Math 404 Advanced Linear Algebra Spring 2007 Exam 2 Feingold

(1) (30 Pts) Let $L : \mathbf{R}_4 \to \mathbf{R}^3$ be given by

$$L\left(\begin{bmatrix} a & b & c & d \end{bmatrix}\right) = \begin{bmatrix} a - b + c - d \\ 2a + b - 3c + d \\ -a + 2b + 2c + 3d \end{bmatrix}.$$

Let S be the standard basis of $\mathbf{R}_4$ and let T be the standard basis of $\mathbf{R}^3$. Let other ordered bases be

$$S' = \{[1\ 2\ 3\ 4], [1\ 1\ 1\ 1], [0\ 1\ 1\ 2], [0\ 0\ 1\ \ -1\} \text{ and } T' = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

(a) (5 pts) Find the matrix $[L]_S^T$ representing L from S to T.

(b) (10 pts) Find the matrix $[L]_{S'}^{T'}$ representing L from S' to T' **without using transition matrices**. (Do it directly.)

(c) (15 pts) Find the transition matrices ${}_S P_{S'}$ and ${}_T Q_{T'}$ and show that $[L]_{S'}^{T'} = ({}_T Q_{T'})^{-1}\ [L]_S^T\ ({}_S P_{S'})$.

(2) (30 Points) Answer (with brief justification) each question separately.

(a) Let $S = \{[1\ 2\ 3], [0\ 1\ 2], [-1\ -1\ -2]\}$ be a basis of $\mathbf{R}_3$ and let $v = [8\ 1\ 9] \in \mathbf{R}_3$. Find the coordinate vector $[v]_S$.

(b) Suppose U and W are subspaces of V with $\dim(U) = 7$, $\dim(W) = 8$ and $\dim(V) = 12$. What is the most you can say about $\dim(U \cap W)$?

(c) Suppose $L : V \to W$ with $\dim(V) = 9$ and $\dim(W) = 5$. What is the most you can say about $\dim(Ker(L))$?

(d) Suppose $A \in \mathbf{F}_n^n$ and $B = P^{-1}AP$ for an invertible matrix P, and $f(t) \in \mathbf{F}[t]$ is any polynomial. What can you say about the relation between $f(A)$ and $f(B)$?

(e) What is $\dim(\mathcal{L}(\mathbf{F}_3^4, \mathbf{F}_5))$? What isomorphism justifies this?

(f) Suppose $L : V \to W$ with $\dim(V) = \dim(W)$ finite, and L is injective. What else can you can say must be true about L? Give as many implied properties of L as you can.

(3) (20 pts) Determine whether each of the following functions is linear, and give all work needed to justify your answer.

(a) (10 pts) $f : \mathbf{R}^2_2 \to \mathbf{R}^2_2$ by $f \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & a-b \\ c+d & cd \end{bmatrix}$.

(b) (10 pts) $g : \mathbf{C}^2 \to \mathbf{C}^2_2$ by $g \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} z+w & \mathbf{i}z - w \\ z + \mathbf{i}w & -\mathbf{i}z - \mathbf{i}w \end{bmatrix}$.

(4) (20 pts) Let $L : \mathbf{R}^2_2 \to \mathbf{R}^3$ and $K : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformations

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b-c \\ b+c-d \\ c+2d \end{bmatrix} \quad \text{and} \quad K\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x-y \\ y-z \end{bmatrix}.$$

Let R, S and T be the standard bases of $\mathbf{R}^2_2$, $\mathbf{R}^3$ and $\mathbf{R}^2$, respectively.

(a) (5 pts) Find the matrix $[L]^S_R$ representing L from R to S.

(b) (5 pts) Find the matrix $[K]^T_S$ representing K from S to T.

(c) (5 pts) Use composition of functions to find the formula for $(K \circ L)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ from the formulas for K and L.

(d) (5 pts) Find matrix $[K \circ L]^T_R$ representing $K \circ L$ from R to T. What should be the relationship between the matrices $[L]^S_R$, $[K]^T_S$ and $[K \circ L]^T_R$? Check that relationship by direct calculation.

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1(a) (5 Pts) $[L]_S^T = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & 1 & -3 & 1 \\ -1 & 2 & 2 & 3 \end{bmatrix}$ is easy to get since S and T are standard.

$L([1\,0\,0\,0]) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $L([0\,1\,0\,0]) = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $L([0\,0\,1\,0]) = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $L([0\,0\,0\,1]) = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$. $[T \mid L(S)] = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 2 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 & 2 & 2 & 3 \\ & T & & & L(S) & & \end{array}\right]$ is already reduced, so the right side is $[L]_S^T$.

