Math 404 Advanced Linear Algebra Spring 2008 Exam 1 Feingold

(1) (20 Points) Let

$$W = \{p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \in \mathbf{F}_3[t] \mid p(0) = 0 \text{ and } p''(1) + p'(1) = 0\}.$$

(a) Prove that $W$ is a subspace of $\mathbf{F}_3[t]$.
(b) Find a basis for $W$.

(2) (30 Points) Answer (with brief justification) each question separately.

(a) If $S = \{v_1, v_2, \ldots, v_n\}$ is an independent set in vector space $V$, what is the relationship between $n$ and $\dim(V)$?
(b) If $T = \{w_1, w_2, \ldots, w_m\}$ spans vector space $W$, what is the relationship between $m$ and $\dim(W)$?
(c) If $T = \{w_1, w_2, \ldots, w_m\}$ spans vector space $W$, and $w \in W$, what is the most you can say about $T \cup \{w\} = \{w_1, \ldots, w_m, w\}$?
(d) If $S = \{v_1, \ldots, v_n\}$ is an independent set in a vector space $V$ and $v \in V$ is not in the span of $S$, then what is the most you can you say about $S \cup \{v\} = \{v_1, \ldots, v_n, v\}$?
(e) If $S = \{v_1, v_2, \ldots, v_n\}$ is an independent set in vector space $V$, and $T = \{w_1, w_2, \ldots, w_m\}$ spans the same vector space $V$, what is the relationship between $m$ and $n$?
(f) Let $A \in \mathbf{F}^m_n$ with $m < n$. What is the most you can say about the dimension of the subspace $W = \{X \in \mathbf{F}^n \mid AX = 0\}$?

(3) (20 Points) Let $S$ be the following subset of $\mathbf{R}^2_2$,

$$S = \left\{ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ 1 & -4 \end{bmatrix} \right\}$$

and let $W = \langle S \rangle$ be the span of $S$.
(a) Is $S$ linearly independent or dependent? If $S$ is dependent, find the non-trivial dependence relations which allow redundant vectors to be removed.
(b) Is $W = \mathbf{R}^2_2$? If not, give conditions on $a, b, c, d$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$.
(c) Find a basis for $W$.
(d) Find $\dim(W)$.

(4) (10 points) Answer each question separately.
(a) Suppose $L : V \to W$ is linear. If $\dim(V) = 7$ and $\dim(W) = 4$, what are the possibilities for $\dim(Ker(L))$ and for $\dim(Range(L))$?
(b) If $U$ and $W$ are subspaces of $V$ with $\dim(U) = 8$, $\dim(W) = 6$ and $\dim(V) = 11$, what is the most you can say about $\dim(U \cap W)$? Hint: $(U + W) \leq V$ and $(U \cap W) \leq W$.

(5) (20 Points) Let $L_A : \mathbf{C}^4 \to \mathbf{C}^3$ be the linear map given by $L_A(X) = AX$ where

$$A = \begin{bmatrix} 1 & -1 & \mathbf{i} & 1+\mathbf{i} \\ \mathbf{i} & -\mathbf{i} & -1 & -1+\mathbf{i} \\ 1+\mathbf{i} & -1-\mathbf{i} & -1+\mathbf{i} & 2\mathbf{i} \end{bmatrix}$$

(a) (5 points) Find the set of all vectors in $\mathrm{Ker}(L)$.
(b) (4 points) Find a basis for $\mathrm{Ker}(L)$.
(c) (1 points) Find $\dim(\mathrm{Ker}(L))$.
(d) (2 points) Is $L$ one-to one? **Explain why!**
(e) (5 points) Find a basis for $\mathrm{Range}(L)$.
(f) (1 points) Find $\dim(\mathrm{Range}(L))$.
(g) (2 points) Is $L$ onto? **Explain why!**

Math 404 Advanced Linear Algebra Spring 2008 Exam 1 Solutions Feingold

1. (20 Points) Let

$$W = \{p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \in \mathbf{F}_3[t] \mid p(0) = 0 \text{ and } p''(1) + p'(1) = 0\}.$$

(a) Prove that $W$ is a subspace of $\mathbf{F}_3[t]$.

