{N}_{k(v)}$. $\square$

2.4. Deformations of isogenies. Let $h \in \mathfrak{E}_0(y)$. Let $\mathcal{Y}(h) : \mathrm{Spec}(\mathcal{R}) \to \mathcal{M}$ be a morphism that is constructed similarly to the composite morphism $\mathcal{Y} : \mathrm{Spec}(\mathcal{R}) \to \mathcal{N} \to \mathcal{M}$ and which lifts the point $y(h) : \mathrm{Spec}(\mathbb{F}) \to \mathcal{M}$. Thus the principally quasi-polarized filtered F-crystal with tensors over $\mathrm{Spec}(\mathcal{R}/p\mathcal{R})$ associated to $\mathcal{Y}(h)$, is (up to $W(\mathbb{F})$-automorphisms of $\mathcal{R}$ that leave invariant the ideal $(x_1, \ldots, x_b)$) of the form

$$\mathcal{C}_z(h) := (h(M) \otimes_{W(k)} \mathcal{R}, F^1(h) \otimes_{W(k)} \mathcal{R}, u_{\mathrm{univ}}(h)(\phi \otimes \Phi), \nabla(h), (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M),$$

where the direct summand $F^1(h)$ of $h(M)$ is of the form $hh_1(F^1)$ for some element $h_1 \in \mathcal{G}(W(k))$ and where $u_{\mathrm{univ}}(h)$ is a suitable universal element associated to the closed subscheme $\mathcal{V}(h) := hh_1\mathcal{V}h_1^{-1}h^{-1}$ of $h\mathcal{G}h^{-1}$.

We assume that there exists a geometric point $\tilde{y}_1 : \mathrm{Spec}(k) \to \mathrm{Spec}(\mathcal{R}/p\mathcal{R}) \to \mathrm{Spec}(\mathcal{R})$ such that $y_1 := \mathcal{Y}(h) \circ \tilde{y}_1 : \mathrm{Spec}(k) \to \mathcal{M}$ factors through $\mathcal{N}$ in such a way that the pull back of $\mathcal{C}_z(h)$ via it is the quadruple $(M_1, \phi_1, (t_{1,\alpha})_{\alpha \in \mathcal{J}}, \psi_{M_1})$ associated naturally to the resulting factorization $y_1 : \mathrm{Spec}(k) \to \mathcal{N}$. Let $\mathcal{G}_1$ be the reductive subgroup of $\boldsymbol{GL}_{M_1}$ which is the Zariski closure of the subgroup of $\boldsymbol{GL}_{M_1[\frac{1}{p}]}$ that fixes $t_{1,\alpha}$ for all $\alpha \in \mathcal{J}$. Let $\bar{\mathcal{Y}}(h) : \mathrm{Spec}(\mathcal{R}/p\mathcal{R}) \to \mathcal{M}$ be the morphism defined naturally by $\mathcal{Y}(h)$.

2.4.1. Fact. *The morphism $\bar{\mathcal{Y}}(h)$ factors uniquely through $\mathcal{N}$ in such way that the resulting factorization $y(h) : Spec(\mathbb{F}) \to \mathcal{N}$ is as in Definition 1.4.1.*

Proof: Let $c := b - \tilde{c}$, where $\tilde{c}$ is the codimension in $\mathrm{Spec}(\mathcal{R}/p\mathcal{R})$ of the point of $\mathrm{Spec}(\mathcal{R}/p\mathcal{R})$ through which y_1 factors. Let $\mathcal{Z} : \mathrm{Spec}(W(k)[[x_1, \ldots, x_c]]) \to \mathrm{Spec}(\mathcal{R})$ be a dominant morphism that lifts $\tilde{y}_1$. Let $\bar{\mathcal{Z}} : \mathrm{Spec}(k[[x_1, \ldots, x_c]]) \to \mathrm{Spec}(\mathcal{R})$ be induced by $\mathcal{Z}$. From properties 2.3 (b) and (c) and Subsection 2.3.2 applied over k, we get that the morphism $\bar{\mathcal{Y}}(h) \circ \bar{\mathcal{Z}}$ factors uniquely through $\mathcal{N}$ in such a way that the resulting morphism $\mathrm{Spec}(k[[x_1, \ldots, x_c]]) \to \mathcal{N}$ lifts y_1. This implies that the morphism $\bar{\mathcal{Y}}(h)$ factors through $\mathcal{N}$ in such a way that $y_1 : \mathrm{Spec}(k) \to \mathcal{N}$ is indeed the resulting factorization of $\tilde{y}_1$. The fact that the resulting factorization $y(h) : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ is as in Definition 1.4.1, follows via specializations from the very smoothness of $\mathcal{N}$ and from the properties enjoyed by y_1. The uniqueness part follows from the uniqueness of the factorization $y(h) : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ (cf. Lemma 2.3.3) and from the uniqueness part of Subsection 2.3 (b). $\square$

2.4.2. Remark. If either $p > 2$ or $p = 2$ and the 2-rank of $y^*(\mathfrak{A})$ is 0, then in fact $\mathcal{Y}(h)$ itself factors uniquely through $\mathcal{N}$. To check this it suffices to show that each lift $\tilde{z}_1 : \mathrm{Spec}(W(k)) \to \mathrm{Spec}(\mathcal{R})$ that lifts $\tilde{y}_1$ gives birth to a morphism $z_1 := \mathcal{Y}(h) \circ \tilde{z}_1 : \mathrm{Spec}(W(k)) \to \mathcal{M}$ that factors uniquely through $\mathcal{N}$. But this follows from Corollary 2.3.1 and [Va11, Part I, Lemma 3.5.2], once we remark that the Hodge filtration of M_1 associated naturally to the lift z_1 of y_1, is defined by a Hodge cocharacter $\mu_1 : \mathbb{G}_m \to \mathcal{G}_1$. For instance, we can take μ_1 to be the inverse of the canonical split cocharacter of (M_1, F_1^1, ϕ_1), where F_1^1 is the Hodge filtration of M_1 associated naturally to z_1.

2.5. Lemma (the Tate–Hodge trick). *Let $y : Spec(\mathbb{F}) \to \mathcal{N}$. We assume that there exists a finite, discrete valuation ring extension V of $W(\mathbb{F})$ and a lift $\tilde{z} : Spec(V) \to \mathcal{N}$ of y such that the Mumford–Tate group of the generic fibre $\mathcal{A}$ of the abelian scheme $\tilde{z}^*(\mathfrak{A})$ over V is naturally identified with a subtorus T of G for which the following property holds:*

(*) *the torus T is the subgroup of $\boldsymbol{GSp}(W, \psi)$ that fixes a semisimple $\mathbb{Q}$–subalgebra of $End(W)$ (this semisimple $\mathbb{Q}$–subalgebra can be taken to be the centralizer of T in $End(W)$).*

Then the Tate–Hodge property holds for $y : Spec(\mathbb{F}) \to \mathcal{N}$. Therefore the Conjecture 1.2 holds for the point $y : Spec(\mathbb{F}) \to \mathcal{N}$ (i.e., regardless of the choice of the family of tensors $(v_\alpha)_{\alpha \in \mathcal{J}}$, each t_α is the crystalline realization of a Hodge cycle $c_{\alpha,A}$ on A).

Proof: It is well known that the Mumford–Tate conjecture holds for abelian varieties over number fields that have complex multiplication. From this and (*) we get that the Tate conjecture holds for $\mathcal{A}$, cf. [Mi6, Thm. 2.6]. From this and [Mi6, Thm. 6.2] we get that the Hodge conjecture holds for $\mathcal{A}$. Thus the Hodge–Tate property holds for $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ and moreover each $z^*(w_\alpha^{\mathfrak{A}})$ is the Hodge cycle associated to an algebraic cycle $c_{\alpha,\mathcal{A}}$ on $\mathcal{A}$. Let $c_{\alpha,A}$ be the algebraic cycle on A which is the specialization of $c_{\alpha,\mathcal{A}}$. The crystalline realization of $c_{\alpha,\mathcal{A}}$ is t_α (cf. the very definition of t_α) and therefore the crystalline realization of $c_{\alpha,A}$ is as well t_α. $\square$

3. Stratifications

We recall that $k(v)$ is the residue field of v. In this Section we will introduce three stratifications of $\mathcal{N}_{k(v)}$ and either recall or prove basic properties of them that are needed in what follows. In particular, we prove a refined general analogue of Oort's result that the supersingular locus of the special fibre of an integral canonical model of a Siegel modular varieties is the only closed Newton polygon stratum in the special fibre (see [Oo1] and Example 3.7). We consider an algebraically closed field k of characteristic p.

3.1. Rational stratification. It is known that there exists a $G(\mathbb{A}_f^{(p)})$-invariant stratification of $\mathcal{N}_{k(v)}$ in reduced, locally closed subschemes defined by the following property:

Two points $y_1, y_2 \in \mathcal{N}(k)$ belong to the same stratum if and only if there exists an isomorphism

$$(M_1[\frac{1}{p}], \phi_1, (t_{1,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_1}) \xrightarrow{\sim} (M_2[\frac{1}{p}], \phi_2, (t_{2,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_2}).$$

We recall (cf. Subsection 2.1), that the quadruple $(M_i, \phi_i, (t_{i,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_i})$ is the analogue of the quadruple $(M, \phi, (t_\alpha)_{\alpha\in\mathcal{J}}, \psi_M)$ but obtained for the k-valued point y_i instead of the $\mathbb{F}$-valued y. This type of stratification of $\mathcal{N}_{k(v)}$ was first introduced in [RR]; we call it the rational stratification of $\mathcal{N}_{k(v)}$.

By the *basic locus* of $\mathcal{N}_{k(v)}$ we mean the closed stratum $\mathfrak{s}_0$ of the rational stratification of $\mathcal{N}_{k(v)}$ whose $\mathbb{F}$-valued points are the basic points, cf. [Va10, Prop. 9.3 (a)] and (1). Thus we have $y \in \mathfrak{s}_0(\mathbb{F})$ if and only if all Newton polygon slopes of $(\mathrm{Lie}(\mathcal{G}_{B(\mathbb{F})}), \phi)$ are 0.

3.1.1. Lemma. *We assume that the adjoint group G^{ad} is simple. Then the basic locus $\mathfrak{s}_0$ is a pro-étale cover of a projective $k(v)$-scheme.*

Proof: To prove this, we can assume that $\mathcal{N}_{k(v)}$ is not a pro-étale cover of a projective $k(v)$-scheme. Thus (G, X) does not have compact factors, cf. [Va4, Cor. 4.3]. Let H_0 be a compact open subgroup of $G(\mathbb{A}_f^{(p)})$. For H_0 small enough, the pull back to $\mathfrak{s}_0$ of $\mathfrak{A}$ is defined over $\mathfrak{s}_0/H_0$. Thus also the pull back to $\mathfrak{s}_0$ of the p-divisible of $\mathfrak{A}$ is defined over $\mathfrak{s}_0/H_0$. For H_0 small enough, the resulting p-divisible group $\mathcal{W}$ over $\mathfrak{s}_0/H_0$ is a direct sum of an étale p-divisible group $\mathcal{W}_0$, of the Cartier dual $\mathcal{W}_1$ of $\mathcal{W}_0$, and of a p-divisible group $\mathcal{W}_{1/2}$ which does not have integral Newton polygon slopes at any point of $\mathfrak{s}_0/H_0$. Moreover, $\mathcal{W}_0$ becomes constant after pull back to $(\mathfrak{s}_0/H_0)_{\mathbb{F}}$ (this is so as the mentioned pull back of $\mathcal{W}_0$ lifts to a constant étale p-divisible group over $\mathcal{N}_{W(\mathbb{F})}$). Let $\bar{\mathfrak{s}}_0/H_0$ be a compactification of $\mathfrak{s}_0/H_0$ coming from a toroidal compactification of $\mathcal{N}/H_0$; thus the natural abelian scheme over $\mathfrak{s}_0/H_0$ extends to a semiabelian scheme over $\bar{\mathfrak{s}}_0/H_0$ whose fibres over points of $\bar{\mathfrak{s}}_0/H_0 \setminus \mathfrak{s}_0/H_0$ are not abelian varieties. Let $\mathrm{Spec}(\bar{V}) \to \bar{\mathfrak{s}}_0/H_0$ be a morphism, where $\bar{V}$ is an $\mathbb{F}$-algebra which is a discrete valuation ring. We assume that the generic point $\mathrm{Spec}(\bar{K})$ of $\mathrm{Spec}(\bar{V})$ maps to a point of $\mathfrak{s}_0/H_0$. Then the pull back to $\mathrm{Spec}(K)$ of $\mathcal{W}$ extends to a p-divisible group over $\mathrm{Spec}(\bar{V})$, cf. the description of $\mathcal{W}$. This implies that the morphism $\mathrm{Spec}(\bar{V}) \to \bar{\mathfrak{s}}_0/H_0$ factors through $\mathfrak{s}_0/H_0$, cf. [dJ, Criterion 2.5]. Thus $\mathfrak{s}_0/H_0$ is a projective $k(v)$-scheme. $\square$

3.1.2. Definition. Let $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ be a basic point. We say that y is a *pivotal point*, if the following two properties hold (to be compared with [Va12, Def. 8.2.1 and Thm. 8.3]):

(i) For all elements $g \in \mathrm{Ker}(\mathcal{G}(W(\mathbb{F})) \to \mathcal{G}(\mathbb{F}))$ there exists $g_1 \in \mathcal{G}(W(\mathbb{F}))$ such that we have $g\phi = g\phi g^{-1}$.

(ii) We have inclusions $p\mathrm{Lie}(\mathcal{G}) \subseteq \{x \in \mathrm{Lie}(\mathcal{G}) | \phi(x) = x\} \subseteq \mathrm{Lie}(\mathcal{G})$.

From property (i) and [Va12, Thms. 1.4 and 8.3] we get that:

(iii) There exists a maximal torus $\mathcal{T}$ of $\mathcal{G}$ such that its Lie algebra is generated by elements fixed by ϕ.

3.1.3. Example. We assume that $\mathcal{N} = \mathcal{M}$. A point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ is basic (resp. pivotal) if and only if A is a supersingular abelian variety (resp. if and only if the p-divisible group of A is a product of p-divisible groups of supersingular elliptic curves).

3.1.4. Example. There exists a special point $z : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}$ whose closed point maps to a pivotal point $y \in \mathfrak{s}_0(\mathbb{F})$, cf. property 3.1.3 (iii) and [Va10, Subsubsect. 9.4.1] (more precisely, this follows from [Va8, Subsubsect. 4.2.2 and the proof of Thm. 5.2.3]).

3.2. Proposition. *Each stratum $\mathfrak{s}$ of the rational stratification of $\mathcal{N}_{k(v)}$ is an open closed subscheme of the Newton polygon stratification of $\mathcal{N}_{k(v)}$ defined by the F-crystal over $\mathcal{N}_{k(v)}$ of the p-divisible group of $\mathfrak{A}_{k(v)}$. Moreover, $\mathfrak{s}_0$ is the reduced scheme of the pull back via the morphism $\mathcal{N}_{k(v)} \to \mathcal{N}^{\mathrm{cl}}_{k(v)}$ of a reduced, closed subscheme $\mathfrak{s}_0^{\mathrm{cl}}$ of $\mathcal{N}^{\mathrm{cl}}_{k(v)}$.*

Proof: The first part is a particular case of [Va6, Sect. 8, Thm. 8]. To prove the second part, let $\mathfrak{s}_0^{\mathrm{cl}}$ be the image of $\mathfrak{s}_0$ in $\mathcal{N}^{\mathrm{cl}}$. As the morphism $\mathcal{N} \to \mathcal{M}_{O_{(v)}}$ is finite, $\mathfrak{s}_0^{\mathrm{cl}}$ is a reduced, closed subscheme of $\mathcal{N}^{\mathrm{cl}}_{k(v)}$. Thus to end the proof, it suffices to show that if $y \in \mathfrak{s}_0(\mathbb{F})$ and $y_1 \in \mathcal{N}(\mathbb{F})$ are two points that map to the same $\mathbb{F}$-valued point y^{cl} of $\mathcal{N}^{\mathrm{cl}}(\mathbb{F})$, then we have $y_1 \in \mathfrak{s}_0(\mathbb{F})$. As y and y_1 map to y^{cl}, we can identify $(A, \lambda_A) = (A_1, \lambda_{A_1})$ and $(M, \phi, \psi_M) = (M_1, \phi_1, \psi_{M_1})$ and thus we can view both $\mathcal{G}$ and $\mathcal{G}_1$ as subgroup schemes of $\boldsymbol{GL}_M$. Let (A_0, λ_{A_0}), $\mathbb{F}_{p^q}$, $(M_0, \phi_0, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_{M_0})$, and π_0 be as in Subsection 2.1. For $q >> 0$, we can assume that each $t_{1,\alpha}$ is a tensor of the tensor algebra of $\mathrm{End}(M_0)[\frac{1}{p}]$. Thus $\mathcal{G}$ and $\mathcal{G}_1$ are pull backs of closed subgroup schemes $\mathcal{G}_0$ and $\mathcal{G}_{1,0}$ of $\boldsymbol{GL}_{M_0}$. As y is basic we have $\pi_0 \in Z^0(G)(\mathbb{Q}) \leqslant Z^0(\mathcal{G}_0)(B(\mathbb{F}_{p^q})) = Z^0(\mathcal{G}_{1,0})(B(\mathbb{F}_{p^q}))$. This implies that all Newton polygon slopes of $(\mathrm{Lie}(\mathcal{G}_{1,0})[\frac{1}{p}], \phi_0)$ are 0. Thus we have $y_1 \in \mathfrak{s}_0(\mathbb{F})$. $\square$

3.3. Lemma. *Let $y, y_1 : Spec(\mathbb{F}) \to \mathcal{N}$ be two points that map to the same $\mathbb{F}$-valued point y^{cl} of $\mathcal{N}^{\mathrm{cl}}$. We assume that the isogeny property holds for both y and y_1. We also assume that the endomorphism property holds for y. Then the endomorphism property holds for y_1. If moreover y is a basic point, then we have $y = y_1$.*

Proof: As both y and y_1 map to y^{cl} and as the endomorphism property holds for y, from the very definition of $\mathcal{E}$ and its analogue $\mathcal{E}_1$ for y_1 we get that $\mathcal{E}_1$ is a subgroup of $\mathcal{E}$. Let Γ_1 (resp. Γ) be the subgroup of $\widetilde{\mathcal{E}_1(\mathbb{Q})}$ (resp. $\widetilde{\mathcal{E}(\mathbb{Q})}$) which normalizes M; we view it as the group of $\mathbb{Z}_{(p)}$-valued points of a suitable flat, affine group scheme over $\mathbb{Z}_{(p)}$ which extends $\mathcal{E}_1$ (resp. $\mathcal{E}$). As the morphism $\mathcal{N} \to \mathcal{N}^{\mathrm{cl}}$ is finite, the fibres of the map $\mathcal{N}(\mathbb{F}) \to \mathcal{N}^{\mathrm{cl}}(\mathbb{F})$ are finite. Thus the natural maps $o(y) \to o^{\mathrm{big}}(y^{\mathrm{cl}})$ and $o(y_1) \to o^{\mathrm{big}}(y^{\mathrm{cl}})$ have finite fibres. From this and the fact that formula (3) holds for both y and y_1 (cf. Corollary 2.3.4), we get that the quotient set Γ/Γ_1 is finite. This implies that $\dim(\mathcal{E}) = \dim(\mathcal{E}_1)$. From this and the fact that $\mathcal{E}_1$ is a subgroup of the connected group $\mathcal{E}$, we get that $\mathcal{E}_1 = \mathcal{E}$. Thus the endomorphism property holds for y_1.

We assume that y is basic. Thus y_1 is also basic, cf. Proposition 3.2. As $\mathrm{Lie}(\mathcal{G})[\frac{1}{p}]$ and $\mathrm{Lie}(\mathcal{G}_1)[\frac{1}{p}]$ have $\mathrm{Lie}(\mathcal{E})$ and $\mathrm{Lie}(\mathcal{E}_1)$ (respectively) as $B(\mathbb{F})$-bases, we get that inside

$\mathrm{End}(M)[\frac{1}{p}] = \mathrm{End}(M_1)[\frac{1}{p}]$ we have $\mathrm{Lie}(\mathcal{G})[\frac{1}{p}] = \mathrm{Lie}(\mathcal{G}_1)[\frac{1}{p}]$. This implies that $\mathcal{G}_{B(\mathbb{F})} = \mathcal{G}_{1,B(\mathbb{F})}$ and thus $\mathcal{G} = \mathcal{G}_1$. From this and Lemma 2.3.3 we get that $y = y_1$. $\square$

3.3.1. Remark. Let $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ be an arbitrary point for which the isogeny property holds. Due to the finiteness of the morphism $\mathcal{N} \to \mathcal{N}^{\mathrm{cl}}$, a similar argument to the one of the proof of Lemma 3.3 shows that the identity component of $\mathcal{E}$ is the identity component of the subgroup of $\boldsymbol{Aut}(A)_{\mathbb{Q}}$ whose $\mathbb{Q}$–valued point are those $\mathbb{Q}$–automorphisms a of A that satisfy only the property (ii) of the Definition 1.4.3.

3.4. Traverso and level 1 stratifications. It is known (see [Va6, Sect. 8.7]) that there exists a $G(\mathbb{A}_f^{(p)})$-invariant stratification of $\mathcal{N}_{k(v)}$ (in the sense of [Va6, Subsect. 8.2]) in regular, equidimensional, locally closed subschemes (of either $\mathcal{N}_{k(v)}$ or of pull backs of $\mathcal{N}_{k(v)}$ to algebraically closed fields that contain $k(v)$ and that have countable transcendental degrees) that is defined by the following property:

Two points $y_1, y_2 \in \mathcal{N}(k)$ factor through the same stratum if and only if there exists an isomorphism $(M_1, \phi_1, (t_{1,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_1}) \tilde{\to} (M_2, \phi_2, (t_{2,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_2})$.

We call such a stratification the Traverso stratification of $\mathcal{N}_{k(v)}$. In what follows we also need the level 1 form of the Traverso stratification. It is known (see [Va6, Sect. 8.6]) that there exists a $G(\mathbb{A}_f^{(p)})$-invariant stratification of $\mathcal{N}_{k(v)}$ in regular, equidimensional, locally closed subschemes (of either $\mathcal{N}_{k(v)}$ or of pull backs of $\mathcal{N}_{k(v)}$ to algebraically closed fields that contain $k(v)$ and that have countable transcendental degrees) that is defined by the following property:

Two points $y_1, y_2 \in \mathcal{N}(k)$ factor through the same stratum if and only if there exists an element $g_{12} \in Ker(\mathcal{G}_2(W(k)) \to \mathcal{G}_2(k))$ which fixes ψ_{M_2} and for which there exists an isomorphism $(M_1, \phi_1, (t_{1,\alpha})_{\alpha\in\mathcal{J}}, g_{12}\psi_{M_1}) \tilde{\to} (M_2, \phi_2, (t_{2,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_2})$.

We call such a stratification the level 1 stratification of $\mathcal{N}_{k(v)}$.