1(b) (10 Pts) $L([1\ 2\ 3\ 4]) = \begin{bmatrix} -2 \\ -1 \\ 21 \end{bmatrix}$, $L([1\ 1\ 1\ 1]) = \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$, $L([0\ 1\ 1\ 2]) = \begin{bmatrix} -2 \\ 0 \\ 10 \end{bmatrix}$,

$L([0\ 0\ 1\ \ -1]) = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}$.

Row reduce $\left[\begin{array}{ccc|cccc} 3 & 1 & 1 & -2 & 0 & -2 & 2 \\ 2 & -1 & 2 & -1 & 1 & 0 & -4 \\ 1 & 1 & 0 & 21 & 6 & 10 & -1 \\ & T' & & & L(S') & & \end{array}\right]$ to $\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & -66 & -19 & -34 & 11 \\ 0 & 1 & 0 & 87 & 25 & 44 & -12 \\ 0 & 0 & 1 & 109 & 32 & 56 & -19 \\ & I_3 & & & [L]_{S'}^{T'} & & \end{array}\right]$

1(c) (15 Pts) ${}_SP_{S'} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 \\ 4 & 1 & 2 & -1 \end{bmatrix}$ and ${}_TQ_{T'} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ since S and T are the standard bases.

To get ${}_{T'}Q_T = ({}_TQ_{T'})^{-1}$, reduce

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ & T' & & & T & \end{array}\right] \text{ to } \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -3 \\ 0 & 1 & 0 & -2 & 1 & 4 \\ 0 & 0 & 1 & -3 & 2 & 5 \\ & I_3 & & & {}_{T'}Q_T & \end{array}\right]$$

$$({}_TQ_{T'})^{-1}[L]_S^T({}_SP_{S'}) = \begin{bmatrix} 2 & -1 & -3 \\ -2 & 1 & 4 \\ -3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & 1 & -3 & 1 \\ -1 & 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 \\ 4 & 1 & 2 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} 3 & -9 & -1 & -12 \\ -4 & 11 & 3 & 15 \\ -4 & 15 & 1 & 20 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 \\ 4 & 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -66 & -19 & -34 & 11 \\ 87 & 25 & 44 & -12 \\ 109 & 32 & 56 & -19 \end{bmatrix} = [L]_{S'}^{T'}$$

checks.

(2) (30 Points) Answer (with brief justification) each question separately.

(a) $[v]_S = \begin{bmatrix} -7 \\ 0 \\ -15 \end{bmatrix}$ since $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 8 \\ 2 & 1 & -1 & 1 \\ 3 & 2 & -2 & 9 \end{array}\right]$ reduces to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -15 \end{array}\right]$

(b) Given: $\dim(U) = 7$, $\dim(W) = 8$ and $\dim(V) = 12$. We know $U \cap W$ is a subspace of U and of W, so $\dim(U \cap W) \leq \dim(U) = 7$. We also know that $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$ and $\dim(U + W) \leq \dim(V) = 12$ so $7 + 8 - \dim(U \cap W) \leq 12$, so $3 \leq \dim(U \cap W)$. The most we can say is that $3 \leq \dim(U \cap W) \leq 7$.

(c) We have $L : V \to W$ with $\dim(V) = 9$ and $\dim(W) = 5$ and we know $\dim(V) = \dim(Ker(L)) + \dim(Range(L))$, where $0 \leq \dim(Range(L)) \leq \dim(W) = 5$, so the most we can say is $4 \leq \dim(Ker(L)) \leq 9$.

(d) If $f(t) = \sum_{i=0}^{m} c_i t^i$, then

$$f(B) = \sum_{i=0}^{m} c_i B^i = \sum_{i=0}^{m} c_i (P^{-1}AP)^i = \sum_{i=0}^{m} c_i (P^{-1}A^i P)$$
$$= P^{-1}\left(\sum_{i=0}^{m} c_i A^i\right) P = P^{-1} f(A) P.$$

So the relationship is $f(B) = f(P^{-1}AP) = P^{-1}f(A)P$.

(e) $\dim(\mathcal{L}(\mathbf{R}_3^4, \mathbf{R}_5)) = (5)(12) = 60$. This is justified by the isomorphism $\mathcal{L}(\mathbf{R}_3^4, \mathbf{R}_5) \cong \mathbf{R}_{12}^5$ sending a linear transformation $L : \mathbf{R}_3^4 \to \mathbf{R}_5$ to the matrix $[L]_S^T$ representing L from S to T.

(f) Since $L : V \to W$ with $\dim(V) = \dim(W)$ finite and L is injective, $\dim(Ker(L)) = 0$, so the dimension formula $\dim(V) = \dim(Ker(L)) + \dim(Range(L))$ says $\dim(V) = \dim(Range(L))$ so $\dim(W) = \dim(Range(L))$ says L is onto (surjective). Therefore, L is bijective, invertible, and an isomorphism.