**SOLUTION:** One way to show $W$ is a subspace is to show the zero polynomial is in $W$ and that $W$ is closed under addition and scalar multiplication. The zero polynomial $\theta(t) = 0 + 0t + 0t^2 + 0t^3$ is in $W$ because $\theta(0) = 0$, and $\theta'(t) = \theta''(t) = \theta(t)$, so $\theta''(1) + \theta'(1) = 0$. If $p(t), q(t) \in W$, then $(p+q)(t) \in W$ because $(p+q)(t) = p(t) + q(t)$ so $(p+q)(0) = p(0) + q(0) = 0 + 0 = 0$ and $(p+q)''(1) + (p+q)'(1) = p''(1) + q''(1) + p'(1) + q'(1) = p''(1) + p'(1) + q''(1) + q'(1) = 0 + 0 = 0$. If $p(t) \in W$ and $b \in \mathbf{F}$ then $(bp)(t) \in W$ since $(bp)(0) = b(p(0)) = b(0) = 0$ and $(bp)''(1) + (bp)'(1) = b(p''(1) + p'(1)) = b(0) = 0$.

(b) Find a basis for $W$.

**SOLUTION:** There is a way of answering both parts (a) and (b) at the same time, as follows. The condition $p(0) = 0$ means $a_0 = 0$. Since $p'(t) = a_1 + 2a_2 t + 3a_3 t^2$ and $p''(t) = 2a_2 + 6a_3 t$, the condition $p''(1) + p'(1) = 0$ means $(2a_2 + 6a_3) + (a_1 + 2a_2 + 3a_3) = 0$, that is, $a_1 + 4a_2 + 9a_3 = 0$. The simplest interpretation of these two equations is: $a_0 = 0$, $a_1 = -4a_2 - 9a_3$, $a_2 = r \in \mathbf{F}$ is free, $a_3 = s \in \mathbf{F}$ is free. Then $W = \{(-4r - 9s)t + rt^2 + st^3 \in \mathbf{F}_3[t] \mid r, s \in \mathbf{F}\}$ is the span of the set of two polynomials $\{-4t + t^2, -9t + t^3\}$, making $W$ a subspace, and this set is independent since the only way to get $\theta(t) = (-4r - 9s)t + rt^2 + st^3$ is when $r = 0$ and $s = 0$. So that set is a basis of $W$.

---

2. (30 points)
(a) $\dim(V) \geq n$ (An independent set can be extended to a basis.)
(b) $\dim(W) \leq m$ (A spanning set can be cut down to a basis.)
(c) $T \cup \{w\}$ is dependent: last vector is a linear combo of previous vectors.
(d) $S \cup \{v\}$ is independent: no vector is a linear combo of previous vectors.
(e) $n \leq m$ since $n \leq dim(V) \leq m$.
(f) Since $m < n$ the linear system has more variables than equations, so it must have nontrivial solutions. When the augmented matrix is row reduced, it can have at most $m$ leading ones, so there are at least $n - m$ columns without leading ones, giving at least $n - m$ free variables in the solution space. Therefore, $dim(W) \geq n - m$.

3. (20 points) (a) Let the 5 vectors of $S$ be denoted $v_1, \cdots, v_5$. Determine if $\sum_{i=1}^{5} a_i v_i = \theta$ has nontrivial solutions. Reduce

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 3 & -2 & 5 & 0 \\ -1 & -2 & 2 & -2 & 1 & 0 \\ 1 & 2 & -4 & 3 & -4 & 0 \end{array}\right] \text{ to } \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = -s \\ x_2 = -\frac{1}{2}r - \frac{1}{2}s \\ x_3 = \frac{1}{2}r - \frac{3}{2}s \\ x_4 = r \in \mathbf{R} \\ x_5 = s \in \mathbf{R} \end{array}$$

which has nontrivial solutions so $S$ is dependent. Each free variable gives a dependence relation. When $r = 1$ and $s = 0$, we get $-\frac{1}{2}v_2 + \frac{1}{2}v_3 + v_4 = \theta$ and when $r = 0$ and $s = 1$ we get $-v_1 - \frac{1}{2}v_2 - \frac{3}{2}v_3 + v_5 = \theta$. These allow the redundant vectors $v_4$ and $v_5$ to be expressed as linear combinations of $v_1$, $v_2$ and $v_3$, so $S$ is spanned by just those three vectors.