3.4.1. Lemma. *Each stratum $\mathfrak{s}$ of the level 1 and Traverso stratifications of $\mathcal{N}_{k(v)}$ is a quasi-affine scheme.*

Proof: The isomorphism class of the principally quasi-polarized Barsotti–Tate group of level 1 associated to a point $y_i : \mathrm{Spec}(k) \to \mathfrak{s}$ (equivalently, the isomorphism class of the reduction modulo p of $(M_i, \phi_i, p\phi_i^{-1}, \psi_i)$), does not depend on the choice of the k-valued point y_i of $\mathfrak{s}$. This implies that the image of $\mathfrak{s}$ in $\mathcal{N}^{\mathrm{cl}}_{k(v)}$ is a reduced, locally closed subscheme of a stratum of the Ekedahl–Oort (i.e., of level 1) stratification of $\mathcal{M}_{k(v)}$ and thus (cf. [Oo2, Thm. 1.2]) it is a quasi-affine scheme. As the morphism $\mathcal{N} \to \mathcal{N}^{\mathrm{cl}}$ is finite, we conclude that $\mathfrak{s}$ is a quasi–affine scheme. $\square$

3.5. Lemma. *Let $\mathfrak{z}$ be the Zariski closure in $\mathcal{N}_{k(v)}$ of the $G(\mathbb{A}_f^{(p)})$-orbit of some point $y : Spec(\mathbb{F}) \to \mathcal{N}$. If $\mathfrak{z}$ is 0 dimensional, then y is a basic point.*

Proof: In what follows, we will view $\boldsymbol{Aut}(A)$ as an affine group scheme over $\mathbb{Z}$. Let H_0 be a compact open subgroup of $G(\mathbb{A}_f^{(p)})$. If $\mathfrak{z}$ is 0 dimensional, then the natural map

$$\boldsymbol{Aut}(A)(\mathbb{Z}_{(p)}) \cap \widetilde{\mathcal{E}(\mathbb{Q})} \backslash [\{1_M\} \times G(\mathbb{A}_f^{(p)})/H_0] \to o^{\mathrm{big}}(y)/H_0$$

has a finite domain. This implies that the set $\widetilde{\mathcal{E}(\mathbb{Q})}\backslash G(\mathbb{A}_f^{(p)})/H_0$ is finite.

We will show that the assumption that y is not a basic point leads to a contradiction. Let A_0, π_0, and $\mathbb{F}_{p^q}$ be as in Subsection 2.1. As y is not basic, the crystalline realization of π_0 is an element $\pi_0 \in \mathcal{G}(B(\mathbb{F}))$ which does not belong to $Z(\mathcal{G})(B(\mathbb{F}))$. As the elements of $Z(G)(\mathbb{Q})$ define naturally $\mathbb{Q}$–endomorphisms of the universal abelian scheme $\mathfrak{A}$ over $\mathcal{N}$, we get that if l is a prime distinct from p, then the l-adic realization of π_0 is an element of $G(\mathbb{Q}_l)$ that does not belong to $Z(G)(\mathbb{Q}_l)$. Therefore $\mathcal{E}_{\mathbb{Q}_l}$ is contained in a reductive subgroup $\mathcal{F}$ of $G_{\mathbb{Q}_l}$ distinct from $G_{\mathbb{Q}_l}$ (we can take $\mathcal{F}$ to be the centralizer in $G_{\mathbb{Q}_l}$ of the mentioned l-adic realization). We now choose l such that both groups $G_{\mathbb{Q}_l}$ and $\mathcal{F}$ are split (equivalently, that both $G_{\mathbb{Q}_l}$ and the Frobenius torus of π_0 are split). As the quotient variety $Q := \mathcal{F}\backslash G_{\mathbb{Q}_l}$ is a smooth, connected, affine variety of positive dimension, it is easy to see that the analytic l-adic subvariety $\mathcal{F}(\mathbb{Q}_l)\backslash G(\mathbb{Q}_l)$ of $Q(\mathbb{Q}_l)$ is not compact. But as the set $\widetilde{\mathcal{E}(\mathbb{Q})}\backslash G(\mathbb{A}_f^{(p)})/H_0$ is finite and as $\mathcal{E}(\mathbb{Q}_l) \leqslant \mathcal{F}(\mathbb{Q}_l)$, one gets that the analytic l-adic subvariety $\mathcal{F}(\mathbb{Q}_l)\backslash G(\mathbb{Q}_l)$ is compact. Contradiction. Thus y is a basic point. $\square$

3.6. Lemma. *We assume that the Shimura pair (G, X) has compact factors (in the sense before Theorem 1.5).*[5] *Let $\mathfrak{z}$ be a closed, reduced, $G(\mathbb{A}_f^{(p)})$-invariant subscheme of $\mathcal{N}_{k(v)}$. Then the intersection $\mathfrak{z} \cap \mathfrak{s}_0$ is non-empty.*

Proof: We consider the stratification of $\mathfrak{z}$ which is the pull back of the Traverso stratification of $\mathcal{N}_{k(v)}$. Let $\mathfrak{z}_0$ be a stratum of this stratification of $\mathfrak{z}$ which is a closed, reduced subscheme of $\mathfrak{z}$. The scheme $\mathfrak{z}_0$ is $G(\mathbb{A}_f^{(p)})$-invariant and therefore (cf. Lemma 3.5) it has either a non-empty intersection with the basic locus or positive dimension. It is known that $\mathcal{N}$ is a pro-étale cover of a projective, smooth $O_{(v)}$-scheme, cf. [Va4, Cor. 4.3]. Thus $\mathfrak{z}_0$ is a pro-étale cover of a projective, smooth $k(v)$-scheme (as this holds for $\mathcal{N}_{k(v)}$) and it is a quasi-affine scheme (cf. Lemma 3.4.1). Therefore $\mathfrak{z}_0$ can not be of positive dimension. We conclude that $\mathfrak{z} \cap \mathfrak{s}_0 \supseteq \mathfrak{z}_0 \cap \mathfrak{s}_0 \neq \emptyset$. $\square$

3.7. Example. Let $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ be a point. Let $\mathfrak{l}$ be the stratum of the Traverso stratification of $\mathcal{N}_{k(v)}$ that contains y. Let $\mathfrak{c}$ be the identity component of $\mathfrak{l}$ that contains y. Let $\mathfrak{t}$ be the smallest closed, reduced subscheme of $\mathfrak{l}$ that contains $\mathfrak{c}$ and is $G(\mathbb{A}_f^{(p)})$-invariant; thus $\mathfrak{c}$ is a connected component of $\mathfrak{t}$. If y is not a basic point, then the dimension of $\mathfrak{t}$ is positive and this implies that $\mathfrak{c}$ has positive dimension.

We assume now that the Shimura pair (G, X) has compact factors. The Zariski closure of $\mathfrak{t}$ in $\mathcal{N}_{k(v)}$ contains points of $\mathfrak{s}_0$, cf. Lemma 3.6. From this and the very definition of $\mathfrak{t}$, we get that the Zariski closure of $\mathfrak{c}$ in $\mathcal{N}_{k(v)}$ contains also points of $\mathfrak{s}_0$.

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5 The proof of this Lemma is the only place in the paper where we require (in this first version) to work with Shimura varieties that have compact factors. Therefore, all the results of the paper continue to hold for all those connected components of strata of the Traverso stratification of $\mathcal{N}_{k(v)}$ which are locally closed subschemes of $\mathcal{N}_{k(v)}$ and whose Zariski closures in $\mathcal{N}_{k(v)}$ contain points of $\mathfrak{s}_0$ (like for all those strata that involve abelian varieties of p-rank 0). One can show that this applies to all cases without compact factors.

3.8. Theorem. *Let $y : Spec(\mathbb{F}) \to \mathcal{N}$ be a point. Let $\mathfrak{c}$ be the connected component of the Traverso stratum which is a locally closed subscheme of $\mathcal{N}_{k(v)}$ and which contains the point y. Let $\mathfrak{r}$ (resp. $\mathfrak{l}$) be the connected component of the rational (resp. level 1) stratum of $\mathcal{N}_{k(v)}$ which contains the point y.*

(a) *If y is not a basic point, then we have* $\dim(\mathfrak{c}) > 0$.

(b) *If y is not a basic point and $\mathfrak{l}$ has a non-trivial intersection with $\mathfrak{s}_0$, then we have* $\dim_y(\mathfrak{r} \cap \mathfrak{l}) \geq 2$ *(here by* $\dim_y$ *we mean the dimension at the point y).*

(c) *If y is a basic point which is not pivotal, then we have* $\dim_y(\mathfrak{r} \cap \mathfrak{l}) \geq 1$.

Proof: Part (a) follows from Example 3.7 (see also below for a second proof). We show that (b) implies (c). As y is not a pivotal point, we have $\dim(\mathfrak{l}) \geq 1$ (cf. [Va12, Cor. 11.2 (c) and Thm. 12.2]). If $\mathfrak{l}$ is contained in $\mathfrak{s}_0$, then y is a basic point and $\mathfrak{r} \cap \mathfrak{l} = \mathfrak{l}$ has positive dimension. Thus we can assume that $\mathfrak{l}$ is not contained in $\mathfrak{s}_0$. But (b) implies that each irreducible component of a stratum of the restriction to $\mathfrak{l} \setminus \mathfrak{s}_0$ of the rational stratification of $\mathcal{N}_{k(v)}$, has dimension at least 2. From this and the fact that the rational stratifications are pure (cf. [Va6, Thm. 8 (c)]), we get that $\dim_y(\mathfrak{s}_0 \cap \mathfrak{l}) \geq 1$. Thus (b) implies (c).

The results (a) to (c) are properties of Shimura F-crystals and thus from now (in order to prove (a) to (c) by induction on $\dim(\mathcal{G}_{\mathbb{F}})$) we will forget about the geometric context provided by $\mathcal{N}$ and we will work only with a Shimura F-crystal $(M, \phi, \mathcal{G})$ over $\mathbb{F}$. The roles of $\mathfrak{r}$, $\mathfrak{c}$, $\mathfrak{l}$ will be replaced by the ones of the corresponding formal subschemes (denoted in the same way) of the formal deformation space $\mathcal{D}$ over $\mathrm{Spf}(\mathbb{F})$ of $(M, \phi, \mathcal{G})$. Here $\mathcal{D}$ is a formally smooth subspace of the formal deformation space over $\mathrm{Spf}(\mathbb{F})$ of the p-divisible group over $\mathbb{F}$ whose Dieudonné module is (M, ϕ): with the notations of Subsection 2.3, we can identify $\mathcal{D} = \mathrm{Spf}(\mathcal{R}/p\mathcal{R})$. We will divide the proof into steps.

Step 1. Let $\mathcal{P}$ be the unique parabolic subgroup scheme of $\mathcal{G}$ such that the following two properties hold (cf. [Va8, Subsect. 2.3]):

(i) the Lie algebra $\mathrm{Lie}(\mathcal{P})[\frac{1}{p}]$ is left invariant by ϕ;

(ii) all Newton polygon slopes of $(\mathrm{Lie}(\mathcal{P})[\frac{1}{p}], \phi)$ are non-positive and all Newton polygon slopes of $(\mathrm{Lie}(\mathcal{G})[\frac{1}{p}]/\mathrm{Lie}(\mathcal{P})[\frac{1}{p}], \phi)$ are positive.

Let $\mathcal{U}$ be the unipotent radical of $\mathcal{P}$. From the non-negative standard form of $(M, \phi, \mathcal{G})$ (see [Va8, Subsects. 3.1 and 3.2]) we get the existence:

(iii) of a maximal torus $\mathcal{T}$ of $\mathcal{P}$ and of a cocharacter $\mu : \mathbb{G}_m \to \mathcal{T}$ with the property that we have a direct sum decomposition $M = F^1 \oplus F^0$ such that $\phi(\frac{1}{p}F^1 \oplus F^0) = M$ and $\mathbb{G}_m$ acts on F^j through μ via the $-j$-th power of the identical character of $\mathbb{G}_m$ (here $j \in \{0, 1\}$);

(iv) of an element $g \in \mathcal{P}(W(\mathbb{F}))$ such that $\varphi := g\phi$ normalizes the Lie algebra of $\mathcal{T}$.

Let $\mathcal{H}$ be the unique Levi subgroup scheme of $\mathcal{P}$ that contains $\mathcal{T}$. We can write $g = u\ell$, where $u \in \mathcal{U}(W(\mathbb{F}))$ and $\ell \in \mathcal{H}(W(\mathbb{F}))$. We have a direct sum decomposition

$$\mathrm{Lie}(\mathcal{U}) = \oplus_{i=-1}^{1} \tilde{F}^i(\mathrm{Lie}(\mathcal{U}))$$

such that $\mathbb{G}_m$ acts on $\tilde{F}^i(\mathrm{Lie}(\mathcal{U})$ through μ and through inner conjugation via the $-i$-th power of the identical character of $\mathbb{G}_m$. We have an identity

$$\mathrm{Lie}(\mathcal{U}) = \oplus_{i=-1}^{1}\phi(p^{-i}\tilde{F}^i(\mathrm{Lie}(\mathcal{U}))). \tag{5}$$

Let $c_i \in \mathbb{N}\cup\{0\}$ be the rank of $\tilde{F}^i(\mathrm{Lie}(\mathcal{U}))$.

Step 2. Let $\mathcal{U}^{\mathrm{l}}$ be the analogue of $\mathcal{U}$ but working with $\boldsymbol{GL}_M$ instead of with $\mathcal{G}$. Thus $\mathrm{Lie}(\mathcal{U}^{\mathrm{l}})$ is the largest direct summand of $\mathrm{End}(M)$ such that ϕ normalizes $\mathrm{Lie}(\mathcal{U}^{\mathrm{l}})[\frac{1}{p}]$ and all Newton polygon slopes of $(\mathrm{Lie}(\mathcal{U}^{\mathrm{l}})[\frac{1}{p}],\phi)$ are negative. Let $\tilde{\mathrm{Lie}}(\mathcal{U}^{\mathrm{l}})$ be the largest $W(\mathbb{F})$-submodule of $\mathrm{Lie}(\mathcal{U}^{\mathrm{l}})$ with the property that $(\tilde{\mathrm{Lie}}(\mathcal{U}^{\mathrm{l}}),\phi)$ is a Dieudonné module. Let $\tilde{\mathrm{Lie}}(\mathcal{U}) := \mathrm{Lie}(\mathcal{U})\cap\tilde{\mathrm{Lie}}(\mathcal{U}^{\mathrm{l}})$. The pair $(\tilde{\mathrm{Lie}}(\mathcal{U}),p\phi)$ is the largest Dieudonné module contained in $(\mathrm{Lie}(\mathcal{U}),p\phi)$. The codimension of the p-divisible group over $\mathbb{F}$ whose Dieudonné module is $(\tilde{\mathrm{Lie}}(\mathcal{U}),p\phi)$, is c_{-1} (cf. [Va7, Lemma 3.1 (a) and (b)]). It is known that $\tilde{\mathrm{Lie}}(\mathcal{U}^{\mathrm{l}})$ is a subalgebra of $\mathrm{Lie}(\mathcal{U}^{\mathrm{l}})$, cf. [Va7, Subsect. 4.1]. Let $\mathcal{V}_{-1}^{\mathrm{l}}$ be the affine, smooth group scheme over $W(\mathbb{F})$ defined by the rule: if $\star$ is a $W(\mathbb{F})$-algebra, then $\mathcal{V}_{-1}^{\mathrm{l}}(\star)$ is the group of invertible elements of $1_{M\otimes_{W(\mathbb{F})}\star}+\tilde{\mathrm{Lie}}(\mathcal{U}^{\mathrm{l}})\otimes_{W(\mathbb{F})}\star$. We have a natural homomorphism $\mathcal{V}_{-1}^{\mathrm{l}}\to\tilde{\mathcal{U}}^{\mathrm{l}}$ whose generic fibre is an isomorphism, to be viewed as a canonical identification. Let $\mathcal{V}_{-1}$ be the Zariski closure of $\mathcal{U}_{B(\mathbb{F})}$ in $\mathcal{V}_{-1}^{\mathrm{l}}$. Let $\mathcal{V}_{-1,0}$ be a closed, smooth subscheme of $\mathcal{V}_{-1}$ which contains the identity section of $\mathcal{V}_{-1}$, which is isomorphic to $\mathrm{Spec}(W(\mathbb{F})[x_1,\ldots,x_{c_{-1,0}}])$ for some non-negative integer $c_{-1,0}\in\{0,\ldots,c_{-1}\}$, and whose tangent space $\mathrm{Lie}(\mathcal{V}_{-1,0})$ at the identity section has the properties that the natural $W(\mathbb{F})$-linear map

$$\mathrm{Lie}(\mathcal{V}_{-1,0})/p\mathrm{Lie}(\mathcal{V}_{-1,0})\to\mathrm{Lie}(\mathcal{U})/[p\mathrm{Lie}(\mathcal{U})+\tilde{F}^0(\mathrm{Lie}(\mathcal{U}))+\tilde{F}^1(\mathrm{Lie}(\mathcal{U}))]$$

is an isomorphism and that it is contained in $\tilde{\mathrm{Lie}}(\mathcal{U})$.

Step 3. We check that we have an inequality $\dim(\mathfrak{c})\geq c_{-1,0}$. Let $\mathrm{Spf}(\mathcal{R}_{c_{-1,0}})$ be the completion of $\mathcal{V}_{-1,0}$ along its identity section. Let $u_{-1,0,\mathrm{univ}}:\mathrm{Spec}(\mathcal{R}_{c_{-1,0}})\to\mathcal{V}_{-1,0}$ be the universal (natural) morphism. Let $\Phi_{-1,0}:=\Phi_{c_{-1,0}}$ be the Frobenius endomorphism of Subsection 2.3. Using $\mathcal{V}_{-1,0}$, $u_{-1,0,\mathrm{univ}}$, $\Phi_{-1,0}$ (instead of $\mathcal{V}$, u_{univ}, Φ), as in Subsection 2.3 (a) and (b) we build up a deformation of $(M,F^1,\phi,(t_\alpha)_{\alpha\in\mathcal{J}},\psi_M)$ over $\mathcal{R}_{c_{-1,0}}/p\mathcal{R}_{c_{-1,0}}$ which is versal and which defines a formally smooth subscheme $\mathcal{D}_{-1,0}$ of $\mathcal{D}$ isomorphic to $\mathrm{Spf}(\mathbb{F}[[x_1,\ldots,x_{c_{-1,0}}]])$. As in [Va7, Lemma 2.4 (b)] we argue that $\mathcal{D}_{-1,0}$ is contained in $\mathfrak{c}$. This implies that $\dim(\mathfrak{c})\geq c_{-1,0}$.

Step 4. If $\mathrm{Lie}(\mathcal{U})^p=0$, then it is easy to see that the exponential map from $\tilde{\mathrm{Lie}}(\mathcal{U})$ to $\mathcal{V}_{-1}^{\mathrm{l}}$ allows us to conclude that $\mathcal{V}_{-1}$ is smooth; this implies that we can choose $\mathcal{Z}_{-1,0}$ such that we have $c_{-1,0}=c_{-1}$. It seems to us that we can always choose $\mathcal{V}_{-1,0}$ such that we have $c_{-1,0}=c_{-1}$. For the time being, we will check that in general we can choose $\mathcal{V}_{-1,0}$ such that we have the following modest inequality $c_{-1,0}\geq\min\{2,c_{-1}\}$.

To check this we can assume that $c_{-1}>0$. Let $Z(\mathcal{U})$ be the largest smooth, closed subgroup scheme of $\mathcal{U}$ that centralizes $\mathcal{U}$. It has positive relative dimension, it is isomorphic to a product of $\mathbb{G}_a$ group schemes, and the quotient group scheme $\mathcal{U}/Z(\mathcal{U})$ exists and is smooth. For $i\in\{-1,0,1\}$, let $c_{-1,1,i}$ be the rank of $\mathrm{Lie}(Z(\mathcal{U}))\cap\tilde{F}^i(\mathrm{Lie}(\mathcal{U}))$. If $c_{-1,1,-1}\geq 2$, then we can choose $\mathcal{V}_{-1,0}$ to be a direct summand of $Z(\mathcal{U})$ which is of relative dimension

$c_{-1,1,-1}$ and which is contained in $\mathcal{V}_{-1}$. Thus to prove that $c_{-1,0} \geq \min\{2, c_{-1}\}$, we can assume that $c_{-1,1,-1} = 1$ and that $c_{-1} \geq 2$.

As $c_{-1,1,-1} = 1$ and as all the Newton polygon slopes of $(\mathrm{Lie}(Z(\mathcal{U}))[\frac{1}{p}], \phi)$ are negative, from (5) we get that we must have $c_{-1,1,1} = 0$ (i.e., $\tilde{F}^1(\mathrm{Lie}(Z(\mathcal{U}))) = 0$) and that all Newton polygon slopes of $(\mathrm{Lie}(Z(\mathcal{U}))[\frac{1}{p}], \phi)$ are equal to $\frac{-1}{1+c_{-1,1,0}}$. This implies that $(\mathrm{Lie}(Z(\mathcal{U})), p\phi)$ is a p-divisible group of codimension 1. Thus $\mathrm{Lie}(Z(\mathcal{U})) \subseteq \tilde{\mathrm{Lie}}(\mathcal{U})$, $Z(\mathcal{U}) \leqslant \mathcal{V}_{-1}$, and moreover $(\tilde{\mathrm{Lie}}(\mathcal{U})/\mathrm{Lie}(Z(\mathcal{U})), p\phi)$ is the largest Dieudonné module contained in $\mathrm{Lie}(\mathcal{U})/\mathrm{Lie}(Z(\mathcal{U})), p\phi)$. Thus by working with the largest smooth, closed subgroup scheme $Z(\mathcal{U}/Z(\mathcal{U}))$ of $\mathcal{U}/Z(\mathcal{U})$ that centralizes $\mathcal{U}/Z(\mathcal{U})$, as in the previous paragraph we argue that $\mathcal{V}_{-1}/Z(\mathcal{U})$ contains a direct summand of $Z(\mathcal{U}/Z(\mathcal{U}))$ which is isomorphic to $\mathbb{G}_a$ and whose special fibre has a Lie algebra whose image in $\mathrm{Lie}(\mathcal{U})/[p\mathrm{Lie}(\mathcal{U})+\mathrm{Lie}(Z(\mathcal{U}))+\tilde{F}^0(\mathrm{Lie}(\mathcal{U})) + \tilde{F}^1(\mathrm{Lie}(\mathcal{U}))]$ is non-trivial. Therefore we can choose $\mathcal{Z}_{-1,0}$ to have relative dimension at least equal to $c_{-1,1,-1} + 1 \geq 2$. This ends the argument that we can choose $\mathcal{V}_{-1,0}$ such that we have an inequality $c_{-1,0} \geq \min\{2, c_{-1}\}$.