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(3) (20 Pts) Determine linearity.

(a) (10 pts) $f : \mathbf{R}_2^2 \to \mathbf{R}_2^2$ by $f\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & a-b \\ c+d & cd \end{bmatrix}$ is not linear because for $r \in \mathbf{R}$, we do not have $f(rM) = rf(M)$. The formula gives $f\left(r\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = f\begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix} = \begin{bmatrix} ra+rb & ra-rb \\ rc+rd & rcrd \end{bmatrix} \neq r\begin{bmatrix} a+b & a-b \\ c+d & cd \end{bmatrix}$ because of the lower right entry.

(3) (20 Pts) Determine linearity.

(b) (10 pts) $g : \mathbf{C}^2 \to \mathbf{C}^2_2$ by $g\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} z+w & \mathbf{i}z - w \\ z + \mathbf{i}w & -\mathbf{i}z - \mathbf{i}w \end{bmatrix}$ is linear because it satisfies $g(v_1+v_2) = g(v_1)+g(v_2)$ and $g(cv) = cg(v)$ for every $v_1, v_2, v \in \mathbf{C}^2$ and $c \in \mathbf{C}$.

$$g\left(\begin{bmatrix} z_1 \\ w_1 \end{bmatrix} + \begin{bmatrix} z_2 \\ w_2 \end{bmatrix}\right) = g\left(\begin{bmatrix} z_1+z_2 \\ w_1+w_2 \end{bmatrix}\right) = \begin{bmatrix} (z_1+z_2)+(w_1+w_2) & \mathbf{i}(z_1+z_2)-(w_1+w_2) \\ (z_1+z_2)+\mathbf{i}(w_1+w_2) & -\mathbf{i}(z_1+z_2)-\mathbf{i}(w_1+w_2) \end{bmatrix}$$

$$= \begin{bmatrix} z_1+w_1 & \mathbf{i}z_1 - w_1 \\ z_1+\mathbf{i}w_1 & -\mathbf{i}z_1 - \mathbf{i}w_1 \end{bmatrix} + \begin{bmatrix} z_2+w_2 & \mathbf{i}z_2 - w_2 \\ z_2+\mathbf{i}w_2 & -\mathbf{i}z_2 - \mathbf{i}w_2 \end{bmatrix} = g\left(\begin{bmatrix} z_1 \\ w_1 \end{bmatrix}\right) + g\left(\begin{bmatrix} z_2 \\ w_2 \end{bmatrix}\right)$$

$$g\left(c\begin{bmatrix} z \\ w \end{bmatrix}\right) = g\left(\begin{bmatrix} cz \\ cw \end{bmatrix}\right) = \begin{bmatrix} cz+cw & \mathbf{i}cz - cw \\ cz+\mathbf{i}cw & -\mathbf{i}cz - \mathbf{i}cw \end{bmatrix} = c\begin{bmatrix} z+w & \mathbf{i}z - w \\ z+\mathbf{i}w & -\mathbf{i}z - \mathbf{i}w \end{bmatrix} = cg\begin{bmatrix} z \\ w \end{bmatrix}$$

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(4) (20 Pts)

(a) (5 Pts) $\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 \\ & S & & & & L(R) & \end{array}\right]$ is reduced so $[L]^S_R = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

(b) (5 Pts) $\left[\begin{array}{cc|ccc} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ T & & & K(S) & \end{array}\right]$ is reduced so $[K]^T_S = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

(c) (5 Pts) $(K \circ L)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = K\left(\begin{bmatrix} a+b-c \\ b+c-d \\ c+2d \end{bmatrix}\right) = \begin{bmatrix} 2(a+b-c)-(b+c-d) \\ (b+c-d)-(c+2d) \end{bmatrix}$

$$= \begin{bmatrix} 2a+b-3c+d \\ b-3d \end{bmatrix}$$

(d) (5 Pts) $\left[\begin{array}{cc|cccc} 1 & 0 & 2 & 1 & -3 & 1 \\ 0 & 1 & 0 & 1 & 0 & -3 \\ T & & & (K\circ L)(R) & & \end{array}\right]$ is reduced so $[K \circ L]^T_R = \begin{bmatrix} 2 & 1 & -3 & 1 \\ 0 & 1 & 0 & -3 \end{bmatrix}$.

The relationship should be $([K]^T_S)([L]^S_R) = [K \circ L]^T_R$.

Check: $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -3 & 1 \\ 0 & 1 & 0 & -3 \end{bmatrix}$.