(b) Since $W$ is spanned by just 3 vectors and $\dim(\mathbf{R}_2^2) = 4$, $W \neq \mathbf{R}_2^2$. Since $W = \langle\{v_1, v_2, v_3\}\rangle$, a matrix is in this span when the following system is consistent:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & a \\ 1 & -1 & 3 & b \\ -1 & -2 & 2 & c \\ 1 & 2 & -4 & d \end{array}\right] \text{ reduces to } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2a + c \\ 0 & 1 & 0 & -a - \frac{3}{2}c - \frac{1}{2}d \\ 0 & 0 & 1 & \frac{1}{2}c - \frac{1}{2}d \\ 0 & 0 & 0 & -3a + b - c + d \end{array}\right]$$

which is consistent iff $0 = -3a + b - c + d$. This is the condition we wanted to find.

(c) From part (a), after removing the redundant vectors from $S$, we have $\{v_1, v_2, v_3\}$ is a basis for $W$. Another possible answer is obtained by using the condition from part (b), which expresses any element of $W$ as

$$\begin{bmatrix} a & b \\ c & 3a - b + c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

so a basis of $W$ is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$.

(d) $\dim(W) = 3$ since we have three vectors in a basis for $W$.

4. (10 points) (a) $L : V \to W$ is linear, $\dim(V) = 7$ and $\dim(W) = 4$, $Ker(L) \leq V$, $Range(L) \leq W$, so $7 = dim(V) = dim(Ker(L)) + dim(Range(L))$, says this sum could only be one of $7 + 0 = 6 + 1 = 5 + 2 = 4 + 3 = 3 + 4$. This says $3 \leq dim(Ker(L)) \leq 7$ and $0 \leq dim(Range(L)) \leq 4$.

   (b) If $U$ and $W$ are of $V$ with $\dim(U) = 8$, $\dim(W) = 6$ and $\dim(V) = 11$, what is the most you can say about $\dim(U \cap W)$? Hint: $(U + W) \leq V$ and $(U \cap W) \leq W$. From the formula $dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$ and the fact that $dim(U + W) \leq dim(V)$, we can say that $8 + 6 - dim(U \cap W) \leq 11$, so $3 = 8 + 6 - 11 \leq dim(U \cap W)$ is a lower bound. Also, since $(U \cap W) \leq W$, we have the upper bound $dim(U \cap W) \leq 6$, so the most we can say is $3 \leq dim(U \cap W) \leq 6$.

---

5. (20 points) To find Ker($L$), row reduce

$$\left[\begin{array}{cccc|c} 1 & -1 & \mathbf{i} & 1+\mathbf{i} & 0 \\ \mathbf{i} & -\mathbf{i} & -1 & -1+\mathbf{i} & 0 \\ 1+\mathbf{i} & -1-\mathbf{i} & -1+\mathbf{i} & 2\mathbf{i} & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & -1 & \mathbf{i} & 1+\mathbf{i} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{r} x_1 = r - \mathbf{i}s - (1+\mathbf{i})t \\ x_2 = r \in \mathbf{C} \\ x_3 = s \in \mathbf{C} \\ x_4 = t \in \mathbf{C} \end{array}.$$

   (a) (5 points) $\text{Ker}(L_A) = \left\{ \begin{bmatrix} r - \mathbf{i}s - (1+\mathbf{i})t \\ r \\ s \\ t \end{bmatrix} \in \mathbf{C}^4 \;\middle|\; r, s, t \in \mathbf{C} \right\}.$

   (b) (4 points) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\mathbf{i} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1-\mathbf{i} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\text{Ker}(L_A)$.

   (c) (1 points) $\dim(\text{Ker}(L_A)) = 3$.

   (d) (2 points) $L_A$ is not one-to-one since $\text{Ker}(L_A)$ is nontrivial.

   (e) (5 points) $\text{Range}(L_A)$ is the span of the columns of $A$, but the three free variables in $\text{Ker}(L_A)$ mean the last three columns of $A$ are redundant vectors, so a basis for $\text{Range}(L)$ is column one of $A$,

$$\left\{ \begin{bmatrix} 1 \\ \mathbf{i} \\ 1+\mathbf{i} \end{bmatrix} \right\}.$$

   (f) (1 points) $\dim(\text{Range}(L_A)) = 1$.

   (g) (2 points) $L_A$ is not onto, $\text{Range}(L_A) \neq \mathbf{C}^3$.