Step 5. We will choose $\mathcal{V}_{-1,0}$ such that we have an inequality $c_{-1,0} \geq \min\{2, c_{-1}\}$. If $(M, \phi, \mathcal{G})$ is not basic, then $\mathcal{P} \neq \mathcal{G}$, $\mathcal{U}$ is of positive dimension, $\mathcal{H} \neq \mathcal{G}$, the Newton polygon slopes of $(\mathrm{Lie}(\mathcal{U}), p\phi)$ belong to the interval $[0, 1)$, we have $c_{-1} > 0$, and we have $\dim(\mathfrak{c}) \geq c_{-1,0} \geq 1$. Thus (a) holds.

Step 6. Next we will prove (b) and (c) by induction on $\dim(\mathcal{G}_{\mathbb{F}})$. Obviously (b) and (c) hold if $\dim(\mathcal{G}_{\mathbb{F}}) \leq 2$. Let $e \in \mathbb{N} \setminus \{1, 2\}$. We assume that (b) and (c) hold if $\dim(\mathcal{G}_{\mathbb{F}}) < e$ and we check that (b) and (c) hold even if $\dim(\mathcal{G}_{\mathbb{F}}) = e$. It suffices to check that (b) holds (as above we argue that (b) implies (c)). Thus $(M, \phi, \mathcal{G})$ is not basic and therefore $\dim(\mathcal{H}_{\mathbb{F}}) < e = \dim(\mathcal{G}_{\mathbb{F}})$. To prove (b) we can assume that $c_{-1} = 1$ (as otherwise we have $\dim(\mathfrak{r} \cap \mathfrak{l}) \geq \dim(\mathfrak{c}) \geq 2$). As $c_{-1} = 1$ and as all the Newton polygon slopes of $(\mathrm{Lie}(\mathcal{U})[\frac{1}{p}], \phi)$ are negative, from (5) we get that we must have $c_0 = 0$ (i.e., $\tilde{F}^1(\mathrm{Lie}(\mathcal{U})) = 0$) and that all Newton polygon slopes of $(\mathrm{Lie}(\mathcal{U})[\frac{1}{p}], \phi)$ are equal to $\frac{-1}{c_0+1}$. This implies that $(\mathrm{Lie}(\mathcal{U}), p\phi)$ is a p-divisible group of codimension 1. Thus, by replacing ϕ with $u_1 \phi u_1^{-1}$, with $u_1 \in \mathcal{U}(W(\mathbb{F}))$, we can assume that $u = 1_M$. Therefore the triple $(M, \phi, \mathcal{H})$ is a basic Shimura F-crystal. Let $\mathfrak{r}_0$, $\mathfrak{c}_0$, $\mathfrak{l}_0$ be the analogues of $\mathfrak{r}$, $\mathfrak{c}$, and $\mathfrak{l}$ (respectively) but for $(M, \phi, \mathcal{H})$ instead of for $(M, \phi, \mathcal{G})$. If $(M, \phi, \mathcal{H})$ is not pivotal, then by induction we have $\dim(\mathfrak{r}_0 \cap \mathfrak{l}_0) \geq 1$. From this and the inequality $c_{-1} = 1$ we get that $\dim(\mathfrak{r} \cap \mathfrak{l}) \geq 1 + c_{-1} \geq 2$ (in other words, one can put together the two 1 dimensional formal smooth subspaces of $\mathcal{D}$ that come from $\mathcal{Z}_{-1,0}$ and from $\mathcal{H}$ to form a 2 dimensional formal smooth subspaces of $\mathcal{D}$ contained in $\mathfrak{e} \cap \mathfrak{l}$). Thus (b) holds if $(M, \phi, \mathcal{H})$ is not pivotal.

We are left to show that the assumption that $(M, \phi, \mathcal{H})$ is pivotal leads to a contradiction. The assumption that $(M, \phi, \mathcal{H})$ is pivotal implies that we can assume that $g = \ell$ is the identity element (to be compared with the property 3.1.2 (iii)). Thus $\phi = \varphi$ normalizes $\mathrm{Lie}(\mathcal{T})$. This implies that there exists a $W(\mathbb{F})$-basis $\{w_1, \ldots, w_e\}$ for $\mathrm{Lie}(\mathcal{G})$ which is the union of $W(\mathbb{F})$-bases for $\mathrm{Lie}(\mathcal{U})$, for $\mathrm{Lie}(\mathcal{H})$, and for the opposite $\mathrm{Lie}(\mathcal{U})^{\mathrm{opp}}$ of $\mathrm{Lie}(\mathcal{U})$ with respect to $\mathcal{T}$, and there exists a permutation τ of the set $\{1, 2, \ldots, e\}$, such that for all $i \in \{1, \ldots, e\}$ we have

$$\phi(w_i) = p^{\varepsilon_i} w_{\tau(i)}$$

for some number $\varepsilon_i \in \{-1, 0, 1\}$. We can assume that the $W(\mathbb{F})$-span of each w_i is normalized by $\mathcal{T}$. As $(\mathrm{Lie}(\mathcal{U}), p\phi)$ is a Dieudonné module of codimension 1, the pair $(\mathrm{Lie}(\mathcal{U})^{\mathrm{opp}}, \phi)$ is a Dieudonné module of dimension 1. From this and the fact that $(M, \phi, \mathcal{H})$ is pivotal, we get that for each cycle $(i_1 \ldots i_s)$ of τ the sequence $(\varepsilon_{i_1}, \ldots, \varepsilon_{i_s})$ does not contain any subsequence of the form $(-1, 0, \ldots, 0, -1)$ (to be compared with [Va12, Thm. 8.3]). This implies that $\dim(\mathfrak{l}) = 0$ (cf. [Va12, Thms. 1.2 and 12.2 (d) and Subsubsect. 4.1.1]) and therefore $(M, \phi, \mathcal{G})$ is pivotal (cf. [Va12, Corollary 11.2 (c) and Thm. 12.2]). This contradicts the fact that $(M, \phi, \mathcal{G})$ is not basic. Thus (b) holds. This ends the induction and thus the proof of the Theorem. $\square$

4. Reduction steps for the isogeny property

In this Section we get reduction steps towards the proof of the isogeny property.

4.1. Theorem. *Let $y : Spec(\mathbb{F}) \to \mathcal{N}$ be a point. Let $\mathfrak{l}$ be the stratum of the Traverso stratification of $\mathcal{N}_{k(v)}$ that contains y. Let $\mathfrak{c}$ be the identity component of $\mathfrak{l}$ that contains y. We assume that the following two conditions hold:*

(i) *the isogeny property holds for each basic point $Spec(\mathbb{F}) \to \mathcal{N}$;*

(ii) *the Zariski closure of $\mathfrak{c}$ in $\mathcal{N}_{k(v)}$ intersects non-trivially the basic locus $\mathfrak{s}_0$.*

Then the isogeny property holds for the point $y : Spec(\mathbb{F}) \to \mathcal{N}$.

Proof: We consider a morphism $y_\ell : \mathrm{Spec}(\mathbb{F}[[x]]) \to \mathcal{N}$ which lifts y and which has the property that the resulting morphism $\mathrm{Spec}(\mathbb{F}((x))) \to \mathcal{N}$ factors through the generic point y_g of $\mathfrak{c}$ (we recall that $\mathfrak{l}$ is regular). We know that y_g specializes to a point $y_0 \in \mathfrak{s}_0(\mathbb{F})$, cf. condition (ii). Let $\mathcal{Q}$ be an integral, normal, noetherian, local ring of residue field $\mathbb{F}$ such that we have a morphism $w : \mathrm{Spec}(\mathcal{Q}) \to \mathcal{N}$ with the properties that the closed point of $\mathrm{Spec}(\mathcal{Q})$ maps to the point y_0 of $\mathfrak{s}_0$ and the generic point of $\mathrm{Spec}(\mathcal{Q})$ maps to the point y_g of $\mathfrak{c}$. Let $\mathfrak{l}^{\mathrm{big}}$ be the reduced, locally closed subscheme of $\mathcal{M}_{\mathbb{F}_p}$ which is a stratum of the Traverso stratification of $\mathcal{M}_{\mathbb{F}_p}$ and which contains the composite point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N} \to \mathcal{M}$. It is known (cf. [Va2, Subsubsect. 5.3.2 (c)]) that there exists an infinite sequence

$$\cdots \to \mathfrak{l}_s^{\mathrm{big}} \to \cdots \to \mathfrak{l}_3^{\mathrm{big}} \to \mathfrak{l}_2^{\mathrm{big}} \to \mathfrak{l}_1^{\mathrm{big}} \to \mathfrak{l}_0^{\mathrm{big}} = \mathfrak{l}^{\mathrm{big}}$$

of faithfully flat, finite morphisms between regular schemes with the property that:

(i) the pull back $(\mathcal{D}_s, \lambda_{\mathcal{D}_s})$ to $\mathfrak{l}_s^{\mathrm{big}}$ of the principally quasi-polarized Barsotti–Tate of level p^s of the universal principally polarized abelian scheme over $\mathcal{M}$ is constant i.e., there exists an isomorphism $\nu_s^{\mathrm{big}} : (A[p^s], \lambda_A[p^s])_{\mathfrak{l}_s^{\mathrm{big}}} \xrightarrow{\sim} (\mathcal{D}_s, \lambda_{\mathcal{D}_s})$.

We can assume that:

(ii) for each $s \in \mathbb{N}$ we have $\nu_{s, \mathfrak{l}_{s+1}^{\mathrm{big}}}^{\mathrm{big}} = \nu_{s+1}^{\mathrm{big}}[p^s]$.

Let $\mathcal{R}_1 := W(\mathbb{F})[[x]]$ and let Φ_1 be the Frobenius endomorphism of $\mathcal{R}_1$ that is compatible with $\sigma_{\mathbb{F}}$ and that takes x to x^p. Let R be the normalization of $\mathbb{F}[[x]] = \mathcal{R}_1/p\mathcal{R}_1$ in an algebraic closure k of $\mathbb{F}((x))$; it is an integral, normal, perfect $\mathbb{F}$-algebra. Let $(\mathcal{I}, \lambda_{\mathcal{I}})$ be the principally quasi-polarized p-divisible group over $\mathbb{F}[[x]]$ of $y_\ell^*(\mathfrak{A}, \lambda_{\mathfrak{A}})$. Let

$$(M_\ell, \phi_{M_\ell}, (t_{\ell,\alpha})_{\alpha\in\mathcal{J}}, \nabla_{M_\ell}, \psi_{M_\ell})$$

be the principally quasi-polarized F-crystal with tensors over $\mathbb{F}[[x]]$ associated naturally to the morphism $y_\ell : \mathrm{Spec}(\mathbb{F}[[x]]) \to \mathcal{N}$. Thus M_ℓ is a free $\mathcal{R}_1$-module of rank $2d$, ∇_{M_ℓ} is an integrable, topologically nilpotent connection on M_ℓ, and each $t_{\ell,\alpha}$ is a tensor of the tensor algebra of $\mathrm{End}(M_\ell)[\frac{1}{p}]$ that is fixed by ϕ_{M_ℓ}, that is annihilated by ∇_ℓ, and that is the crystalline realization of the Hodge cycle $z_\ell^*(w_\alpha^{\mathfrak{A}})$ on $z_\ell^*(\mathfrak{A})$, where $z_\ell : \mathrm{Spec}(\mathcal{R}_1) \to \mathcal{N}$ is an arbitrary morphism that lifts y_ℓ.

Due to the properties (i) and (ii), there exists an isomorphism

$$\iota : (A[p^\infty], \lambda_A[p^\infty])_R \xrightarrow{\sim} (\mathcal{I}, \lambda_{\mathcal{I}})_R.$$

We can assume that this isomorphism ι lifts the identity automorphism of $(A[p^\infty], \lambda_A[p^\infty])$. The crystalline realization of ι is an isomorphism (denoted in the same way)

$$\iota : (M_\ell, \phi_{M_\ell}, (t_{\ell,\alpha})_{\alpha\in\mathcal{J}}, \nabla_{M_\ell}, \psi_{M_\ell})_R \xrightarrow{\sim} (M \otimes_{W(\mathbb{F})} \mathcal{R}_1, \phi \otimes \Phi_1, (t_\alpha)_{\alpha\in\mathcal{J}}, \delta, \psi_M)_R$$

of principally quasi-polarized F-crystals with tensors over R (here δ is the flat connection on $M \otimes_{W(\mathbb{F})} \mathcal{R}_1$ that annihilates $M \otimes 1$ and the lower left index R means a pull back of F-crystals to R). The fact that ι maps each $t_{\ell,\alpha}$ to t_α follows from the fact that there exists no element of the tensor algebra of $\mathrm{End}(M) \otimes_{W(\mathbb{F})} \mathrm{Ker}(W(R) \twoheadrightarrow R)[\frac{1}{p}]$ which is fixed by $\phi \otimes \Phi_R$, where Φ_R is the Frobenius endomorphism of $W(R)[\frac{1}{p}]$.

Let $h \in \mathfrak{E}_0(y)$. By composing ι with the natural $\mathbb{Z}[\frac{1}{p}]$-isogeny

$$(M \otimes_{W(\mathbb{F})} \mathcal{R}_1, \phi \otimes \Phi_1, (t_\alpha)_{\alpha\in\mathcal{J}}, \delta, \psi_M)_R \cdots\cdots > (h(M) \otimes_{W(\mathbb{F})} \mathcal{R}_1, \phi \otimes \Phi_1, (t_\alpha)_{\alpha\in\mathcal{J}}, \delta, \psi_M)_R,$$

we get a $\mathbb{Z}[\frac{1}{p}]$-isogeny

$$\iota(h) : (M_\ell, \phi_{M_\ell}, (t_{\ell,\alpha})_{\alpha\in\mathcal{J}}, \nabla_{M_\ell}, \psi_{M_\ell})_R \cdots\cdots > (h(M) \otimes_{W(\mathbb{F})} \mathcal{R}_1, \phi \otimes \Phi_1, (t_\alpha)_{\alpha\in\mathcal{J}}, \delta, \psi)_R$$

which is induced naturally by an analogous isogeny between principally quasi-polarized p-divisible groups over R. Let $i_\ell^{\mathrm{large}} : \mathrm{Spec}(R) \to \mathcal{N}$ be the morphism defined naturally by y_ℓ and let $j_\ell^{\mathrm{large}}(h) : \mathrm{Spec}(R) \to \mathcal{M}$ be the morphism obtained naturally from y_ℓ via the $\mathbb{Z}[\frac{1}{p}]$-isogeny $\iota(h)$ (in the same way we constructed $y(h) : \mathrm{Spec}(\mathbb{F}) \to \mathcal{M}$ from $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$). Such a $\mathbb{Z}[\frac{1}{p}]$-isogeny is defined over a discrete valuation ring which is a finite, normal $\mathbb{F}[[x]]$-subalgebra of R. As such a discrete valuation ring is isomorphic to $\mathbb{F}[[x]]$, not to introduce extra notations we can assume that $j_\ell^{\mathrm{large}}(h) : \mathrm{Spec}(R) \to \mathcal{M}$ factors naturally through a morphism $j_\ell(h) : \mathrm{Spec}(\mathbb{F}[[x]]) \to \mathcal{M}$. The morphism $j_\ell(h)$ lifts $y(h)$, cf. constructions (here it is critical that $(\mathcal{I}, \lambda_{\mathcal{I}})$ is geometrically constant and not only of constant Newton polygon). From Subsection 2.3 (b) and Fact 2.4.1 we get that $y(h)$ factors through $\mathcal{N}$ as in Definition 1.4.1 if and only the point $j_{\ell,g}(h) : \mathrm{Spec}(k) \to \mathcal{M}$ induced naturally by $j_\ell(h)$ factors through $\mathcal{N}$ in a way analogous to Definition 1.4.1. But we can choose (enlarge) $\mathcal{Q}$ such that the morphism $j_{\ell,g}(h) : \mathrm{Spec}(k) \to \mathcal{M}$ is the composite of the natural morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(\mathcal{Q})$ with a morphism $w(h) : \mathrm{Spec}(\mathcal{Q}) \to \mathcal{M}$ obtained from w via a natural $\mathbb{Z}[\frac{1}{p}]$-isogeny. As above, from Subsection 2.3 (b) and Fact 2.4.1 we get that $w(h)$ factors through $\mathcal{N}$ in a way analogous to Definition 1.4.1 if and only if the point $y_0(h) : \mathrm{Spec}(\mathbb{F}) \to \mathcal{M}$ induced naturally by $w(h)$ factors through $\mathcal{N}$ as in Definition 1.4.1.

We have $y_0(h) = y_0(h_0)$ for a suitable element $h_0 \in \mathfrak{E}_0^{\mathrm{big}}(y_0)$, where $\mathfrak{E}_0^{\mathrm{big}}(y_0)$ is as in Subsubsection 2.1.1. We assume that we can choose h_0 to be an element of $\mathfrak{E}_0(y_0)$. From this assumption and the condition (i), we get that $y_0(h)$ factors through $\mathcal{N}$ as in Definition 1.4.1. We conclude that $y(h)$ factors through $\mathcal{N}$ as in Definition 1.4.1. As $h \in \mathfrak{E}_0(y)$ was arbitrary, we get that the isogeny property holds for y.

Thus to end the proof of the Theorem, it suffices to show that we can choose h_0 to be an element of $\mathfrak{E}_0(y_0)$. The proof of this is essentially the same as of [Va11, Part I, Subsubsects. 5.1 to 5.4]. We recall the details. We consider a morphism $y_s : \mathrm{Spec}(\mathbb{F}[[x]]) \to \mathrm{Spec}(\mathcal{Q})$ which lifts $y_0(h)$ and which is dominant. Let $(M_s, \phi_{M_s}, (t_{s,\alpha})_{\alpha\in\mathcal{J}}, \nabla_s, \psi_{M_s})$ be the principally quasi-polarized F-crystal with tensors over $\mathbb{F}[[x]]$ associated naturally to the morphism $y_s : \mathrm{Spec}(\mathbb{F}[[x]]) \to \mathrm{Spec}(\mathcal{Q})$.

Let $\mathcal{K}_1$ be the field of fractions of $\mathcal{R}_1$ and let $\mathcal{O}_1$ be the local ring of $\mathcal{R}_1$ which is a discrete valuation ring of mixed characteristic $(0,p)$. We have $\mathcal{O}_1/p\mathcal{O}_1 = \mathbb{F}((x))$. Let $\mathcal{G}_s$ be the Zariski closure in $\boldsymbol{GL}_{M_1}$ of the subgroup of $\boldsymbol{GL}_{M_1\otimes_{\mathcal{R}_1}\mathcal{K}_1}$ that fixes each $t_{s,\alpha}$ with $\alpha \in \mathcal{J}$. If $\mathcal{R}_1 \to \mathcal{O}_1 \to W(k)$ is the Teichmüller lift that lifts the natural embedding $\mathbb{F}[[x]] \hookrightarrow k$, then there exists an isomorphism

$$(7a)\qquad (M_s, (t_{s,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_s}) \otimes_{\mathcal{R}_1} W(k) \xrightarrow{\sim} (h(M), (t_\alpha)_{\alpha\in\mathcal{J}}, \psi_M) \otimes_{W(\mathbb{F})} W(k),$$

cf. constructions. This implies that $\mathcal{G}_{s,\mathcal{O}_1}$ is a reductive subgroup scheme of $\boldsymbol{GL}_{M_s\otimes_{\mathcal{R}_1}\mathcal{O}_1}$.

From this and the crystalline reductiveness principle, as in [Va11, Part I, Thm. 5.2] we get that $\mathcal{G}_s$ itself is a reductive subgroup scheme of $\boldsymbol{GL}_{M_s}$. As in [Va11, Part I, Subsect. 5.3] we argue that we have a cocharacter $\mu_s : \mathbb{G}_m \to \mathcal{G}_s$ such that there exists a direct sum decomposition $M_s = F_s^1 \oplus F_s^0$ with the properties that for $i \in \{0,1\}$ the group scheme $\mathbb{G}_m$ acts on F_s^i through μ_s via the $-i$-th power of the identical character of $\mathbb{G}_m$ and that F_s^1/pF_s^1 is the kernel of ϕ_s modulo p.

We consider the morphism $\mathcal{R}_1 \to W(\mathbb{F})$ that maps x to 0. We have

$$(7b)\qquad (h(M_0), \phi_0, (t_{0,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_0}) = (M_s, \phi_s, (t_{s,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_s}) \otimes_{\mathcal{R}_1} W(\mathbb{F})$$

(cf. the fact that y_s specializes to $y_0(h)$). Let $F_0^1 := F_s^1 \otimes_{W(k)} W(\mathbb{F})$.

As in Subsection 2.3 (a) and (b), Faltings deformation theory implies that the quintuple $(M_s, \phi_s, F_s^1, (t_{s,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_s})$ is induced from a deformation of $(h(M_0), F_0^1, \phi_0, (t_{0,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_0})$ to $\mathcal{R}_b/p\mathcal{R}_b$ constructed as in Subsection 2.3 (a). In particular, we have an isomorphism

$$(7c)\qquad (h(M_0), (t_{0,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_0}) \otimes_{W(\mathbb{F})} W(\mathbb{F})[[x]] \xrightarrow{\sim} (M_s, (t_{s,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_s}).$$

From (7a) to (7c) and from (1) we get that there exists an isomorphism

$$(M_0, (t_\alpha)_{\alpha\in\mathcal{J}}, \psi_{M_0}) \xrightarrow{\sim} (h(M_0), (t_{0,\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_0}).$$

This implies that we can assume that $h \in \mathcal{G}_0(B(\mathbb{F}))$ fixes ψ_{M_0}. The fact that we have in fact $h \in \mathfrak{C}_0(y_0)$ follows from the existence of the cocharacter μ_s. $\square$

4.2. Proposition. *We assume that the Conjecture 1.2 holds for the point $y : Spec(\mathbb{F}) \to \mathcal{N}$. Then the endomorphism property holds for $y : Spec(\mathbb{F}) \to \mathcal{N}$.*

Proof: We consider the direct sum decomposition

$$\mathrm{End}(W) = \mathrm{Lie}(G) \oplus \mathrm{Lie}(G)^{\perp}, \tag{8}$$

where $\mathrm{Lie}(G)^{\perp}$ is the perpendicular on $\mathrm{Lie}(G)$ with respect to the trace form on $\mathrm{End}(W)$. Let Π be the projector of $\mathrm{End}(W)$ on $\mathrm{Lie}(G)$ along $\mathrm{Lie}(G)^{\perp}$; we identify it as well with an element of $\mathrm{End}(W) \otimes_{\mathbb{Q}} \mathrm{End}(W)^* = \mathrm{End}(W) \otimes_{\mathbb{Q}} \mathrm{End}(W) \subseteq \mathcal{T}(\mathrm{End}(W))$ fixed by G. Thus we can assume that there exists an element $\alpha_0 \in \mathcal{J}$ such that we have $v_{\alpha_0} = \Pi$.

Let A_0, $\mathbb{F}_{p^q}$, $(M_0, \phi_0, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_{M_0})$, and π_0 be as in Subsection 2.1. The centralizer $\mathcal{C}$ of π_0 in $\mathrm{End}(M[\frac{1}{p}])$ is the $B(\mathbb{F})$-subalgebra generated by crystalline realizations of $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Therefore we can identify naturally

$$\mathcal{C} = \mathrm{End}(A)^{\mathrm{opp}} \otimes_{\mathbb{Z}} B(\mathbb{F}). \tag{9}$$

As π_0 fixes t_{α_0} and due to (8), we have a direct sum decomposition of $B(\mathbb{F})$-vector spaces

$$\mathcal{C} = (\mathcal{C} \cap \mathrm{Lie}(\mathcal{G})[\frac{1}{p}]) \oplus (\mathcal{C} \cap \mathrm{Lie}(\mathcal{G})[\frac{1}{p}]^{\perp}), \tag{10}$$

where $\mathrm{Lie}(\mathcal{G})[\frac{1}{p}]^{\perp}$ is the perpendicular on $\mathrm{Lie}(\mathcal{G})[\frac{1}{p}]$ with respect to the trace form on $\mathrm{End}(M[\frac{1}{p}])$. But t_{α_0} is the crystalline realization of an algebraic cycle $c_{\alpha_0,A}$ on A, cf. the assumption that the Conjecture 1.2 holds for y. From this, (9), and (10), we get that we have a direct sum decomposition of $\mathbb{Q}$–vector spaces

$$\mathrm{End}(A)^{\mathrm{opp}} \otimes_{\mathbb{Z}} \mathbb{Q} = (\mathrm{End}(A)^{\mathrm{opp}} \otimes_{\mathbb{Z}} \mathbb{Q} \cap \mathrm{Lie}(\mathcal{G})[\frac{1}{p}]) \oplus (\mathrm{End}(A)^{\mathrm{opp}} \otimes_{\mathbb{Z}} \mathbb{Q} \cap \mathrm{Lie}(\mathcal{G})[\frac{1}{p}]^{\perp}).$$

In other words, there is a Lie subalgebra $\mathrm{End}(A)^{\mathrm{opp}} \otimes_{\mathbb{Z}} \mathbb{Q} \cap \mathrm{Lie}(\mathcal{G})[\frac{1}{p}]$ of $\mathrm{End}(A)^{\mathrm{opp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ such that a $\mathbb{Q}$–basis for it is also a $\mathbb{Q}_p$-basis for $\{x \in \mathrm{Lie}(\mathcal{G})[\frac{1}{p}] | \phi(x) = x\}$. This implies that there is a reductive subgroup $\mathcal{E}$ of the group of invertible elements of $\mathrm{End}(A)^{\mathrm{opp}} \otimes_{\mathbb{Z}} \mathbb{Q}$ whose Lie algebra is precisely $\mathrm{End}(A)^{\mathrm{opp}} \otimes_{\mathbb{Z}} \mathbb{Q} \cap \mathrm{Lie}(\mathcal{G})[\frac{1}{p}]$, cf. [Va11, Part I, App., Lemma A2 (b)]. Due to the fact that the cycle Π is algebraic, for each prime $l \neq p$ the l-adic realization of an arbitrary element of $\mathcal{E}(\mathbb{Q})$ is an element of $G(\mathbb{Q}_l)$. Therefore $\mathcal{E}$ is a subgroup of the group defined in Definition 1.4.3. We get that $\mathcal{E}$ is indeed the group defined in Definition 1.4.3 and is reductive of the same dimension as $\{x \in \mathrm{Lie}(\mathcal{G})[\frac{1}{p}] | \phi(x) = x\}$. Thus the endomorphism property holds for $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$. Thus (a) holds. $\square$

4.3. Proposition. *Let $y : Spec(\mathbb{F}) \to \mathcal{N}$ be a basic point. We assume that the following six properties hold:*

(i) *the derived group G^{der} is simply connected;*

(ii) *the isogeny property holds for y;*

(iii) *the endomorphism property holds for y;*

(iv) *there exists a semisimple $\mathbb{Z}_{(p)}$-subalgebra of $End(L_{(p)})$ such that its centralizer in $\boldsymbol{GSp}(L_{(p)},\psi)$ is a reductive group scheme $G_{1,\mathbb{Z}_{(p)}}$ with the properties that $Z^0(G)=Z^0(G_1)$ and that the monomorphism $G^{\mathrm{der}}_{W(\mathbb{F})}\to G^{\mathrm{der}}_{1,W(\mathbb{F})}$ is either an isomorphism or a product of monomorphisms of one of the following three forms: $\boldsymbol{Sp}_{2n}\hookrightarrow \boldsymbol{SL}_{2n}$, $\boldsymbol{Spin}_{2n-1}\hookrightarrow \boldsymbol{SL}_{2^{n-1}}$, $\boldsymbol{Spin}_{2n}\hookrightarrow \boldsymbol{SL}_{2^{n-2}}\times \boldsymbol{SL}_{2^{n-2}}$;*

(v) *the Hasse principle holds for the tori G^0/G^{der} and G^0_1/G^{der}_1, where $G^0=G/\cap \boldsymbol{Sp}(W,\psi)$ and $G^0_1:=G_1\cap \boldsymbol{Sp}(W,\psi)$;*

(vi) *either the abelian variety A is supersingular or we have $G=G_1$.*

Then we have $\mathfrak{s}_0(\mathbb{F})=o(y)$.

Proof: Let G_1 be the generic fibre of $G_{1,\mathbb{Z}_{(p)}}$. We have injective maps $(G,X)\hookrightarrow(G_1,X_1)\hookrightarrow(\boldsymbol{GSp}(W,\psi),S)$ of Shimura pairs, cf. property (iv). Let $y_0:\mathrm{Spec}(\mathbb{F})\to\mathcal{N}$ be a basic point. To prove the Proposition it suffices to show that $y_0\in o(y)$. To prove this we can replace y_0 by a point in the $G(\mathbb{A}_f^{(p)})$-orbit of y_0. Thus we can assume that both points $y,y_0:\mathrm{Spec}(\mathbb{F})\to\mathcal{N}$ factor through the same connected component of $\mathcal{N}$, cf. [Va1, Lemma 3.3.]. For the sake of clarity we will number the steps of the proof.

Step 1. We know that there exist elements $h\in\mathfrak{E}_0(y)$ and $t\in G_1(\mathbb{A}_f^{(p)})$ such that we have an identity $y_0=y(h)t$ of $\mathbb{F}$-valued points of $\mathcal{M}$, cf. [Va10, Prop. 9.3 (b)] (here we are using properties (i) and (v)). As both points $y,y_0:\mathrm{Spec}(\mathbb{F})\to\mathcal{N}$ factor through the same connected component of $\mathcal{N}$, we have $t\in G^0_1(\mathbb{A}_f^{(p)})$. If $G=G_1$, then (due to property (iv)) it is well known that $\mathcal{N}=\mathcal{N}^{\mathrm{cl}}$ and thus we have $y_0\in o(y)$.

Thus from now on we can assume that $G\neq G_1$; therefore A is a supersingular abelian variety (cf. property (vi)). Let $t_0=t^{-1}$. Due to the property (ii), we can assume that $h=1_M$. Thus we have $y=y_0t_0$ and we can identify $(M_0,\phi_0,\psi_{M_0})=(M,\phi,\psi_M)$ and $(A_0,\lambda_{A_0})=(A,\lambda_A)$.

Let $b:=\dim(X)$ and $\mathcal{R}_b$ be as in Subsection 2.3. Let $\mathcal{D}_y$ be the formal deformation space of (A,λ_A) corresponding to $y:\mathrm{Spec}(\mathbb{F})\to\mathcal{N}$ and let $\mathcal{D}^{\mathrm{big}}_y$ be the formal deformation space of (A,λ_A) corresponding to the composite point $y:\mathrm{Spec}(\mathbb{F})\to\mathcal{N}\to\mathcal{M}$. We fix identifications $\mathcal{D}_y=\mathrm{Spf}(\mathcal{R}_b)$ and $\mathcal{D}^{\mathrm{big}}_y=\mathrm{Spf}(\mathcal{R}_{\frac{d(d+1)}{2}})$. Similarly we define $\mathcal{D}_{y_0}$ and its translation $\mathcal{D}_{y_0,t_0}$ via t_0. Both $\mathcal{D}_y$ and $\mathcal{D}_{y_0,t_0}$ are formal deformation subspaces of $\mathcal{D}^{\mathrm{big}}_y$. For $n\in\mathbb{N}$ let $\mathcal{D}_{y,n}:=\mathrm{Spf}(\mathcal{R}_b/(p,x_1,\ldots,x_b)^n)$. Similarly we define $\mathcal{D}^{\mathrm{big}}_{y,n}$ and $\mathcal{D}_{y_0,t_0,n}$.

Let $\mathcal{E}^0$ be the subgroup of $\mathcal{E}$ that fixes λ_A (equivalently ψ_M). Let $\mathcal{E}_1$ and $\mathcal{E}^0_1$ be the analogues of $\mathcal{E}$ and $\mathcal{E}^0$ (respectively) but associated to the point $y_1:\mathrm{Spec}(\mathbb{F})\to\mathcal{N}\to\mathcal{N}_1$ defined by y, where $\mathcal{N}_1$ is the integral canonical model of $(G_1,X_1,G_{1,\mathbb{Z}_{(p)}}(\mathbb{Z}_p),v_1)$ (with v_1 as the prime of $E(G_1,X_1)$ divided by v) and where $\mathcal{N}\to\mathcal{N}_1$ is the functorial morphism.

Let $g\in\mathcal{E}^0_1(\mathbb{Q}_p)$ be such that it takes $t_{0,\alpha}$ to t_α for all $\alpha\in\mathcal{J}$, cf. proof of [Va10, Prop. 9.3 (b)]. For each $n\in\mathbb{N}$ there exists a $\mathbb{Z}_{(p)}$-automorphism g_n of $(A_0,\lambda_{A_0})=(A,\lambda_A)$ such that g and g_n are congruent modulo p^n. Let t_n be the image of g_n in $G^0_1(\mathbb{A}_f^{(p)})$.

Step 2. We will show that the two formal subschemes of $\mathcal{D}_y^{\mathrm{big}}$ corresponding to the basic loci of $\mathcal{D}_{y_0,t_0}$ and $\mathcal{D}_y$ are permuted under the natural action of a $\mathbb{Z}_{(p)}$-automorphism g_0 of (A,λ_A) on $\mathcal{D}_y^{\mathrm{big}}$. For this part of the proof we will allow h to be an arbitrary element of $\mathfrak{E}_0^{\mathrm{big}}(y)\cap G_1(B(\mathbb{F}))$. Let $n_0\in\mathbb{N}$ be such that each $\mathbb{Z}_{(p)}$-automorphism of (A,λ_A) acts trivially on the supersingular locus of $\mathcal{D}_y^{\mathrm{big}}$. Briefly, speaking n is the largest number such that we have $\frac{1}{p^{n_0}}\mathbb{Z}$–isogenies between the suitable extensions of (A,λ_A) and the pull backs of $(\mathfrak{A},\lambda_{\mathfrak{A}})$ via arbitrary generic points of the supersingular locus of $\mathcal{D}_y$ (here the isogenies should be compatible with the natural families of tensors indexed by $\mathcal{J}$). Let $g_0:=g_{n_0}$. Let $n\in\mathbb{N}$ with $n\geq n_0$. By replacing t_0 with t_nt_0, h with g_nh, and each $t_{0,\alpha}$ with $g_n(t_{0,\alpha})$, we can assume that g_n is the identity element. Such a replacement is allowed as g_n and g_{n_0} act in the same way on the supersingular locus of $\mathcal{D}_y^{\mathrm{big}}$. As g_n is the identity element, the reductive group schemes $\mathcal{G}_{0,W_n(\mathbb{F})}$ and $\mathcal{G}_{W_n(\mathbb{F})}$ coincide. From this and Faltings deformation theory (see Subsection 2.3 (a)) we easily get that $\mathcal{D}_{y,p^n}=\mathcal{D}_{y_0,t_0,p^n}$. As $n\geq n_0$ is arbitrary, by passing to limit $n\to\infty$ we conclude that to show that g_0 has the desired property we can assume that $\mathcal{D}_y=\mathcal{D}_{y_0,t_0}$ and this case is obvious.

Step 3. Let $\mathfrak{c}_0^{\mathrm{cl}}$ be the image in $\mathcal{N}_{k(v)}^{\mathrm{cl}}$ of the connected component $\mathfrak{c}_0$ of the basic locus $\mathfrak{s}_0$ which contains the point y_0. Similarly we define $\mathfrak{c}$ and $\mathfrak{c}^{\mathrm{cl}}$ but working with y. We view both $\mathfrak{c}_0^{\mathrm{cl}}$ and $\mathfrak{c}^{\mathrm{cl}}$ as reduced, closed subschemes of $\mathcal{N}_{k(v)}^{\mathrm{cl}}$.

Due to Step 2, we get that $\mathfrak{c}_0^{\mathrm{cl}}t_{n_0}t_0$ is isomorphic to $\mathfrak{c}^{\mathrm{cl}}$ under the isomorphism defined by g_0. As A is supersingular, it is easy to see that each $\mathbb{F}$-valued point of $\mathfrak{c}_0^{\mathrm{cl}}$ is of the form $y_0(h_0)$ for some element $h_0\in\mathfrak{E}_0(y_0)$. Similarly, each $\mathbb{F}$-valued point of $\mathfrak{c}$ belongs to $o(y)$. We conclude that $\mathfrak{c}(\mathbb{F})G(\mathbb{A}_f^{(p)})$ is a $G(\mathbb{A}_f^{(p)})$-invariant subset of $o(y)$ (and thus also of $o^{\mathrm{big}}(y)$, cf. Corollary 2.3.5 and the property (iii)) which is equal to the subset $\mathfrak{c}_0^{\mathrm{cl}}(\mathbb{F})t_{n_0}t_0G(\mathbb{A}_f^{(p)})$ of $o^{\mathrm{big}}(y)$.

Based on the previous paragraph, a simple count on points (cf. formula (3)) shows (as in the proof of Lemma 3.3) that the endomorphism property holds for y_0 and in fact that for each prime $l\neq p$, $\mathcal{E}_{\mathbb{Q}_l}$ is the conjugate of $\mathcal{E}_{0,\mathbb{Q}_l}$ under the element $t_nt_0\in G_1^0(\mathbb{A}_f^{(p)})$.

Step 4. Due to the property (iv), the torus $Z^0(G_1^0)$ contains $Z(G^{\mathrm{der}})$ and G^{ad} is its own normalizer in G_1^{der}. From this and the property (vi) we get that G^0 is its own normalizer in G_1^0. We consider the torsor of G^0 that parametrizes the existence of an element $j\in\mathcal{E}_1^0(\mathbb{Q})$ that takes under inner conjugation $\mathcal{E}$ to $\mathcal{E}_0$. This makes sense as G^0 is its own normalizer in G_1^0. This torsor defines a class $\gamma\in H^1(\mathbb{Q},G^0)$ which is trivial at all primes $l\neq p$, cf. end of Step 3. As $\mathcal{E}_{\mathbb{R}}^{\mathrm{der}}$ and $\mathcal{E}_{0,\mathbb{R}}^{\mathrm{der}}$ are compact subgroups of $\mathcal{E}_{1,\mathbb{R}}^0$, from the property (iv) we easily get that they are $\mathcal{E}_1^0(\mathbb{R})$-conjugate. Thus γ is also trivial at the infinite place of $\mathbb{Q}$. The existence of the element g implies that the class γ is also trivial at the prime p. The Hasse principle holds for G^{der} (cf. property (i)) and for G^0/G^{der} (cf. property (iv)) and thus it also holds for G^0 itself. This implies that the class γ is trivial. Thus the mentioned torsor is trivial and therefore there exists an element $j\in\mathcal{E}_1^0(\mathbb{Q})$ such that we have $j\mathcal{E}_0j^{-1}=\mathcal{E}$.

Step 5. The existence of j allows us to redo Step 1 in a better way. More precisely, j defines a $\mathbb{Q}$–isogeny from (A,λ_A) to (A_0,λ_{A_0}) under which we can identify $(M[\frac{1}{p}],\phi,\psi_M)=(M_0[\frac{1}{p}],\phi_0,\psi_{M_0})$ in such a way that we have $\mathcal{E}_0=\mathcal{E}$ and there exists an element $\tilde{g}\in\mathcal{E}_1(\mathbb{Q}_p)$

which fixes ψ_M and which takes each t_α to $t_{0,\alpha}$. As $\mathcal{E}_0=\mathcal{E}$ we also have $\mathcal{G}_{B(\mathbb{F})}=\mathcal{G}_{0,B(\mathbb{F})}$. Thus $\tilde g$ normalizes $\mathcal{G}_{B(\mathbb{F})}$ and therefore we have $\tilde g\in\mathcal{G}(B(\mathbb{F}))$. This implies that $\tilde g\in\mathcal{E}(\mathbb{Q}_p)$. Thus we have $t_{0,\alpha}=t_\alpha$. As we have assumed that the element $h\in\mathfrak{E}_o(y)$ of Step 1 is the identity, we can choose $\tilde g$ such that in fact we have $\tilde g(M)=M_0$. By replacing y with $y(\tilde g)$, we can assume that $\tilde g=1_M$. This implies we have $t_\alpha=t_{0,\alpha}$ for all $\alpha\in\mathcal{J}$ and that there exists an element $t\in G(\mathbb{A}_f^{(p)})$ such that we have an identity $y_0=yt$ of $\mathbb{F}$-valued points of $\mathcal{M}$ (more precisely, as $\mathcal{E}_0=\mathcal{E}$ such an element $t\in G_1(\mathbb{A}_f^{(p)})$ normalizes $G(\mathbb{A}_f^{(p)})$ and thus it belongs to $G(\mathbb{A}_f^{(p)})$). As we have $t_\alpha=t_{0,\alpha}$ for all $\alpha\in\mathcal{J}$, from Lemma 2.3.3 we get that we have an identity $y_0=yt$ of $\mathbb{F}$-valued points of $\mathcal{N}$. Thus $y_0\in o(y)$. $\square$

4.4. Proposition. *Let $y:Spec(\mathbb{F})\to\mathcal{N}$ be a basic point. We assume that one of the following two properties hold:*

(i) *either A is supersingular and the Conjecture 1.2 holds for y*

(ii) *or the adjoint group G^{ad} is simple and there exists a semisimple $\mathbb{Z}_{(p)}$-subalgebra of $End(L_{(p)})$ such that its centralizer in $\boldsymbol{GSp}(L_{(p)},\psi)$ is $G_{\mathbb{Z}_{(p)}}$.*

Then the isogeny property holds for y.

Proof: We first assume that the property (i) holds. We consider the moduli space $\mathfrak{s}$ over $\mathbb{F}$ of quadruples of the form $(\tilde A,\lambda_{\tilde A},(c_{\alpha,\tilde A})_{\alpha\in\mathcal{J}},(\eta_{N,\tilde A})_{N\in\mathbb{N}\setminus p\mathbb{N}})$, where $(\tilde A,\lambda_{\tilde A})$ is a principally polarized supersingular abelian variety endowed with a family of algebraic cycles (up to numerical equivalence) $(c_{\alpha,\tilde A})_{\alpha\in\mathcal{J}}$ and with compatible level-N symplectic similitude structures $\eta_{N,\tilde A}$ for all $N\in\mathbb{N}$ relatively prime to p, which satisfies the following two axioms modeled on $\mathcal{N}$ and $\mathfrak{s}_0$:

(a) For each prime l, the l-adic realization of $c_{\alpha,\tilde A}$ corresponds to v_α via the isomorphism $W\otimes_{\mathbb{Q}}\mathbb{Q}_l\xrightarrow{\sim}(\text{proj.lim.}_{n\in\mathbb{N}}\tilde A[l^n])\otimes_{\mathbb{Z}_l}\mathbb{Q}_l$ induced by the $\eta_{l^n,\tilde A}$'s with $n\in\mathbb{N}$.

(b) There exists a $\mathbb{Q}$–isogeny $(\tilde A,\lambda_{\tilde A},(c_{\alpha,\tilde A})_{\alpha\in\mathcal{J}})\cdots>(A,\lambda_A,(c_{\alpha,A})_{\alpha\in\mathcal{J}})$ under which the principally quasi-polarized Dieudonné module of $(\tilde A,\lambda_{\tilde A})$ endowed with the family of tensors that are the crystalline realizations of $(c_{\alpha,\tilde A})_{\alpha\in\mathcal{J}})$, can be identified with a quadruple of the form $(h(M),\phi,(t_\alpha)_{\alpha\in\mathcal{J}},\psi_M)$ for some element $h\in\mathfrak{E}_0(y)$.

The moduli space $\mathfrak{s}$ is a reduced, closed, $G(\mathbb{A}_f^{(p)})$-invariant subscheme of $\mathcal{M}_{\mathbb{F}}$ and it a pro-étale cover of a projective $\mathbb{F}$-scheme. The set $\mathfrak{s}(\mathbb{F})$ is exactly the set of $\mathbb{F}$-valued points of $\mathcal{M}$ defined by the points $y(h)G(\mathbb{A}_f^{(p)})$ with $h\in\mathfrak{E}_0(y)$.

Let $\mathfrak{c}$ be the connected component of $\mathfrak{s}$ to which y belongs. As A is supersingular, it is easy to see that $\mathfrak{c}$ is a closed subscheme of both $\mathcal{N}_{k(v)}$ and $\mathcal{N}^{\mathrm{cl}}_{k(v)}$ with the property that each point of $\mathfrak{c}(\mathbb{F})$ is of the from $y(h)$ for some element $h\in\mathfrak{E}_0(y)$. Thus to prove the Proposition in the case when (i) holds, it suffices to show that the connected components of $\mathfrak{s}$ are permuted transitively by $G(\mathbb{A}_f^{(p)})$.

But this follows from the following two facts:

(c) Each connected component $\mathfrak{c}_0$ of $\mathfrak{s}$ has an $\mathbb{F}$-valued pivotal point (in the sense of Definition 3.1.2).

(d) The $\mathbb{F}$-valued pivotal points of $\mathfrak{s}$ form one orbit under the action of $G(\mathbb{A}_f^{(p)})$.

To check (c), we first recall that each connected component $\mathfrak{c}_0$ is a pro-étale cover of a projective $\mathbb{F}$-scheme. Based on this and Lemma 3.4.1, we get that there exists a stratum $\mathfrak{l}$ of the level 1 stratification of $\mathcal{N}_{k(v)}$ whose intersection with $\mathfrak{c}_0$ is non-empty and has dimension 0. But from Theorem 3.8 (c) we get that the points of $\mathfrak{l}\cap\mathfrak{c}_0$ are pivotal (though Theorem 3.8 is stated in terms only of $\mathbb{F}$-valued points of $\mathcal{N}$, we have provided a proof of it that applies to all formal deformation spaces of Shimura F-crystals). We conclude that $\mathfrak{l}$ is the stratum corresponding to pivotal points and thus the property (c) holds.

Property (d) is a direct consequence of the fact that the set of lattices $h(M)$ of $M[\frac{1}{p}]$ with the property that $h\in\mathfrak{E}_0(y)$ and that the quadruple $(h(M),\phi,\psi_M,h(\mathcal{G}h^{-1})$ is pivotal (i.e., is $\mathcal{E}^0(\mathbb{Q}_p)$ isomorphic to the one associated to a pivotal point of $\mathfrak{s}_0(\mathbb{F})$), are conjugate under $\mathcal{E}^0(\mathbb{Q}_p)$ and thus also under the subgroup $\mathcal{E}^0(\mathbb{Q})$ of $\widetilde{\mathcal{E}(\mathbb{Q})}$. Here $\mathcal{E}^0$ is the subgroup of $\mathcal{E}$ that fixes λ_A (equivalently ψ_M).

Thus if (i) holds we conclude that $\mathfrak{s}=\mathfrak{s}_0=\mathfrak{c}G(\mathbb{A}_f^{(p)})$.

We now assume that (ii) holds. The same proof applies, with the only modification that instead of algebraic cycles we have to use endomorphisms. We only have to add that the resulting moduli space $\mathfrak{s}$ is a disjoint union of a finite number of basic loci and thus it is a pro-étale cover of a projective $\mathbb{F}$-scheme, cf. Lemma 3.1.1. $\square$

5. Proof of the Basic Theorem 1.5

In this Section we prove the basic Theorem 1.5. If E is a number field unramified over p, let $E_{(p)}$ be the normalization of $\mathbb{Z}_{(p)}$ in E; it is a finite, étale $\mathbb{Z}_{(p)}$-algebra. Let the simple, adjoint Shimura pair (G_0,X_0) and the prime p be as in Theorem 1.5.

5.1. Constructions. We begin the proof of Theorem 1.5 by constructing the injective map $(G,X)\hookrightarrow(\boldsymbol{GSp}(W,\psi),S)$ and the $\mathbb{Z}$-lattice L of W. For the sake of clarity, we itemize the steps of the construction. We will construct only $(G,X)\hookrightarrow(\boldsymbol{GSp}(W,\psi),S)$ and the $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W (implicitly L will be an arbitrary $\mathbb{Z}$-lattice of W which extends $L_{(p)}$ and which is self dual with respect to ψ).

(a) Let F_0 be a totally real number field such that we have $G_0=\mathrm{Res}_{F_0/\mathbb{Q}}J_0$, with J_0 an absolutely simple, adjoint group over $\mathbb{F}_0$, cf. [De2, Subsubsect. 2.3.4 (a)]. The field F_0 is unique up to isomorphisms (i.e., up to $\mathrm{Gal}(\mathbb{Q})$-conjugation). As the group $G_{0,\mathbb{Q}_p}=\mathrm{Res}_{F_0\otimes_{\mathbb{Q}}\mathbb{Q}_p/\mathbb{Q}_p}J_0\times_{F_0}\mathbb{Q}_p$ is unramified, the $\mathbb{Q}_p$-algebra $F_0\otimes_{\mathbb{Q}}\mathbb{Q}_p/\mathbb{Q}_p$ is unramified. Thus the field F_0 is unramified above p.

(b) If (G_0,X_0) is of A_1, B_n, C_n, or $D_n^{\mathbb{R}}$ type, let E_0 be an arbitrary totally imaginary quadratic extension of F_0 for which the following two properties hold:

(b.i) There exists a prime $l\neq p$ which splits in F_0 and E_0 has exactly $[F_0:\mathbb{Q}]$ primes that divide l, all but one of them being unramified over l.

(b.ii) It is unramified above p and moreover there exists no prime of F_0 that divides p and that splits in E_0.

If (G_0,X_0) is of A_n type with $n\geq 2$, let E_0 be the unique totally imaginary quadratic extension of F_0 such that the $\mathrm{Gal}(\mathbb{Q})$-set $\mathrm{Hom}(E_0,\overline{\mathbb{Q}})$ is naturally identified with the

$\mathrm{Gal}(\mathbb{Q})$-set defined by the ending notes of the Dynkin diagram of $G_{0,\overline{\mathbb{Q}}}$ (see either [De2] or [Va3, Subsect. 2.2] for the action of $\mathrm{Gal}(\mathbb{Q})$ on the Dynkin diagram of $G_{0,\overline{\mathbb{Q}}}$). As $G_{0,\mathbb{Q}_p}$ splits over an unramified finite field extension of $\mathbb{Q}_p$, the field E_0 is unramified over p.

(c) Let H_0 be an arbitrary hyperspecial subgroup of $G_0(\mathbb{Q}_p)$. Let $G_{0,\mathbb{Z}_{(p)}}$ be the unique adjoint group scheme over $\mathbb{Z}_{(p)}$ such that we have $G_{0,\mathbb{Z}_{(p)}}(\mathbb{Z}_p) = H_0$, cf. [Va5, Lemma 2.3 (a)]. Let $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}}$ be the simply connected semisimple group scheme cover of $G_{0,\mathbb{Z}_{(p)}}$.

(d) We assume that (G_0, X_0) is of A_n type. We take $(G, X) \hookrightarrow (\boldsymbol{GSp}(W, \psi), S)$ and L to be as in [Va5, Prop. 3.2]. Thus the property (*) holds, $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}}$ is the derived group of $G_{\mathbb{Z}_{(p)}}$, the centralizer $\mathcal{B}_{(p)}$ of $G_{\mathbb{Z}_{(p)}}$ in $\boldsymbol{GL}_{L_{(p)}}$ is a semisimple $\mathbb{Z}_{(p)}$-algebra, and $G_{\mathbb{Z}_{(p)}}$ is the centralizer of $\mathcal{B}_{(p)}$ in $\boldsymbol{GSp}(L_{(p)}, \psi)$. In particular, (G, X) is a Shimura variety of PEL type and the embedding $(G, X) \hookrightarrow (\boldsymbol{GSp}(W, \psi), S)$ is a PEL type embedding. We recall from [Va5, Prop. 3.2] and its proof, that $L_{(p)}$ has a natural structure of an $E_{0,(p)}$-module and that $Z(G_{\mathbb{Z}_{(p)}})$ is connected and a subtorus of $\mathrm{Res}_{E_{0,(p)}/\mathbb{Z}_{(p)}}\mathbb{G}_m$. More precisely, $Z(G_{\mathbb{Z}_{(p)}})$ is the largest subtorus $T_{\mathbb{Z}_{(p)}}$ of $\mathrm{Res}_{E_{0,(p)}/\mathbb{Z}_{(p)}}\mathbb{G}_m$ that is the extension of $Z(\boldsymbol{GL}_{L_{(p)}})$ by a torus $T_{c,\mathbb{Z}_{(p)}}$ which over $\mathbb{R}$ is compact (and which is isogenous to $\mathrm{Res}_{F_{0,(p)}/\mathbb{Z}_{(p)}}[\mathrm{Res}_{E_{0,(p)}/F_{0,(p)}}\mathbb{G}_m]/\mathbb{G}_m$). Let

$$W \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{j \in \mathrm{Hom}(E_0, \mathbb{C})} W_j$$

be the direct sum decomposition such that W_j is the largest complex subspace of $W \otimes_{\mathbb{Q}} \mathbb{C}$ on which E_0 acts via scalar multiplication through the embedding $j : E_0 \hookrightarrow \mathbb{C}$. If $i : F_0 \hookrightarrow \mathbb{R}$ is such that the group $J_0 \times_{F_0,i} \mathbb{R}$ is compact and if $j_{i,1}, j_{i,2} : E_0 \hookrightarrow \mathbb{C}$ are the two embeddings that extend i, then one has an (independent) choice to make on the Hodge types of $W_{j_{i,1}}$ and $W_{j_{i,2}}$ defined by a (any) element $x \in X$ (cf. proof of [Va5, Prop. 3.2]):

(d.i) If $W \otimes_{\mathbb{Q}} \mathbb{C} = F^{-1,0}_x \oplus F^{0,-1}_x$ is the Hodge decomposition defined by x, then either we have either $W_{j_{i,1}} \subseteq F^{-1,0}_x$ and $W_{j_{i,2}} \subseteq F^{0,-1}_x$ or we have $W_{j_{i,1}} \subseteq F^{0,-1}_x$ and $W_{j_{i,2}} \subseteq F^{-1,0}_x$.

For most applications, it is irrelevant which choices we make. However, in Subsection 5.3 (f) below we will impose a technical condition (***) which might or might not hold, depending on the choices we make. Thus, if possible, the choices will be such that the technical condition (***) holds.

(e) We assume that (G_0, X_0) is of B_n or $D^{\mathbb{R}}_n$ (resp. of C_n) type and that has compact factors. We will take $(G, X) \hookrightarrow (\boldsymbol{GSp}(W, \psi), S)$ as in [De2, Prop. 2.3.10] and [Va1, Subsects. 6.5 and 6.6]. To make this very explicit, we will introduce an extra number field E which contains F_0, which is unramified over p, and for which the group scheme $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} E_{(p)}$ is split. Let $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} E_{(p)} \hookrightarrow \boldsymbol{GL}_{\tilde{W}_{(p)}}$ be the standard spin (resp. symplectic) representation. Thus $\tilde{W}_{(p)}$ is a free $E_{(p)}$-module of rank 2^{n-1} (resp. $2n$). Let $L_{(p)} := \tilde{W}_{(p)} \otimes_{\mathbb{F}_{0,(p)}} E_{0,(p)}$ but viewed as a free $\mathbb{Z}_{(p)}$-module. The semisimple group scheme $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}}$ is naturally a closed subgroup scheme of $\boldsymbol{GL}_{L_{(p)}}$. Based on [Va5, Lemma 2.3 (b)], we can speak about the reductive group scheme $\tilde{G}_{\mathbb{Z}_{(p)}}$ which is the closed subgroup scheme of $\boldsymbol{GL}_{L_{(p)}}$ generated by $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}}$ and by the torus $T_{\mathbb{Z}_{(p)}}$ obtained as in (d). If (G_0, X_0) is not of

$D_n^{\mathbb{R}}$ type, then we take $G_{\mathbb{Z}_{(p)}} := \tilde{G}_{\mathbb{Z}_{(p)}}$. If (G_0, X_0) is of $D_n^{\mathbb{R}}$ type, then we take $G_{\mathbb{Z}_{(p)}}$ to be generated by $\tilde{G}_{\mathbb{Z}_{(p)}}$ and by the largest torus of $\boldsymbol{Sp}(L_{(p)}, \psi)$ which centralizes $G_{\mathbb{Z}_{(p)}}$. Let G and $\tilde{G}$ be the generic fibres of G and $\tilde{G}$ (respectively). Let $Z^{00}(G) := Z^0(\tilde{G}) = T_{\mathbb{Z}_{(p)}}$. As in [Va1, Subsects. 6.5 and 6.6], [Va5, Prop. 3.2, end of proof], or [Va11, Part I, proof of Lemma 4.2.1] we argue that we can choose ψ such that it restricts to a perfect alternating form on $L_{(p)}$. As (G_0, X_0) has compact factors, it is easy to see that the property (b.i) implies (cf. Lemma 2.2.1 applied to $(T, T_1) := (\tilde{G}/G^{\mathrm{der}}, \mathbb{G}_m)$) that:

(e.i) The smallest subtorus $\diamond$ of $Z^0(G)$ such that each element of X factors through the extension to $\mathbb{R}$ of the subgroup of G generated by G^{der} and $\diamond$, is $Z^{00}(G)$ (thus it is $Z^0(G)$ itself if and only if (G_0, X_0) is not of $D_n^{\mathbb{R}}$ type).

As $Z^{00}(G)$ is the extension of a torus isogenous to $\mathrm{Res}_{F_{0,(p)}/\mathbb{Z}_{(p)}}[\mathrm{Res}_{E_{0,(p)}/F_{0,(p)}}\mathbb{G}_m]/\mathbb{G}_m$ by $Z(\boldsymbol{GL}_W)$, from the property (b.ii) we get that:

(e.ii) The torus $Z^{00}(G)_{\mathbb{Q}_p}$ is the extension of an anisotropic torus over $\mathbb{Q}_p$ by $Z(\boldsymbol{GL}_W)_{\mathbb{Q}_p}$.

(f) We assume that (G_0, X_0) is of A_n, B_n, C_n, or $D_n^{\mathbb{R}}$ type and that all simple factors of $G_{0,\mathbb{R}}$ are non-compact. If (G_0, X_0) is of A_n type, we also assume that n is odd and that $G_{\mathbb{R}}^{\mathrm{der}}$ is a product of $\boldsymbol{SU}(\frac{n+1}{2}, \frac{n+1}{2})$ groups. Then in (d) and (e), we could take $G_{\mathbb{Z}_{(p)}}$ to be the reductive subgroup scheme of $\boldsymbol{GL}_{L_{(p)}}$ generated by $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}}$ and by $Z(\boldsymbol{GL}_{L_{(p)}})$ (this makes sense, cf. [Va5, Lemma 2.3 (b)]). But for the sake of uniformity, we will take $G_{\mathbb{Z}_{(p)}}$ to be the reductive subgroup scheme of $\boldsymbol{GL}_{L_{(p)}}$ generated by $G^{\mathrm{sc}}_{\mathbb{Z}_{(p)}}$ and by $Z(\boldsymbol{GL}_{L_{(p)}})$ only in the case when (G_0, X_0) is of C_n type. The reflex field is $E(G, X) = \mathbb{Q}$, cf. [De2, Rms. 2.3.12 and 2.3.13]. Let $Z^{00}(G) := Z(\boldsymbol{GL}_{L_{(p)}})$; the analogue of the property (e.i) obviously holds.

If (G_0, X_0) is of C_n type, then the monomorphism $G^{\mathrm{der}}_{\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} W(\mathbb{F}) \hookrightarrow \boldsymbol{Sp}(L_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(\mathbb{F}), \psi)$ is a direct sum of standard composite monomorphisms of the form $\boldsymbol{Sp}_{2n} \hookrightarrow \boldsymbol{Sp}_{2n}^s \hookrightarrow \boldsymbol{Sp}_{2ns}$ (with $s := \frac{d}{n[F_0:\mathbb{Q}]}$); from this and the identity $Z^0(G) = \mathbb{G}_m$ we get that the centralizer $\mathcal{B}_{(p)}$ of $G_{\mathbb{Z}_{(p)}}$ in $\boldsymbol{GL}_{L_{(p)}}$ is a semisimple $\mathbb{Z}_{(p)}$-algebra and that $G_{\mathbb{Z}_{(p)}}$ is the centralizer of $\mathcal{B}_{(p)}$ in $\boldsymbol{GSp}(L_{(p)}, \psi)$. Thus $(G, X) \to (\boldsymbol{GSp}(W, \psi), S)$ is a PEL type embedding.

(g) We will check that the properties 4.3 (i), (iv), and (v) hold. The derived group G^{der} is connected and thus the property 4.3 (i) holds. Let $\mathcal{B}_{(p)}$ be the centralizer of $G_{\mathbb{Z}_{(p)}}$ in $\mathrm{End}(L_{(p)})$. It is a semisimple $\mathbb{Z}_{(p)}$-algebra. Let $G_{1,\mathbb{Z}_{(p)}}$ be the reductive subgroup scheme of $\boldsymbol{GSp}(L_{(p)}, \psi)$ that is the centralizer of $\mathcal{B}_{(p)}$ in $\mathrm{End}(L_{(p)})$. It is easy to see that the condition 4.3 (iv) holds. Let $G^0 := G \cap \boldsymbol{Sp}(W, \psi)$ and $G_1^0 := G_1 \cap \boldsymbol{Sp}(W, \psi)$. We will check that the Hasse principle holds for G^0/G^{der} and G_1^0/G_1^{der}. It is easy to see that G^0/G^{der} and G_1^0/G_1^{der} are isomorphic and thus we can work only with G^0/G^{der}. We will only consider the A_n, B_n, and C_n types, as the $B_n^{\mathbb{R}}$ type is very much the same. The torus $Z^0(G^0)$ is the maximal compact subtorus of $\mathrm{Res}_{E_0/\mathbb{Q}}\mathbb{G}_m$ and thus the Hasse principle holds for it. The torus G^0/G^{der} is isomorphic to $Z^0(G^0)$ (more precisely, it is the quotient of $Z^0(G^0)$ by $Z^0(G^0)[m]$, where m is 2 if (G_0, X_0) is of B_n or C_n type and is $n+1$ if (G_0, X_0) is of A_n type). From the last two sentences we get that the Hasse principle holds for G^0/G^{der}. Thus the property 4.3 (v) holds as well.

(h) Regardless of what the type of (G_0, X_0) is, properties (a) and (b) of Theorem 1.5 hold. In the next six Subsections we will check one by one that all the properties (c) to (h) of Theorem 1.5 hold.

5.2. Proof of Theorem 1.5 (c). In this Subsection we assume that (G_0, X_0) is not of A_n type. Let $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ be a basic point and let $z : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}$ be an arbitrary lift of it. We will use the canonical identification $Z^0(G)_{B(\mathbb{F})} = Z^0(\mathcal{G}_{B(\mathbb{F})})$ defined by crystalline realizations of $\mathbb{Q}$–endomorphisms of $\mathfrak{A}$ that correspond naturally to $\mathbb{Q}$–valued points of $Z^0(G)$. Let A_0, $\mathbb{F}_{p^q}$, $(M_0, \phi_0, (t_\alpha)_{\alpha\in\mathcal{J}}, \psi_{M_0})$, and π_0 be as in Subsection 2.1. As y is basic, we can identify π_0 with a $\mathbb{Q}$–valued point of $Z^0(G)$ and thus also with an element of $Z^0(\mathcal{G})(B(\mathbb{F}))$.

But $Z^0(G)$ is a subtorus of $\mathrm{Res}_{E_0/\mathbb{Q}}\mathbb{G}_m$ (see Subsection 5.1). As $\mathrm{Res}_{E_0/\mathbb{Q}}\mathbb{G}_m$ acts on W via its natural structure of an E_0-vector space, we have canonical monomorphisms $\mathbb{Q}[\pi_0] \hookrightarrow E_0 \hookrightarrow \mathrm{End}(W)$ of simple $\mathbb{Q}$–algebras. This implies that the double centralizer of $\mathbb{Q}[\pi_0]$ in $\mathrm{End}(W)$ is $\mathbb{Q}[\pi_0]$. Therefore the Lefschetz group of A is equal to the center $T_0 := \mathrm{Res}_{\mathbb{Q}[\pi_0]/\mathbb{Q}}\mathbb{G}_m$ of the group scheme of invertible elements of $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Thus to prove that the Tate conjecture holds for A, it suffices to show that the group $\{\pi_0^s | s \in \mathbb{Z}\}$ is dense in $T_0(\mathbb{Q})$ (cf. [Mi6, Sect. 2]). The easiest way to prove this is to show that in fact we have $\mathbb{Q}[\pi_0] = \mathbb{Q}$ i.e., that the abelian variety A is supersingular. If $Z^{00}(G) = \mathbb{G}_m$, then the fact that $\mathbb{Q}[\pi_0] = \mathbb{Q}$ is obvious. Thus we can assume that $Z^{00}(G) \neq \mathbb{G}_m$. As (G_0, X_0) is of B_n, C_n, or $D_n^{\mathbb{R}}$ type and as $Z^{00}(\mathbb{G}_m) \neq \mathbb{G}_m$, from Subsection 5.1 (f) we get that (G_0, X_0) has compact factors. Therefore $Z^{00}(G)_{\mathbb{Q}_p}$ is the extension of an anisotropic torus by $Z(\boldsymbol{GL}_W)_{\mathbb{Q}_p} = \mathbb{G}_m$, cf. property 5.1 (e.ii). Thus the only split subtorus of $Z^{00}(G)_{\mathbb{Q}_p}$ is $Z(\boldsymbol{GL}_W)_{\mathbb{Q}_p}$.

For Newton polygon slope quasi-cocharacters we refer to either [Pi] or [Va8, Subsect. 2.2]. The Newton polygon slope quasi-cocharacter of (M, ϕ) is a quasi-cocharacter of $Z^{00}(\mathcal{G}_{B(\mathbb{F})}) := Z^{00}(G)_{B(\mathbb{F})}$ which is the extension to $B(\mathbb{F})$ of a quasi-cocharacter of $Z^0(G)_{\mathbb{Q}_p}$ and thus it factors through the only split subtorus $Z(\boldsymbol{GL}_W)_{\mathbb{Q}_p}$ of $Z^{00}(G)_{\mathbb{Q}_p}$. Therefore the Newton polygon slope quasi-cocharacter of (M, ϕ) factors through $Z(\boldsymbol{GL}_{M[\frac{1}{p}]})$. Thus all Newton polygon slopes of (M, ϕ) are equal and therefore are $\frac{1}{2}$. Thus A is a supersingular abelian variety. This ends the proof of Theorem 1.5 (c). [6] Thus the property 4.3 (vi) holds as well. □

5.3. Proof of Theorem 1.5 (d). In this Subsection we prove Theorem 1.5 (d). For the sake of clarity, we itemize the six main tricks (ideas) we use to prove Theorem 1.5 (d). In the first five tricks we assume that (G_0, X_0) is not of A_n type. The last trick explains the modifications required to be made in order to handle the A_n type case as well.

(a) The diagonal trick. We assume that (G_0, X_0) is of B_n, C_n, or $D_n^{\mathbb{R}}$ type. Let $m \in \mathbb{N}$. To prove Theorem 1.5 (d) we can replace A by a power A^m of it. In the crystalline context, this corresponds to a replacement of M with M^m and to a replacement of each

6 We assume that (G_0, X_0) is of A_n type with $n > 1$ and that there is no prime of F_0 which divides p and which splits in E_0. Then a similar argument shows that the abelian variety A is supersingular and therefore that the Tate conjecture holds for A.

$t_\alpha \in \mathcal{T}(\mathrm{End}(M[\frac{1}{p}]))$ with a *diagonal* tensor $t_\alpha^{(m)} \in \mathcal{T}(\mathrm{End}(M^m[\frac{1}{p}]))$. Like if $t_\alpha \in \mathrm{End}(M[\frac{1}{p}])$, then $t_\alpha^{(m)} := (t_\alpha, \ldots, t_\alpha) \in \mathrm{End}(M[\frac{1}{p}])^m \subseteq \mathrm{End}(M^m[\frac{1}{p}])$. At the level of Betti homologies with $\mathbb{Z}_{(p)}$ coefficients, this corresponds to a replacement of $L_{(p)}$ by $L_{(p)}^m$ and of v_α by $v_\alpha^{(m)}$. We recall from Subsection 5.1 (e), that we have $L_{(p)} = \tilde{W}_{(p)} \otimes_{\mathbb{F}_{0,(p)}} E_{0,(p)}$. It is convenient to introduce a totally real number field F_1 which is unramified over p, which contains F_0, and for which we have $[F_1 : F_0] = m$. Thus $F_{1,(p)}$ is a free $F_{0,(p)}$-module of rank m and therefore we can identify

$$L_{(p)}^m = \tilde{W}_{(p)} \otimes_{\mathbb{F}_{0,(p)}} \mathbb{F}_{1,(p)} \otimes_{\mathbb{F}_{0,(p)}} E_{0,(p)} = L_{(p)} \otimes_{F_{0,(p)}} F_{1,(p)} \tag{11}$$

as $G_{\mathbb{Z}_{(p)}}$-modules. Under this identification we have

$$v_\alpha^{(m)} = v_\alpha \otimes 1 \in \mathcal{T}(\mathrm{End}(W)) \otimes_{F_0} F_1 = \mathcal{T}(\mathrm{End}(W \otimes_{F_0} F_1)).$$

(b) The relative PEL trick. Let $G_{1,\mathbb{Z}_{(p)}}^{\mathrm{sc}} := \mathrm{Res}_{F_{1,(p)}/\mathbb{Z}_{(p)}} J_{0,F_{1,(p)}}^{\mathrm{sc}}$, where $J_{0,F_{0,(p)}}^{\mathrm{sc}}$ is the simply connected semisimple group scheme cover over $F_{0,(p)}$ whose generic fibre has J_0 as its adjoint group and for which we have $G_{\mathbb{Z}_{(p)}}^{\mathrm{sc}} = \mathrm{Res}_{F_{0,(p)}/\mathbb{Z}_{(p)}} J_{0,F_{0,(p)}}^{\mathrm{sc}}$ and where $J_{0,F_{1,(p)}}^{\mathrm{sc}}$ is the simply connected semisimple group scheme cover of the extension to $F_{1,(p)}$ of $J_{0,F_{0,(p)}}$. We view naturally $G_{1,\mathbb{Z}_{(p)}}^{\mathrm{sc}}$ as a simply connected semisimple subgroup scheme of $\boldsymbol{GL}_{L_{(p)}^m}$. Let $G_{1,\mathbb{Z}_{(p)}}$ be the reductive subgroup scheme of $\boldsymbol{GL}_{L_{(p)}^m}$ generated by $G_{1,\mathbb{Z}_{(p)}}^{\mathrm{sc}}$ and $Z^0(G_{\mathbb{Z}_{(p)}})$, cf. [Va5, Lemma 2.3 (b)]. Let $G_1 := G_{1,\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} \mathbb{Q}$. Let X_1 be the $G_1(\mathbb{R})$-conjugacy class of the composite of the natural monomorphism $G_\mathbb{R} \hookrightarrow G_{1,\mathbb{R}}$ with any monomorphism $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \hookrightarrow G_\mathbb{R}$ that defines an element of X. We have natural injective maps of Shimura pairs

$$(G, X) \hookrightarrow (G_1, X_1) \hookrightarrow (\boldsymbol{GSp}(W^m, \psi_m), S_m),$$

where ψ_m is a perfect, alternating form on L^m such that we get naturally a Siegel pair $(\boldsymbol{GSp}(W^m, \psi_m), S_m)$.[7] We remark that the composite monomorphisms $G_{\mathbb{Z}_{(p)}} \hookrightarrow G_{1,\mathbb{Z}_{(p)}} \hookrightarrow \boldsymbol{GL}_{L_{(p)}^m}$ form a relative PEL situation (embedding) in the sense of [Va1, Subsubsect. 4.3.16] i.e., we have (cf. loc. cit.):

(b.i) The group scheme $G_{\mathbb{Z}_{(p)}}$ is the closed subgroup scheme of $G_{1,\mathbb{Z}_{(p)}}$ that fixes a semisimple $\mathbb{Z}_{(p)}$-subalgebra of $\mathrm{End}(L_{(p)}^m)$.

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7 One can always modify the construction of the injective map $(G, X) \hookrightarrow (\boldsymbol{GSp}(W, \psi), S)$ such that it factors through the natural injective map $(G, X) \hookrightarrow (G_1, X_1)$ (for this, one would only need to enlarge the field E such that it also contains F_1). Thus for the proof of Theorem 1.5, one can always assume that $m = 1$ (but in such a case one would either have to write A as an m-th power of an abelian variety B or work with the weaker form of the Conjecture 1.2 in which one replaces "regardless of the choice" by "we can choose". However, for the sake of better results and generality and of future applications, we preferred not to assume that $m = 1$.

(c) The A_1 twisting trick. By enlarging the finite field extension F_1 of F, we can assume that there exist injective maps

$$(G_2, X_2) \hookrightarrow (G_1, X_1) \hookrightarrow (\boldsymbol{GSp}(W^m, \psi_m), S_m)$$

of Shimura pairs, such that the following two properties hold:

(c.i) The Zariski closure $G_{2,\mathbb{Z}_{(p)}}$ of G_2 in $G_{1,\mathbb{Z}_{(p)}}$ is a reductive group scheme such that we have $Z^0(G_{2,\mathbb{Z}_{(p)}}) = Z^{00}(G_{\mathbb{Z}_{(p)}})$, the derived group G_2^{der} is simply connected, and G_2^{ad} is an adjoint group of the form $\mathrm{Res}_{F_0/\mathbb{Q}} J_2$, where J_2 is an absolutely simple, adjoint group of A_1 Dynkin type.

(c.ii) The reductive group scheme $G_{2,\mathbb{Z}_{(p)}}$ is the closed subgroup scheme of $\boldsymbol{GSp}(L^m_{(p)}, \psi_m)$ that fixes a semisimple $\mathbb{Z}_{(p)}$-subalgebra of $\mathrm{End}(L^m_{(p)})$. Moreover the derived group scheme $G^{\mathrm{der}}_{2,\mathbb{Z}_p}$ is a product of Weil restrictions of $\boldsymbol{SL}_2$ group schemes.

The existence of the injective map $(G_2, X_2) \hookrightarrow (G_1, X_1)$ is argued using the same twisting arguments as in [Va3, Subsect. 4.5]. In fact, as J_2 we can take any absolutely simple, adjoint group over F_0 which has the following three properties:

(c.iii) It is unramified over $\mathbb{Q}_p$.

(c.iv) For each embedding $i_0 : F_0 \hookrightarrow \mathbb{R}$ such that the group $J_0 \times_{F_0, i_0} \mathbb{R}$ is non-compact (resp. it is compact), the group $J_2 \times_{F_0, i_0} \mathbb{R}$ is a $\boldsymbol{PGL}_2$ group (resp. it is compact).

If (G_0, X_0) is of B_n type, let $s := 2n+1$. If (G_0, X_0) is of $D_n^{\mathbb{R}}$ type, let $s := 2n$. Let F_2^{sc} be the simply connected semisimple group cover of J_2. The monomorphism $G_2^{\mathrm{der}} \to G_1^{\mathrm{der}}$ is a composite of the natural monomorphism $G_2^{\mathrm{der}} = \mathrm{Res}_{F_0/\mathbb{Q}} J_2^{\mathrm{sc}} \hookrightarrow \mathrm{Res}_{F_1/\mathbb{Q}} J^{\mathrm{sc}}_{2,F_1}$ with a monomorphism of the form $\mathrm{Res}_{F_1/\mathbb{Q}} J^{\mathrm{sc}}_{2,F_1} \hookrightarrow G_1^{\mathrm{der}}$ which is an inner twist of the Weil restriction from F_1 to $\mathbb{Q}$ of a composite monomorphism of the form:

(c.v) $\boldsymbol{Spin}_{2,1} \hookrightarrow \boldsymbol{Spin}_{s-2,2}$, if (G_0, X_0) is of B_n or $D_n^{\mathbb{R}}$ type;

(c.vi) $\boldsymbol{SL}_2 \hookrightarrow \boldsymbol{SL}_2^n \hookrightarrow \boldsymbol{Sp}_{2n}$ (the first one being a diagonal monomorphism), if (G_0, X_0) is of C_n type.

The fact that such an inner twist exists is guaranteed by [Va3, Lemmas 4.4 and 4.6].

Let $T_{2,\mathbb{Z}_{(p)}}$ be a maximal torus of $G_{2,\mathbb{Z}_{(p)}}$ which over $\mathbb{R}$ has $\mathbb{R}$-rank 1 and whose extension to $\mathbb{Z}_p$ has an image in $G^{\mathrm{der}}_{2,\mathbb{Z}_p}$ which is an anisotropic torus. Let $T_2 := T_{2,\mathbb{Z}_{(p)}} \times_{\mathbb{Z}_{(p)}} \mathbb{Q}$. We can choose T_2 such that there exists a prime $\tilde{l}$ distinct from p and from the prime l of the property (b.ii) for which the following property holds:

(c.vii) The prime $\tilde{l}$ splits in E_0 and the image of $T_{2,\mathbb{Q}_{\tilde{l}}}$ in $G^{\mathrm{der}}_{2,\mathbb{Q}_{\tilde{l}}}$ is a product of anisotropic tori of dimension 1, all but one of them being split after extension to the unramified quadratic extension of $\mathbb{Q}_{\tilde{l}}$.

Let $x_2 : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \hookrightarrow T_{2,\mathbb{R}}$ be a homomorphism that defines an element of X_2, cf. [Ha, Lemma 5.5.3]. Let $\mathcal{T}_2$ be the integral canonical model of the Shimura quadruple $(T_2, \{x_2\}, T_{2,\mathbb{Z}_{(p)}}(\mathbb{Z}_p), v_2^{T_2})$, where $v_2^{T_2}$ is a prime of $E(T_2, \{x_2\})$ that divides v.

(d) The Tate–Hodge trick. We check that the smallest subtorus of T_2 such that x_2 factors through its extension to $\mathbb{R}$, is T_2 itself. If (G_0, X_0) does not have compact factors, then $Z^{00}(G)=\mathbb{G}_m$ (cf. Subsection 5.1 (f))) and the statement follows easily from the property (c.vii), cf. Lemma 2.2.1 applied to $(T,T_1):=(T_2,Z^{00}(G))$. If (G_0,X_0) has compact factors, then the statement follows from properties 5.1 (e.i) and (c.vii) (cf. Lemma 2.2.1 applied to $(T,T_1):=(T_2,Z^{00}(G))$).

From the previous paragraph we get that T_2 is the Mumford–Tate group of every abelian variety $\mathcal{A}_2$ over $B(\mathbb{F})$ which is associated naturally to a composite morphism $z_2 : \operatorname{Spec}(W(\mathbb{F}))\to\mathcal{T}_2\to\mathcal{N}_m\to\mathcal{M}_m$, where $\mathcal{M}_m$ and $\mathcal{N}_m$ are the integral canonical models of $(\boldsymbol{GSp}(W^m,\psi_m),S_m,\boldsymbol{GSp}(L^m,\psi_m)(\mathbb{Z}_p),p)$ and $(G_1,X_1,G_{1,\mathbb{Z}_{(p)}}(\mathbb{Z}_p),v_1)$ (respectively) and where $\mathcal{T}_2\to\mathcal{N}_m\to\mathcal{M}_m$ are the functorial morphisms (see [Va5, Subsubsect. 2.4.2]). Here v_1 is the prime of $E(G_1,X_1)$ divided by $v_2^{T_2}$; it divides v. Due to the property (c.ii), the torus T_2 is the subtorus of $\boldsymbol{GSp}(W^m,\psi_m)$ that fixes a semisimple $\mathbb{Q}$–subalgebra of $\operatorname{End}(W^m)$. From this and Lemma 2.5 we get that the Hodge–Tate property holds for the point $y_2:\operatorname{Spec}(\mathbb{F})\to\mathcal{T}_2\to\mathcal{N}_m$ defined naturally by z_2.

(e) The basic point trick. The point $y_2:\operatorname{Spec}(\mathbb{F})\to\mathcal{T}_2\to\mathcal{N}_m$ is basic (cf. the anisotropic part of (c) above; to be compared with [Va8, Subsubsect. 4.2.2]). To the point $y_2:\operatorname{Spec}(\mathbb{F})\to\mathcal{N}_m$ corresponds a septuple $(A_2,\lambda_{A_2},M_2,\phi_2,(t^{(m)}_{2,\alpha})_{\alpha\in\mathcal{J}},\psi_{M_2},(\eta_{N,y_2})_{N\in\mathbb{N}\setminus p\mathbb{N}})$ analogous to the sextuple $(A,\lambda_A,M,\phi,(t_\alpha)_{\alpha\in\mathcal{J}},\psi_M,(\eta_{N,y})_{N\in\mathbb{N}\setminus p\mathbb{N}})$. We know that there exists an algebraic cycle $c_{\alpha^{(m)},A_2}$ on A_2 whose crystalline realization is $t^{(m)}_{2,\alpha}$, cf. Lemma 2.5. Thus Conjecture 1.2 holds for the point $y_2:\operatorname{Spec}(\mathbb{F})\to\mathcal{N}_m$. Therefore the endomorphism property holds for y_2 (cf. Proposition 4.2) and the isogeny property holds for y_2 (cf. Proposition 4.4). Thus all the six properties 4.3 (i) to (vi) hold in the context of $\mathcal{N}_m$, (G_2,X_2), $y_2:\operatorname{Spec}(\mathbb{F})\to\mathcal{N}_m$, and $y:\operatorname{Spec}(\mathbb{F})\to\mathcal{N}_m$ (above we have checked that the properties 4.3 (i), (iv), (v), and (vi) hold in the context of (G,X) but the same arguments show that they hold as well in the context of (G_2,X_2)). Thus from Proposition 4.3 we get that all the basic points $\operatorname{Spec}(\mathbb{F})\to\mathcal{N}_m$ form one isogeny set $o(y_2)$. In particular, the point $y:\operatorname{Spec}(\mathbb{F})\to\mathcal{N}_m$ (defined naturally by $y:\operatorname{Spec}(\mathbb{F})\to\mathcal{N}$) belongs to the isogeny set $o(y_2)$ and the isogeny property holds for the point $y:\operatorname{Spec}(\mathbb{F})\to\mathcal{N}_m$.

The fact that $y:\operatorname{Spec}(\mathbb{F})\to\mathcal{N}_m$ belongs to $o(y_2)$ implies that each $t^{(m)}_\alpha$ is the crystalline realization of an algebraic cycle $c_{\alpha^{(m)},A^m}$ on A^m. From this and (a) we get that Conjecture 1.2 holds for the basic point $y:\operatorname{Spec}(\mathbb{F})\to\mathcal{N}$. This ends the proof of Theorem 1.5 (d) for the B_n, C_n, and $D_n^{\mathbb{R}}$ types.

(f) The A_n type variation trick. We now assume that (G_0,X_0) is of A_n type and we explain the modifications needed to be made in (a) to (e) above in order to get that Theorem 1.5 (d) continues to holds for the A_n type. If $n=1$, it is more practical to think of the A_1 type as a C_1 type and thus this case is handled by (a) to (e) above. Thus until Subsubsection 5.3.1 we can assume that $n>1$. The technical condition mentioned in Theorem 1.5 (d) is as follows.

(***) If neither n is odd nor $G^{\mathrm{der}}_{\mathbb{R}}$ is a product of $\boldsymbol{SU}(\frac{n+1}{2},\frac{n+1}{2})$ groups, then we assume that there exists a subfield F_2 of F_0 and a totally imaginary quadratic extension E_2 of F_2 contained in E_0, such that the smallest subtorus $\diamond$ of $Z^0(G)$ with the property that each element of X factors through the extension to $\mathbb{R}$ of the subgroup of G generated by G^{der} and $\diamond$, is the subtorus $Z^{00}(G)$ of $\operatorname{Res}_{E_0/\mathbb{Q}}\mathbb{G}_m$ generated by $\mathbb{G}_m$ and by the maximal compact subtorus of $\operatorname{Res}_{E_2/\mathbb{Q}}\mathbb{G}_m$.

For instance, if $F_0 = \mathbb{Q}$ then we can take $(F_2, E_2) := (F_0, E_0)$. Let $m \in \mathbb{N}$. We will choose F_1 such that the properties listed in (a) hold. The only modification required to be made to (a), is provided by the single identity

$$L^m_{(p)} = L \otimes_{F_{0,(p)}} F_{1,(p)}$$

(to be compared with (11)) of $G_{\mathbb{Z}_{(p)}}$-modules. No modification is required to be made to (b). Referring to (c), one requires to take (G_2, X_2) to be of A_n type, to have $G_2^{\mathrm{ad}} = \mathrm{Res}_{F_2/\mathbb{Q}} J_2$, where J_2 is an absolutely simple group of A_n Dynkin type, to have $E(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}}) = E(G_0, X_0)$, and to have $Z^0(G_2) = Z^{00}(G)$. In fact, as J_2 we can take any absolutely simple, adjoint group over F_2 which has the following three properties:

(f.i) It is unramified over $\mathbb{Q}_p$.

(f.ii) The number field E_2 is the analogue of E_0 but working with $(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$ instead of with (G_0, X_0) (i.e., the $\mathrm{Gal}(\mathbb{Q})$-set $\mathrm{Hom}(E_2, \overline{\mathbb{Q}})$ is naturally identified with the $\mathrm{Gal}(\mathbb{Q})$-set defined by the ending notes of the Dynkin diagram of $G^{\mathrm{ad}}_{2,\overline{\mathbb{Q}}}$).

(f.iii) For each embedding $i_0 : F_0 \hookrightarrow \mathbb{R}$, by denoting $i_2 : F_2 \to \mathbb{R}$ the restriction of i_0 to F_2, the group $J_2 \times_{F_2, i_2} \mathbb{R}$ is isomorphic to $J_0 \times_{F_0, i_0} \mathbb{R}$.

The main thing that defines F_1 is the requirement that the groups $J_0 \times_{F_0} F_1$ and $J_2 \times_{F_2} F_1$ are isomorphic. By assuming that the technical condition (***) holds, we are indirectly imposing the following condition on choices pertaining to the property 5.1 (d.i).

(f.iv) If $i_2 : F_2 \hookrightarrow \mathbb{R}$ is an embedding such that the group $J_2 \times_{F_2, i_2} \mathbb{R}$ is compact and if $j_2 : E_2 \hookrightarrow \mathbb{C}$ is a fixed extension of i_2, then (due to (***)) for every embedding $j : E_0 \hookrightarrow \mathbb{R}$ which is the restriction to E_0 of an embedding $F_1 \otimes_{F_0} E_0 \hookrightarrow \mathbb{C}$ that extends j_2, the Hodge type of W_j is the same (for instance, for all such j we have $W_j \subseteq F_x^{0,-1}$ for any $x \in X$).

Properties (c.i) and (c.ii) continue to hold but with $\boldsymbol{SL}_2$ replaced by $\boldsymbol{SL}_{n+1}$.
We will choose the maximal torus T_2 of G_2 such that there exists a prime $\tilde{l} >> 0$ distinct from p and such that the following property (analogous to (c.vii)) holds:

(f.v) The prime $\tilde{l}$ splits in E_2 and the image of $T_{2,\mathbb{Q}_{\tilde{l}}}$ in $G^{\mathrm{der}}_{2,\mathbb{Q}_{\tilde{l}}}$ is a product of tori of dimension n which are anisotropic and have no proper subtori, all but one of these tori being isomorphic.

Such tori over $\mathbb{Q}_{\tilde{l}}$ are constructed as follows. We choose a Galois extension $Q_{\tilde{l}}$ of $\mathbb{Q}_{\tilde{l}}$ whose Galois group is S_{n+1}. Let $\tilde{Q}_{\tilde{l}}$ be the fixed subfield of an S_n subgroup of S_{n+1}. Then the torus $[\mathrm{Res}_{\tilde{Q}_{\tilde{l}}/\mathbb{Q}_l} \mathbb{G}_m]/\mathbb{G}_m$ over $\mathbb{Q}_{\tilde{l}}$ has dimension n, has no proper subtorus, and up to an isogeny is a maximal torus of $\boldsymbol{SL_{n+1}}$.
Only the first paragraph of (d) needs to be modified in the case when (G_0, X_0) does not have compact factors and $G_{0,\mathbb{R}}$ is not isomorphic to $[F_0; \mathbb{Q}]$ copies of $\boldsymbol{SU}(\frac{n+1}{2}, \frac{n+1}{2})^{\mathrm{ad}}$. In such a case, from (***) and (f.v) one gets that the smallest subtorus of T_2 such that x_2 factors through its extension to $\mathbb{R}$, is T_2 itself (cf. Lemma 2.2.1 applied to $(T, T_1) := (T_2, Z^{00}(G))$). There is no modification required to be made to (e). Thus if the technical condition (***) holds (resp. does not hold), then Theorem 1.5 (d) holds (resp. Theorem

1.5 (d) follows from the PEL type property of Subsection 5.1 (d)). This ends the proof of Theorem 1.5 (d) for the A_n type with $n > 1$. Thus Theorem 1.5 (d) holds. $\square$

5.3.1. Remark. Suppose that (G_0, X_0) is of B_n or C_n type. Then J_0 has no outer automorphisms and based on this one can check that we can assume that $m = 1$ (and thus that $F_1 = F_0$). Therefore the Tate–Hodge property holds for basic points $\mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$, cf. Subsection 5.3 (d) and (e).

5.4. Proof of Theorem 1.5 (e). We first assume that $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ is a basic point. The fact that the endomorphism property holds for y follows from Theorem 1.5 (d) and Proposition 4.2. The fact that the isogeny property holds for y and that we have an identity $\mathfrak{s}_0(\mathbb{F}) = o(y)$, is argued in the same way we argued in Subsection 5.3 (e) that the isogeny property holds for the point $y_2 : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}_m$ and that all $\mathbb{F}$-valued basic points of $\mathcal{N}_m$ form one isogeny set $o(y_2)$. We now check that the unramified CM lift property holds for y. We know that there exists a special point $\tilde{z} : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}$ whose closed point maps to a point $\tilde{y} \in \mathfrak{s}_0(\mathbb{F}) = o(y)$, cf. [Va10, Subsubsect. 9.4.1] (more precisely, this follows from [Va8, Subsubsect. 4.2.2 and the proof of Thm. 5.2.3]). As $o(\tilde{y}) = o(y)$ and as we have an identity $\mathfrak{s}_0(\mathbb{F}) = o(y)$, we get that the unramified CM lift property holds for y. Thus the first part of Theorem 1.5 (e) holds.

We now assume that (G_0, X_0) has compact factors and that y is not a basic point. Let $\mathfrak{l}$ and $\mathfrak{c}$ be as in Example 3.7. The Zariski closure of $\mathfrak{c}$ in $\mathcal{N}_{k(v)}$ contains points of $\mathfrak{s}_0$, cf. Example 3.7. From this and Theorem 4.1 we get that the isogeny property holds for y. Thus Theorem 1.5 (e) holds. $\square$

5.5. Proof of Theorem 1.5 (f). Let H_0 be a compact, open subgroup of $G(\mathbb{A}_f^{(p)})$ such that the natural morphism $f : \mathcal{N} \to \mathcal{N}^{\mathrm{cl}}$ is a pro-étale cover of the morphism $f_{H_0} : \mathcal{N}/H_0 \to \mathcal{N}^{\mathrm{cl}}/H_0$. As f_{H_0} is an isomorphism in characteristic 0, we easily get that there exists a smallest reduced, closed $k(v)$-subscheme $\mathfrak{U}^{\mathrm{cl}}/H_0$ of $\mathcal{N}^{\mathrm{cl}}/H_0$ such that f_{H_0} is an isomorphism outside $\mathfrak{U}^{\mathrm{cl}}/H_0$. Let $\mathfrak{U}$ and $\mathfrak{U}^{\mathrm{cl}}$ be the reduced schemes of the inverse images of $\mathfrak{U}^{\mathrm{cl}}/H_0$ to $\mathcal{N}$ and $\mathcal{N}^{\mathrm{cl}}$ (respectively). As the morphism $\mathcal{N} \to \mathcal{M}_{O_{(v)}}$ induces formally closed embeddings at the level of completions of local rings of residue field $\mathbb{F}$ (cf. Corollary 2.3.1), we get that:

(i) The set $\mathfrak{U}(\mathbb{F})$ is the union of those fibres of the map $f(\mathbb{F}) : \mathcal{N}(\mathbb{F}) \to \mathcal{N}^{\mathrm{cl}}(\mathbb{F})$ which have at least two elements (points).

Let $\mathfrak{s}_0$ be the basic locus of $\mathcal{N}_{k(v)}$. From the identity $\mathfrak{s}_0 = o(y)$ and the fact that the endomorphism property holds for y (cf. Theorem 1.5 (e)), we get (see Corollary 2.3.5):

(ii) The intersection $\mathfrak{U} \cap \mathfrak{s}_0$ is the empty scheme.

Thus we can take $\mathcal{L} := \mathcal{N}_{k(v)} \setminus \mathfrak{U}$.

We now assume that (G_0, X_0) has compact factors. Obviously $\mathfrak{U}$ is $G(\mathbb{A}_f^{(p)})$-invariant. If the scheme $\mathfrak{U}$ is non-empty, then the intersection $\mathfrak{U} \cap \mathfrak{s}_0$ is non-empty (cf. Lemma 3.6) and therefore we reached a contradiction to the property (ii). Therefore $\mathfrak{U}$ is the empty scheme. This implies that the morphism $f : \mathcal{L} = \mathcal{N} \to \mathcal{N}^{\mathrm{cl}}$ is an isomorphism. $\square$

5.6. Proof of Theorem 1.5 (g). Theorem 1.5 (g) follows from Subsection 1.4.1, cf. works of Zink and Milne (see [Zi] and [Mi1-5]).

For reader's convenience, (already in this first version) we will recall below the essence of [Zi] and [Mi1-5]. We know that the isogeny property holds for all points $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ (cf. Theorem 1.5 (e)) and thus we can speak about the isogeny sets $o(y)$ for which the formulas (2) and (3) hold. We will use the notations of Subsections 1.4.2 to 1.4.6. Let T be a maximal torus of G such that we have a short exact sequence $0 \to Z(\boldsymbol{GL}_{W\otimes_{\mathbb{Q}}\mathbb{R}}) \hookrightarrow T_{\mathbb{R}} \hookrightarrow T_c \to 0$, where the torus T_c is compact. Let $x \in X$ be such that its image factors through $T_{\mathbb{R}}$. Thus we have an injective map $(T, \{x\}) \hookrightarrow (G, X)$ of Shimura pairs. Let $\mathfrak{T}$ be the $\overline{\mathbb{Q}}/\mathbb{Q}$–groupoid associated to T. We have three basic steps.

Step 1. To the Shimura pair $(T, \{x\})$ one associates naturally a homomorphism $t_x : \mathfrak{P} \to \mathfrak{T}$ of $\overline{\mathbb{Q}}/\mathbb{Q}$–groupoids, cf. [Mi3]. By composing t_x with the natural monomorphism $\mathfrak{T} \hookrightarrow \mathfrak{G}$, one gets a homomorphism $j_x : \mathfrak{P} \to \mathfrak{G}$. Any $G(\overline{\mathbb{Q}})$-conjugate of such a homomorphism j_x is called an admissible homomorphism (here we are using the fact that G^{der} is simply connected, cf. [Mi3]). Let $\mathfrak{J}$ be the set of isomorphism classes (up to $G(\overline{\mathbb{Q}})$-conjugation) of such admissible homomorphisms. To j_x one associates a $\mathbb{Z}\Theta$-set $X_p(j_x)$, a $G(\mathbb{A}_f^{(p)})$-torsor $X^p(j_x)$ (under a right action), and the group $\mathcal{E}_{j_x}$ of elements of $G(\overline{\mathbb{Q}})$ which fix j_x under inner conjugation. The Langlands–Rapoport conjecture predicts the existence of a $\mathbb{Z}\Theta \times G(\mathbb{A}_f^{(p)})$-equivariant bijection

$$\mathcal{N}(\mathbb{F}) = \coprod_{[j_x]\in\mathfrak{J}} \mathcal{E}_{j_x}\backslash[X_p(j_x) \times X^p(j_x)]. \tag{12}$$

Step 2. We take $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ to be a point which is the specialization of a point in the image of the natural functorial morphism $\mathrm{Sh}(T, \{x\}) \to \mathrm{Sh}_{H_p}(G, X)$. The choice of y allows us:

- **(i)** to identify $X_p(j_x)$ with the set $\mathfrak{X}_p(y)$ of those $W(\mathbb{F})$-lattices of $M[\frac{1}{p}]$ that are of the form $h(M)$ with $h \in \mathfrak{E}(y)$;
- **(ii)** to identify $X^p(j_x)$ with $G(\mathbb{A}_f^{(p)})$;
- **(iii)** to identify $\mathcal{E}_{j_x}$ with $\mathcal{E}(\mathbb{Q})$ in such a away that the actions on $X_p(j_x) \times X^p(j_x)$ and $\mathfrak{X}_p(y) \times G(\mathbb{A}_f^{(p)})$ coincide (this is possible due to the assumption that the endomorphism property holds for $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$).

These identifications are compatible with the $\mathbb{Z}\Theta \times G(\mathbb{A}_f^{(p)})$-actions. Therefore we have an identification

$$\mathcal{E}_{j_x}\backslash[X_p(j_x) \times X^p(j_x)] = \mathcal{E}(\mathbb{Q})\backslash[\mathfrak{X}_p(y) \times G(\mathbb{A}_f^{(p)})] \tag{13}$$

of $\mathbb{Z}\Theta \times G(\mathbb{A}_f^{(p)})$-sets. One can check that we have $\mathfrak{X}_p(y) = \mathcal{E}(\mathbb{Q})\mathfrak{X}_{0,p}(y)$ (for instance, this is trivial if the group $\mathcal{E}(\mathbb{Q})$ acts on ψ_M via arbitrary integral powers of p). It is easy to see that if $h \in \mathcal{E}(\mathbb{Q})$ is such that $h\mathfrak{X}_{0,p}(y) \cap \mathfrak{X}_{0,p}(y) \neq \emptyset$, then in fact we have $h \in \widetilde{\mathcal{E}(\mathbb{Q})}$. From this and (4) we get that

$$\widetilde{\mathcal{E}(\mathbb{Q})}\backslash[\mathfrak{X}_{0,p}(y) \times G(\mathbb{A}_f^{(p)})] = \mathcal{E}(\mathbb{Q})\backslash[\mathfrak{X}_p(y) \times G(\mathbb{A}_f^{(p)})]. \tag{14}$$

From (13) and (14) we get that we have a natural identification (bijection)

$$\mathcal{E}_{j_x}\backslash[X_p(j_x)\times X^p(j_x)]=\widetilde{\mathcal{E}(\mathbb{Q})}\backslash[\mathfrak{X}_{0,p}(y)\times G(\mathbb{A}_f^{(p)})] \tag{15}$$

of $\mathbb{Z}\Theta\times G(\mathbb{A}_f^{(p)})$-sets.

Step 3. Based on formulas (2), (3), and (15), to prove that (12) holds it suffices to show that the natural map

$$\mathfrak{I}\to\mathfrak{I}$$

that takes $[j_x]$ to $o(y)$ is a bijection. The fact that this map is surjective follows from the fact that the CM property holds for all $\mathbb{F}$-valued points of $\mathcal{N}$. The fact that this map is injective is well known. $\square$

5.7. Proof of Theorem 1.5 (h). If $m=1$, then the Tate–Hodge property holds for basic points $\mathrm{Spec}(\mathbb{F})\to\mathcal{N}$ and thus from Theorem 1.5 (g) we get that the unconditional form of the Langlands–Rapoport conjecture holds for the basic locus of $\mathcal{N}_{k(v)}$. If $m>1$, then a similar argument shows that the unconditional form of the Langlands–Rapoport conjecture holds for the basic locus of $\mathcal{N}^m_{k(v)}$. But as we have $Z^0(G)=Z^0(G_1)$ and due to Theorem 1.5 (g), the fact that the unconditional form of the Langlands–Rapoport conjecture holds for the basic locus of $\mathcal{N}^m_{k(v)}$ implies that the unconditional form of the Langlands–Rapoport conjecture holds for the basic locus of $\mathcal{N}_{k(v)}$ (to be compared with [Mi5, p. 7]; the obstruction is in the cohomology group $H^1(\mathbb{Q},Z^0(G_1))$). Thus Theorem 1.5 (h) holds. This ends the proof of Theorem 1.5. $\square$

6. Main results

In this Section we prove our main results, as Corollaries of the previous Sections and of earlier works. Let (G_1,X_1) be a Shimura pair. Let p be a prime such that the group $G_{1,\mathbb{Q}_p}$ is unramified. In all that follows, each Shimura quadruple of the form (G_1,X_1,H_1,v_1) will involve a hyperspecial subgroup H_1 of $G_1(\mathbb{Q}_p)$ and a prime v_1 of the reflex field $E(G_1,X_1)$ that divides p. Let $k(v_1)$ be the residue field of v_1 and let Θ_1 be the Frobenius automorphism of $\mathbb{F}$ that fixes $k(v_1)$.

We will assume that each simple factor of $(G_1^{\mathrm{ad}},X_1^{\mathrm{ad}})$ is of A_n, B_n, C_n, or $D_n^{\mathbb{R}}$ type (we emphasize that the group G_1 could be a torus; in such a case the adjoint G_1^{ad} is trivial). Let $\mathcal{N}_1$ be the integral canonical model of (G_1,X_1,H_1,v_1) (i.e., of $\mathrm{Sh}_{H_1}(G_1,X_1)$ over the local ring $O_{(v_1)}$ of v_1). Let $\mathcal{N}_1^{\mathrm{ad}}$ be the integral canonical model of $(G_1^{\mathrm{ad}},X_1^{\mathrm{ad}},H_1^{\mathrm{ad}},v_1^{\mathrm{ad}})$. See [Va5, Subsubsect. 2.4.2] for the functorial morphisms between integral canonical models. We recall the following basic property (cf. [Va1] and [Va5, Thm. 5.1]; for $p\in\{2,3\}$ see also [Va11, Parts I and II]).

6.1. Proposition. *The functorial morphism $\mathcal{N}_1\to\mathcal{N}^{\mathrm{ad}}_{1,O_{(v_1)}}$ is a pro-étale cover of its image.*

6.2. Main Corollary A. *We assume that (G_1, X_1) has compact factors. Then the Langlands–Rapoport conjecture holds for the integral canonical model $\mathcal{N}_1$ of the Shimura quadruple (G_1, X_1, H_1, v_1) (i.e., for the set $\mathcal{N}_1(\mathbb{F})$ acted upon by the group $\mathbb{Z}\Theta_1 \times G_1(\mathbb{A}_f^{(p)})$).*

Proof: We proceed by induction on the dimension of X_1. If $\dim(X_1) = 0$, then G_1 is a torus and it is well known that the classical field theory implies that the Langlands–Rapoport conjecture holds for the set $\mathcal{N}_1(\mathbb{F})$.

The inductive step goes as follows. Based on [Mi3] we can assume that G_1^{der} is simply connected. Based on [Pf], to prove the Corollary we can replace the Shimura quadruple (G_1, X_1, H_1, v_1) by another one (G, X, H, v) with the properties that: (i) $G^{\mathrm{der}} = G_1^{\mathrm{der}}$ is simply connected and (ii) we have an identity $(G^{\mathrm{ad}}, X^{\mathrm{ad}}, H^{\mathrm{ad}}, v^{\mathrm{ad}}) = (G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}, H_1^{\mathrm{ad}}, v_1^{\mathrm{ad}})$ between adjoint Shimura quadruples. Therefore (as in [Va5, proof of Thm. 5.1]) we can take (G, X, H, v) to be a product of Shimura quadruples whose adjoints are simple. By induction, we can assume that $(G_0, X_0) := (G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is a simple, adjoint Shimura pair. Therefore we can assume that we are in the context of Theorem 1.5 and accordingly we will use $\mathcal{N}$ instead of $\mathcal{N}_1$. Based on Theorem 1.5 (g), it suffices to show that the endomorphism property and the unramified CM lift property hold for every point $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$. To show this, we can assume that y is not a basic point (cf. Theorem 1.5 (e)).

If A_0, $\mathbb{F}_{p^q}$, and π_0 are as in Subsection 2.1, then the fact that y is not a basic point implies that $\pi_0 \in \mathcal{G}(B(\mathbb{F})) \setminus Z(\mathcal{G})(B(\mathbb{F}))$. Let $\tilde{\mathcal{C}}_{B(\mathbb{F})}$ be the centralizer of π_0 in $\mathcal{G}_{B(\mathbb{F})}$. The group $\tilde{\mathcal{C}}_{B(\mathbb{F})}$ is a reductive subgroup of $\mathcal{G}_{B(\mathbb{F})}$ different from $\mathcal{G}_{B(\mathbb{F})}$. Let $\tilde{\mathcal{C}}$ be the Zariski closure of $\tilde{\mathcal{C}}_{B(\mathbb{F})}$ in $\mathcal{G}$.

We know that the isogeny property holds for y, cf. Theorem 1.5 (e). Thus to check that the endomorphism property and the unramified CM lift property hold for $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$, we can replace y by an arbitrary point in the set $o(y)$. Therefore, by replacing y with $y(h)$ for some element $h \in \mathfrak{C}_0(y)$ we can assume that $\tilde{\mathcal{C}}$ is a reductive group scheme over $W(\mathbb{F})$ and that there exists a Hodge cocharacter $\tilde{\mu} : \mathbb{G}_m \to \tilde{\mathcal{C}}$ of $(M, \phi, \tilde{C})$. We recall that by Hodge cocharacter we mean that we have a direct sum decomposition $M = F^1 \oplus F^0$ such that $M = \phi(\frac{1}{p}F^1 \oplus F^0)$ and for each $i \in \{0, 1\}$ the group $\mathbb{G}_m$ acts on F^i through $\tilde{\mu}$ via the $-i$-th power of the identical character of $\mathbb{G}_m$. Let $z : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}$ be a lift of y such that the Hodge filtration of M defined by the abelian scheme $\mathcal{A} := z^*(\mathfrak{A})$ over $W(\mathbb{F})$, is F^1 itself. If $p > 2$ or if $p = 2$ and the p-rank of A is 0, then such a lift exists and is unique (see [Va11, Part I, Lemma 3.5.2]). If $p = 2$ and the 2-rank of A is positive, then we take such a lift with the property that the 2-divisible group of $\mathcal{A}$ is a direct sum of p-divisible groups whose special fibres are isoclinic (one can easily check that such a lift exists after a natural passage to an isogeny that could require the replacement of y by another point in the isogeny set $o(y)$).

Let $\tilde{\mathcal{Z}}$ be the subtorus of $Z(\tilde{\mathcal{C}})$ whose generic fibre is the extension to $B(\mathbb{F})$ of the Frobenius torus of π_0. The Frobenius endomorphism π_0 leaves invariant F^1 and thus it lifts to an endomorphism of $\mathcal{A}$ (for $p = 2$, cf. also the choice of z). This implies that the Mumford–Tate group of the generic fibre of $\mathcal{A}$ can be identified naturally with a reductive subgroup $G_{\mathcal{A}}$ of G which is contained in the centralizer $\tilde{C}$ in G of the generic fibre of a torus $Z_{\mathbb{Z}_{(p)}}$ of $G_{\mathbb{Z}_{(p)}}$ whose crystalline realization is precisely $\tilde{\mathcal{Z}}$. The centralizer $\tilde{C}_{\mathbb{Z}_{(p)}}$ of $Z_{\mathbb{Z}_{(p)}}$ in $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme whose generic fibre is $\tilde{C}$. Let $x_{\mathcal{A}} \in X$ be a composite monomorphism $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}G_m \hookrightarrow G_{\mathcal{A},\mathbb{R}} \hookrightarrow G_{\mathbb{R}}$ which is the Hodge monomorphism associated naturally to $\mathcal{A}$. Let $\tilde{G}$ be the largest normal subgroup of $\tilde{C}$ such that the pair $(\tilde{G},\tilde{X})$ is a Shimura pair, where $\tilde{X}$ is the $\tilde{G}(\mathbb{R})$-conjugacy class of $x_{\mathcal{A}}$ viewed as a monomorphism $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \hookrightarrow \tilde{G}_{\mathbb{R}}$. Thus $\tilde{G}^{\mathrm{ad}}$ is the product of those simple factors of $\tilde{C}^{\mathrm{ad}}$ which do not normalize $x_{\mathcal{A}}$ (i.e., of those simple factor whose extensions to $\mathbb{R}$ contain a non-trivial image of $x_{\mathcal{A}}$). As a conclusion, we have constructed an injective map $(\tilde{G},\tilde{X}) \hookrightarrow (G,X)$ of Shimura pairs such that the following three properties hold:

(i) there exists an element $\pi_0 \in Z^0(\tilde{G})(\mathbb{Q})$ which is the Betti realization of the endomorphism π_0 of $\mathcal{A}$ and which does not belong to $Z(G)(\mathbb{Q})$;

(ii) the group $\tilde{G}$ is a normal subgroup of the centralizer $\tilde{C}$ of π_0 in G and the quotient group $\tilde{C}/\tilde{G}$ is semisimple;

(iii) the Zariski closures of $\tilde{C}$ and $\tilde{G}$ in $\boldsymbol{GL}_{L_{(p)}}$ (equivalently, in $G_{\mathbb{Z}_{(p)}}$) are reductive group schemes $\tilde{C}_{\mathbb{Z}_{(p)}}$ and $\tilde{G}_{\mathbb{Z}_{(p)}}$ (respectively) over $\mathbb{Z}_{(p)}$.

The part of the property (iii) pertaining to $\tilde{G}_{\mathbb{Z}_{(p)}}$ follows from the part of the property (iii) pertaining to $\tilde{C}$ and from the fact that $\tilde{G}$ is a normal subgroup of $\tilde{C}$. Let $\tilde{H}_p := \tilde{G}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$. Let $\tilde{v}$ be a prime of $E(\tilde{G},\tilde{X})$ that divides v. Let $\tilde{\mathcal{N}}$ be the integral canonical model of the Shimura quadruple $(\tilde{G},\tilde{X},\tilde{H}_p,\tilde{v})$. We will choose $\tilde{v}$ such that $z : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}$ factors naturally through the functorial morphism $\tilde{\mathcal{N}} \to \mathcal{N}$. We denote by $\tilde{z} : \mathrm{Spec}(W(\mathbb{F})) \to \tilde{\mathcal{N}}$ and $\tilde{y} : \mathrm{Spec}(\mathbb{F}) \to \tilde{\mathcal{N}}$ the corresponding factorizations of z and y (respectively).

Due to the property (i) and the fact that G^{der} is simply connected, the group $\tilde{G}^{\mathrm{der}}$ is simply connected and we have $\dim(\tilde{X}) < \dim(X)$. Thus each simple factor of $(\tilde{G}^{\mathrm{ad}},\tilde{X}^{\mathrm{ad}})$ is of A_n, B_n, C_n, or $D_n^{\mathbb{R}}$ type. Moreover, due to the property (i), the group $G^{\mathrm{der}} \cap \tilde{G}$ is of the form $\mathrm{Res}_{F_0/\mathbb{Q}}\tilde{J}_0$, for some reductive subgroup $\tilde{J}_0$ of the simply connected semisimple group cover of J_0, where F_0 and J_0 are as in Subsection 5.1 (a). From this and the fact that (G_0,X_0) has compact factors, we easily get that each simple factor of $(\tilde{G}^{\mathrm{ad}},\tilde{X}^{\mathrm{ad}})$ has compact factors. Based on this and the fact that $\dim(\tilde{X}) < \dim(X)$, by induction we get that the conditional Langlands–Rapoport conjecture holds for $\tilde{\mathcal{N}}$.

Due to the property (ii), we have a direct sum decomposition

$$\mathrm{Lie}(\tilde{C}) = \mathrm{Lie}(Z^0(\tilde{G})) \oplus \mathrm{Lie}(\tilde{G}^{\mathrm{der}}) \oplus \mathrm{Lie}(\tilde{C}/\tilde{G}). \tag{16}$$

Due to the property (ii), the reductive subgroup of $\tilde{C}$ generated by $Z^0(\tilde{G})$ and by the semisimple, normal subgroup of $\tilde{C}$ whose adjoint is $\tilde{C}/\tilde{G}$, is naturally a subgroup of $\mathcal{E}$. Due to this and (16), to prove that the endomorphism property holds for $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ it suffices to show that the endomorphism property holds for $\tilde{y} : \mathrm{Spec}(\mathbb{F}) \to \tilde{\mathcal{N}}$. But by induction, the Langlands–Rapoport conjecture holds for the set $\tilde{\mathcal{N}}(\mathbb{F})$ and therefore the endomorphism and isogeny properties holds for $\tilde{y} : \mathrm{Spec}(\mathbb{F}) \to \tilde{\mathcal{N}}$.[8] Thus the endomorphism

8 If $(\tilde{G},\tilde{X},\tilde{H},\tilde{v}) \hookrightarrow (\boldsymbol{GSp}(W,\psi),S,K_p,p)$ and $(\tilde{G}_1,\tilde{X}_1,\tilde{H}_1,\tilde{v}_1) \hookrightarrow (\boldsymbol{GSp}(W_1,\psi_1),S_1,K_{1,p},p)$ are two injective maps such that $\tilde{G}^{\mathrm{der}} = \tilde{G}_1^{\mathrm{der}}$ and $(\tilde{G}^{\mathrm{ad}},\tilde{X}^{\mathrm{ad}},\tilde{H}^{\mathrm{ad}},\tilde{v}^{\mathrm{ad}}) = (\tilde{G}_1^{\mathrm{ad}},\tilde{X}_1^{\mathrm{ad}},\tilde{H}_1^{\mathrm{ad}},\tilde{v}_1^{\mathrm{ad}})$, then the easiest way to show that the endomorphism property enjoyed by $\mathbb{F}$-valued points

property holds for $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$. Moreover, it is easy to see that Theorem 1.5 (e) implies that we have $o(\tilde{y}) = \tilde{\mathfrak{s}}_0(\mathbb{F})$, where $\tilde{\mathfrak{s}}_0$ is the basic locus of $\tilde{\mathcal{N}}_{k(\tilde{v})}$ (in other words, the changes of Shimura quadruples performed in the second paragraph of this proof preserve the property that the $\mathbb{F}$-valued points of basic loci form only one isogeny set).

Let $\tilde{z}_1 : \mathrm{Spec}(W(\mathbb{F})) \to \tilde{\mathcal{N}}$ be a special point whose closed point maps to a point $\tilde{y}_1 \in \tilde{\mathfrak{s}}_0(\mathbb{F}) = o(\tilde{y})$, cf. [Va10, Subsubsect. 9.4.1] (more precisely, this follows from [Va8, Subsubsect. 4.2.2 and the proof of Thm. 5.2.3]). As the points in the set $o(\tilde{y})$ map (via the functorial morphism $\tilde{\mathcal{N}} \to \mathcal{N}$) to points in the set $o(y)$, we conclude that the composite of $\tilde{z}_1$ with the functorial morphism $\tilde{\mathcal{N}} \to \mathcal{N}$ is a special point $z_1 : \mathrm{Spec}(W(\mathbb{F})) \to \mathcal{N}$ that lifts a point $y_1 \in o(y)$. Thus the unramified CM lift property holds for y.

We conclude that the endomorphism property and the unramified CM lift property hold for all points $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$. Thus the Langlands–Rapoport conjecture holds for $\mathcal{N}$, cf. Theorem 1.5 (g). This ends the induction and the proof of the Corollary.

6.3. On Conjecture 1.2. Let (G_0, H_0) be an arbitrary simple, adjoint Shimura pair of A_n type with $n > 1$. We assume that the group $G_{0,\mathbb{Q}_p}$ is unramified and that (G_0, X_0) has compact factors. In this Subsection we will assume that the injective map $(G, X) \hookrightarrow (\boldsymbol{GSp}(W, \psi), S)$ and the $\mathbb{Z}$-lattice L of W are as in Theorem 1.5 (i.e., as constructed in Subsection 5.1 (d)). In particular, we have a natural identity $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) = (G_0, X_0)$. We will use the basic notations of the proof of Corollary 6.2. Let $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ be a point that is not basic. We will also use the notations of Subsection 6.2. We would like to get a criterion for when the Conjecture 1.2 holds for y. As the isogeny property holds for $y : \mathrm{Spec}(\mathbb{F}) \to \mathcal{N}$ (cf. Theorem 1.5 (e)), by replacing y with $y(h)$ for some element $h \in \mathfrak{E}_0(y)$, we can assume that $\tilde{\mathcal{C}}$ is a reductive group scheme over $W(\mathbb{F})$ and that there exists a Hodge cocharacter $\tilde{\mu} : \mathbb{G}_m \to \tilde{\mathcal{C}}$ which gives birth to a direct sum decomposition $M = F^1 \oplus F^0$ such that we have $M = \phi(\frac{1}{p}F^1 \oplus F^0)$ (cf. proof of Corollary 6.2). Let $\tilde{T}$ be the smallest subtorus of $Z^0(\tilde{G})$ with the property that each element of $\tilde{X}$ factors through the extension to $\mathbb{R}$ of the subgroup of $\tilde{G}$ generated by $\tilde{G}^{\mathrm{der}}$ and $\tilde{T}$.

6.3.1. Criterion. *We assume that $\tilde{T}$ is $Z^0(\tilde{G})$ itself. Then the Conjecture 1.2 holds for $y : Spec(\mathbb{F}) \to \mathcal{N}$.*

Proof: The proof of this is very much the same as the proof of Theorem 1.5 (d) for the A_n type, with the assumption that $\tilde{T}$ is $Z^0(\tilde{G})$ as a substitute for the technical condition (***). As in Subsection 5.3, we will allow the replacement of A and W with A^m and W^m, with $m \in \mathbb{N}$. Not to complicate the notations, in what follows we will not mention each time that we are first performing such a replacement. One only has to show the existence of a maximal torus $\tilde{U}$ of $\tilde{G}$ that has the following four basic properties (to be compared with Lemma 2.2.1 and Subsection 5.3 (d) and (e)):

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of the integral canonical model $\tilde{\mathcal{N}}_1$ of $(\tilde{G}_1, \tilde{X}_1, \tilde{H}_1, \tilde{v}_1)$ transfers automatically to $\mathbb{F}$-valued points of $\tilde{\mathcal{N}}$ is via the counting of the $\mathbb{F}$-valued points in the isogeny classes of a Hodge quasi product injective map $(\tilde{G}, \tilde{X}, \tilde{H}, \tilde{v}) \times^{\mathcal{H}} (\tilde{G}_1, \tilde{X}_1, \tilde{H}_1, \tilde{v}_1) \hookrightarrow (\boldsymbol{GSp}(W \oplus W_1, \varepsilon\psi \oplus \varepsilon_1\psi_1), S_I, K_I, p)$ with $\varepsilon, \varepsilon_1 \in \{-1, 1\}$ (the argument is the same as the one used in the first paragraph of the proof of Lemma 3.3).

(i) the Zariski closure of $\tilde{U}$ in $\boldsymbol{GL}_{L_{(p)}}$ is a torus over $\mathbb{Z}_{(p)}$;

(ii) the image of $\tilde{U}_{\mathbb{Q}_p}$ in $\tilde{G}^{\mathrm{ad}}_{\mathbb{Q}_p}$ is anisotropic;

(iii) there exists a point $\tilde{x} \in \tilde{X}$ such that the smallest subgroup $\ominus$ of $\tilde{G}$ with the property that $\tilde{x}$ factors through $\ominus_{\mathbb{R}}$, is $\tilde{U}$;

(iv) the torus $\tilde{U}$ is the subtorus of $\boldsymbol{GSp}(W, \psi)$ that fixes a semisimple $\mathbb{Q}$–subalgebra of $\mathrm{End}(W)$.

Let s be the number of simple factors of $(\tilde{G}^{\mathrm{ad}}, \tilde{X}^{\mathrm{ad}})$. If $s = 0$ (i.e., if $\tilde{G}$ is a torus), then we can take $\tilde{U} = \tilde{T} = \tilde{G} = Z^0(\tilde{G})$. Property (iii) holds (cf. the identity $\tilde{T} = Z^0(\tilde{G})$), property (ii) is vacuous, property (i) holds due to the property 6.2 (iii), and property (iv) follows from Subsection 5.1 (d) and the property 6.2 (ii).

Suppose now that $s > 0$. Then by induction on $t \in \{0, \ldots, s\}$ we show that there exists a reductive subgroup $\tilde{G}_t$ of $\tilde{G}$ such that the following five properties hold:

(v) the Zariski closure of $\tilde{G}_t$ in $\boldsymbol{GL}_{L_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$;

(vi) we have $Z^0(\tilde{G}) \leqslant \tilde{G}_t$ and the image of $Z^0(\tilde{G}_t)_{\mathbb{Q}_p}$ in $\tilde{G}^{\mathrm{ad}}_{\mathbb{Q}_p}$ is anisotropic;

(vii) the derived subgroup $\tilde{G}_t^{\mathrm{der}}$ is a normal subgroup of $\tilde{G}^{\mathrm{der}}$ and $\tilde{G}_t^{\mathrm{ad}}$ is a product of $s - t$ simple factors of $\tilde{G}^{\mathrm{ad}}$;

(viii) there exists a point $\tilde{x}_t \in \tilde{X}$ such that the smallest subgroup $\ominus$ of $\tilde{G}_t$ with the property that $\tilde{x}$ factors through $\ominus_{\mathbb{R}}$, is $\tilde{G}_t$;

(ix) the group $\tilde{G}_t$ is the subgroup of $\tilde{G}$ that fixes $Z^0(\tilde{G}_t)$ (thus $\tilde{G}_t$ is the subgroup of $\boldsymbol{GSp}(W, \psi)$ that fixes a semisimple $\mathbb{Q}$–subalgebra of $\mathrm{End}(W)$).

The existence of $\tilde{G}_t$ is argued as in Subsection 5.3 (f). Taking $\tilde{U} = \tilde{G}_s$, we get that all the four properties (i) to (iv) hold (cf. properties (v) to (ix)). $\square$

6.4. Main Corollary B. *We assume that (G_1, X_1) has compact factors and that we have an injective map $(G_1, X_1, H_1, v_1) \hookrightarrow (\boldsymbol{GSp}(W, \psi), S, K_p, p)$ of Shimura quadruples, where $(\boldsymbol{GSp}(W, \psi), S, K_p, p)$ is the Shimura quadruple of a Siegel modular pair $(\boldsymbol{GSp}(W, \psi), S)$. Let $\mathcal{M}$ be the integral canonical model of $(\boldsymbol{GSp}(W, \psi), S, K_p, p)$ and let $\mathcal{N}_1 \to \mathcal{M}_{O_{(v_1)}}$ be the functorial morphism. We also assume that there exists a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ which is self dual with respect to ψ, for which the Zariski closure of G_1 in $\boldsymbol{GL}_{L_{(p)}}$ is a reductive group scheme, and for which we have $\boldsymbol{GSp}(L_{(p)}, \psi)(\mathbb{Z}_p) = K_p$. Then we have:*

(a) *The isogeny property, the endomorphism property, and the unramified CM lift property hold for all points Spec$(\mathbb{F}) \to \mathcal{N}_1$.*

(b) *The $\mathbb{F}$-valued points of the basic locus $\mathfrak{s}_{1,0}$ of $\mathcal{N}_{1,k(v_1)}$ form one isogeny set.*

(c) *The functorial morphism $\mathcal{N}_1 \to \mathcal{M}_{O_{(v_1)}}$ is a closed embedding.*

Proof: Part (a) follows from the proof of Corollary 6.2. To prove (b) we have to show that for each point $y_1 \in \mathfrak{s}_{1,0}(\mathbb{F})$, we have an identity $\mathfrak{s}_{1,0}(\mathbb{F}) = o(y_1)$. To check this identity we can assume that $(G_0, X_0) := (G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}})$ is a simple, adjoint Shimura pair and that the injective map $(G_1, X_1, H_1, v_1) \hookrightarrow (\boldsymbol{GSp}(W, \psi), S, K_p, p)$ is obtained as in Theorem 1.5 (cf.

proof of Corollary 6.2 and Proposition 6.1). Therefore the identity $\mathfrak{s}_{1,0}(\mathbb{F}) = o(y_1)$ follows from Theorem 1.5 (e). Thus (b) holds.

To check part (b), we can assume that G_1 is not a torus (as the case of a torus is trivial). It is known that $\mathcal{N}_1$ is the normalization of the Zariski closure $\mathcal{N}_1^{\mathrm{cl}}$ of $\mathrm{Sh}_{H_1}(G_1, X_1)$ in $\mathrm{Sh}_{K_p}(\boldsymbol{GSp}(W,\psi), S)_{E(G_1,X_1)}$, cf. end of Subsection 1.1.2. The inverse image via the map $\mathcal{N}_1(\mathbb{F}) \to \mathcal{N}_1^{\mathrm{cl}}(\mathbb{F})$ of the set $\mathrm{Im}(\mathfrak{s}_{1,0}(\mathbb{F}) \to \mathcal{N}_1^{\mathrm{cl}}(\mathbb{F}))$ is the set $\mathfrak{s}_{1,0}(\mathbb{F})$ itself, cf. Proposition 3.2. Based on this and Lemma 3.6, the proof of (c) is the same as of Subsection 5.5, provided we know that the map $\mathfrak{s}_{1,0}(\mathbb{F}) \to \mathcal{M}(\mathbb{F}) = \mathcal{M}_{O_{(v_1)}}(\mathbb{F})$ is injective.

Let $o^{\mathrm{big}}(y_1)$ be the isogeny set of y_1 but viewed as an $\mathbb{F}$-valued point of $\mathcal{M}$. As $\mathfrak{s}_{1,0}(\mathbb{F}) = o(y_1)$, to end the proof of (c) it suffices to show that the natural map of sets

$$o(y) \to o^{\mathrm{big}}(y)$$

is injective. But this follows from Corollary 2.3.5. Thus (c) holds as well. □

6.5. Corollary. *Let $(G_1, X_1, H_1, v_1) \hookrightarrow (G_2, X_2, H_2, v_2)$ be an injective map of Shimura quadruples. We assume that each simple factor of $(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$ has compact factors and is of A_n, B_n, C_n, or $D_n^{\mathbb{R}}$ type (this implies that (G_1, X_1) also has compact factors). We also assume that the monomorphism $H_1 \hookrightarrow H_2$ is the $\mathbb{Z}_p$-valued points monomorphism associated to a monomorphism $G_{1,\mathbb{Z}_{(p)}} \hookrightarrow G_{2,\mathbb{Z}_{(p)}}$ between reductive group schemes over $\mathbb{Z}_{(p)}$ that extends the monomorphism $G_1 \hookrightarrow G_2$. Let $\mathcal{N}_1 \to \mathcal{N}_{2,O_{(v_1)}}$ be the functorial morphism, where $\mathcal{N}_2$ is the integral canonical model of (G_2, X_2, H_2, v_2). Then the morphism $\mathcal{N}_1 \to \mathcal{N}_{2,O_{(v_1)}}$ is a closed embedding.*

Proof: We can assume that G_2 is not a torus (as this case is trivial). The generic fibre of the morphism $\mathcal{N}_1 \to \mathcal{N}_{2,O_{(v_1)}}$ is a closed embedding. Thus to prove that $\mathcal{N}_1 \to \mathcal{N}_{2,O_{(v_1)}}$ itself is a closed embedding, we can consider pull backs to $W(\mathbb{F})$ and we can work in the pro-étale cover topology of a fixed connected component of $\mathcal{N}_{2,W(\mathbb{F})}$. Based on this and Proposition 6.1, as in the proof of Corollary 6.2 we argue that we can assume that we have an injective map $(G_2, X_2, H_2, v_2) \hookrightarrow (G_3, X_3, H_3, v_3)$ such that the following three properties hold:

- **(i)** We have an identity $G_2^{\mathrm{der}} = G_3^{\mathrm{der}}$ between simply connected semisimple groups.
- **(ii)** The quadruple (G_3, X_3, H_3, v_3) is a product $\prod_{i \in I}(G_{3,i}, X_{3,i}, H_{3,i}, v_{3,i})$ of Shimura quadruples whose adjoints $(G_{3,i}^{\mathrm{ad}}, X_{3,i}^{\mathrm{ad}}, H_{3,i}^{\mathrm{ad}}, v_{3,i}^{\mathrm{ad}})$ are simple and which admit injective maps $(G_{3,i}, X_{3,i}, H_{3,i}, v_{3,i}) \hookrightarrow (\boldsymbol{GSp}(W_i, \psi_i), S_i, K_{i,p}, p)$ into Shimura quadruples of Siegel modular varieties. Moreover, for each $i \in I$ there exists a $\mathbb{Z}_{(p)}$-lattice $L_{i,(p)}$ of W_i which is self dual with respect to ψ_i and with the property that the Zariski closure of $G_{3,i}$ in $\boldsymbol{GL}_{L_{i,(p)}}$ is a reductive group scheme.
- **(iii)** The torus $Z^0(G_2)$ is the extension of a torus which over $\mathbb{R}$ is compact by $\mathbb{G}_m$.

Thus using a Hodge quasi product $\prod_{i\in I}^{\mathcal{H}}(G_{3,i}, X_{3,i}, H_{3,i}, v_{3,i})$ as in [Va3, Subsect. 2.4] and an injective map of the form

$$\prod_{i\in I}^{\mathcal{H}}(G_{3,i}, X_{3,i}, H_{3,i}, v_{3,i}) \hookrightarrow (\boldsymbol{GSp}(\oplus_{i\in I} W_i, \oplus_{i\in I}\varepsilon_i\psi_i), S_I, K_{I,p}, p),$$

where we have $\varepsilon_i \in \{-,1\}$ for all $i \in I$, based on the properties (i) and (ii) we can assume that the injective map $(G_2, X_2, H_2, v_2) \hookrightarrow (G_3, X_3, H_3, v_3)$ factors through such a Hodge quasi product $\prod_{i\in I}^{\mathcal{H}}(G_{3,i}, X_{3,i}, H_{3,i}, v_{3,i})$. We conclude that we can assume that we have an injective map $(G_2, X_2, H_2, v_2) \hookrightarrow (\boldsymbol{GSp}(W,\psi), S, K_p, p)$ of Shimura quadruples such that there exists a $\mathbb{Z}_{(p)}$-lattice $L_{(p)}$ of W which is self dual with respect to ψ and with the property that the Zariski closure of G_2 in $\boldsymbol{GL}_{L_{(p)}}$ is a reductive group scheme. One can make all the necessary arrangements so that the last reductive group scheme is $G_{2,\mathbb{Z}_{(p)}}$ itself. Therefore the Zariski closure of G_1 in $\boldsymbol{GL}_{L_{(p)}}$ is the reductive group scheme $G_{1,\mathbb{Z}_{(p)}}$. Thus the Corollary follows from Corollary 6.4 (c) applied to the functorial composite morphism $\mathcal{N}_1 \to \mathcal{N}_{2,O_{(v_1)}} \to \mathcal{M}_{O_{(v_1)}}$. $\square$

6.5.1. Remark. Based on Corollary 6.5, one can easily get a functorial version of Corollary 6.2.

References

[BM] P. Berthelot and W. Messing, *Théorie de Dieudonné crystalline. III,* The Grothendieck Festschrift, Vol. I, pp. 173–247, Progr. Math., Vol. **86**, Birkhäuser Boston, Boston, MA, 1990.

[De1] P. Deligne, *Travaux de Shimura,* Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389, Lecture Notes in Math., Vol. **244**, pp. 123–165, Springer-Verlag, Berlin, 1971.

[De2] P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques,* Automorphic forms, representations and L-functions (Oregon State Univ., Corvallis, OR, 1977), Part 2, pp. 247–289, Proc. Sympos. Pure Math., Vol. **33**, Amer. Math. Soc., Providence, RI, 1979.

[De3] P. Deligne, *Hodge cycles on abelian varieties,* Hodge cycles, motives, and Shimura varieties, Lecture Notes in Math., Vol. **900**, pp. 9–100, Springer-Verlag, Berlin-New York, 1982.

[dJ] J. de Jong, *Homomorphisms of Barsotti–Tate groups and crystals in positive characteristic,* Invent. Math. **134** (1998), pp. 301–333.

[Fa] G. Faltings, *Integral crystalline cohomology over very ramified valuation rings,* J. of Amer. Math. Soc. **12** (1999), no. 1, pp. 117–144.

[Ha] G. Harder, *Über die Galoiskohomologie halbeinfacher Matrizengruppen II,* Math. Z. **92** (1966), pp. 396–415.

[LR] R. Langlands and M. Rapoport, *Shimuravarietäten und Gerben,* J. reine angew. Math. **378** (1987), pp. 113–220.

[Mi1] J. S. Milne, *Points on Shimura varieties mod p,* Automorphic forms, representations and L-functions (Oregon State Univ., Corvallis, OR, 1977), Part 2, pp. 165–184, Proc. Sympos. Pure Math., Vol. **33**, Amer. Math. Soc., Providence, RI, 1979.

[Mi2] J. S. Milne, *The conjecture of Langlands and Rapoport for Siegel modular varieties,* Bull. Amer. Math. Soc. (N.S.) **24** (1991), no. 2, pp. 335–341.

[Mi3] J. S. Milne, *The points on a Shimura variety modulo a prime of good reduction,* The Zeta functions of Picard modular surfaces, pp. 153–255, Univ. Montréal, Montreal, Quebec, 1992.

[Mi4] J. S. Milne, *Shimura varieties and motives,* Motives (Seattle, WA, 1991), Part 2, pp. 447–523, Proc. Sympos. Pure Math., Vol. **55**, Amer. Math. Soc., Providence, RI, 1994.

[Mi5] J. S. Milne, *Towards a proof of the conjecture of Langlands and Rapoport,* http://arxiv.org/abs/0707.3177.

[Mi6] J. S. Milne, *The Tate conjecture over finite fields,* http://arxiv.org/abs/0709.3040.

[MFK] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory. Third edition,* Ergebnisse der Math. und ihrer Grenzgebiete (2), Vol. **34**, Springer-Verlag, 1994.

[Oo1] F. Oort, *Newton polygons and formal groups: conjectures by Manin and Grothendieck,* Ann. of Math. (2) **152** (2000), no. 1, pp. 183–206.

[Oo2] F. Oort, *Newton polygon strata in the moduli of abelian varieties,* Moduli of abelian varieties (Texel Island, 1999), pp. 417–440, Progr. Math., **195**, Birkhäuser, Basel, 2001.

[Pf] M. Pfau, *The conjecture of Langlands and Rapoport for certain Shimura varieties of non-rational weight,* J. reine angew. Math. **471** (1996), pp. 165–199.

[Pi] R. Pink, *l-adic algebraic monodromy groups, cocharacters, and the Mumford–Tate conjecture,* J. reine angew. Math. **495** (1998), pp. 187–237.

[RR] M. Rapoport and M. Richartz, *On the classification and specialization of F-isocrystals with additional structure,* Compositio Math. **103** (1996), no. 2, pp. 153–181.

[Va1] A. Vasiu, *Integral canonical models of Shimura varieties of preabelian type,* Asian J. Math., Vol. **3** (1999), no. 2, pp. 401–518.

[Va2] A. Vasiu, *Crystalline boundedness principle*, Ann. Sci. l'École Norm. Sup. **39** (2006), no. 2, pp. 245–300.

[Va3] A. Vasiu, *Some cases of the Mumford-Tate conjecture and Shimura varieties,* Indiana Univ. Math. J. **57** (2008), no. 1, pp. 1–76.

[Va4] A. Vasiu, *Projective integral models of Shimura varieties of Hodge type with compact factors,* 25 pages, J. reine angew. Math. **518** (2008), pp. 51–75.

[Va5] A. Vasiu, *Integral canonical models of unitary Shimura varieties*, Asian J. Math., Vol. **12** (2008), no. 2, pp. 151–176 (see also http://arxiv.org/abs/math/0505507).

[Va6] A. Vasiu, *Geometry of Shimura varieties of Hodge type over finite fields,* to appear in the NATO volume of the Proceedings of the Summer School "Higher dimensional geometry over finite fields", June 25 - July 06, 2007, Math. Institut, Georg-August-Universität Göttingen, Germany; see http://arxiv.org/abs/0712.1840.

[Va7] A. Vasiu, *Deformation subspaces of p-divisible groups as formal Lie group structures associated to p-divisible groups,* http://arxiv.org/abs/math.NT/0607508.

[Va8] A. Vasiu, *Manin problems for Shimura varieties of Hodge type,* http://arxiv.org/abs/math/0209410.

[Va9] A. Vasiu, *A motivic conjecture of Milne,* http://arxiv.org/abs/math/0308202.

[Va10] A. Vasiu, *CM lifts for isogeny classes of Shimura F-crystals over finite fields,* preprint, MPI, Bonn, Germany, July 2007.

[Va11] A. Vasiu *Good reductions of Shimura varieties of Hodge type in arbitrary unramified mixed characteristic, Parts I and II,* http://xxx.arxiv.org/abs/0707.1668 and http://arxiv.org/abs/0712.1572 (Part I available also as a preprint, MPI, Bonn, Germany, July 2007).

[Va12] A. Vasiu, *Mod p classification of Shimura F-crystals,* http://arxiv.org/abs/math.NT/0304030.

[Wi] J.-P. Wintenberger, *Un scindage de la filtration de Hodge pour certaines varietés algebriques sur les corps locaux,* Ann. of Math. (2) **119** (1984), no. 3, pp. 511–548.

[Zi] T. Zink, *Isogenieklassen von Punkten von Shimuramannigfaltigkeiten mit Werten in einem endlichen Körper,* Math. Nachr. **112** (1983), pp. 103–124.

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