Math 404 Advanced Linear Algebra Spring 2007 Exam 3 Feingold

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1. (40 Pts) For each $A \in \mathbf{R}^n_n$ given below define $L : \mathbf{R}^n \to \mathbf{R}^n$ by $L(X) = AX$.
   (1) Find the **characteristic polynomial** of L, $\Delta_L(t) = \Delta_A(t) = \det(A - tI_n)$, and find the **eigenvalues** of L, and their **algebraic multiplicities**.
   (2) Can L can be diagonalized? **If not, give reasons why**. If it can, **find a basis** T of $\mathbf{R}^n$ and a **diagonal matrix** $D = [L]^T_T$ representing L from T to T. Also **find the transition matrix** $P = {}_SP_T$ and **check that** $D = P^{-1}AP$.

$$(a) \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix} \qquad (b) \begin{bmatrix} -7 & -4 \\ 1 & -3 \end{bmatrix}$$

2. (15 Pts) For each matrix A in problem 1 find the minimal polynomial $m_A(t)$ and verify that $m_A(A) = 0$.

3. (30 points, 6 points each) Answer each of the following questions separately.
   (a) Let $L : V \to V$ and suppose that $L^n = 0$ for some integer $n > 1$. Show that the only possible eigenvalue for L is $\lambda = 0$.
   (b) Determine whether $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is positive definite.
   (c) Let $V = \mathbf{R}_4$ with the standard real inner product $\langle u, v \rangle = u \cdot v$ and let $S = \{[1 \quad 2 \quad 4 \quad 5], [2 \quad 3 \quad 5 \quad 6]\}$. Find a basis for $S^\perp = \{v \in \mathbf{R}_4 \mid \langle u, v \rangle = 0, \forall u \in S\}$.
   (d) Let $V = \mathbf{C}_3$ with the standard complex inner product $\langle u, v \rangle = u \cdot \bar{v}$. Find the projection of the vector $v = [1 \quad \mathbf{i} \quad (\mathbf{i} + 1)]$ onto the subspace $W = \{[z_1 \quad z_2 \quad z_3] \in \mathbf{C}_3 \mid z_1 + z_2 + z_3 = 0\}$.
   (e) If $A \in \mathbf{R}^n_n$ is an orthogonal matrix, that is, $A^T = A^{-1}$, what is the most you can say about $\det(A)$?

4. (15 points) Let $L : \mathbf{R}^2_2 \to \mathbf{R}^2_2$ be $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & a+b \\ a+b+c & a+b+c+d \end{bmatrix}$, let $v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and let $S = \{v, L(v), L^2(v), L^3(v)\}$.
   (a) Find S and show it is independent, so it is a basis of $\mathbf{R}^2_2$.
   (b) Find $L^4(v)$ and express it as a linear combination of the vectors in S.
   (c) Using the answers to parts (a) and (b) find the companion matrix $C = [L]^S_S$ that represents L from S to S and find its characteristic polynomial.

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1. (40 Points)

(a) (30 Points) (1) The characteristic polynomial is $\Delta_L(t) = \det(A - tI_4) =$

$$\det\begin{bmatrix} 18-t & -20 & -20 & -20 \\ 5 & -7-t & -5 & -5 \\ 5 & -5 & -7-t & -5 \\ 5 & -5 & -5 & -7-t \end{bmatrix} = \det\begin{bmatrix} -t-2 & 0 & 0 & 4t+8 \\ 0 & -t-2 & 0 & t+2 \\ 0 & 0 & -t-2 & t+2 \\ 5 & -5 & -5 & -7-t \end{bmatrix}$$

$$= (t+2)^3 \ \det\begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 5 & -5 & -5 & -7-t \end{bmatrix} = (t+2)^3 \ \det\begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -5 & -5 & -t+13 \end{bmatrix}$$

$$= -(t+2)^3 \ \det\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ -5 & -5 & -t+13 \end{bmatrix} = (t+2)^3 \ \det\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & -5 & -t+8 \end{bmatrix}$$

$$= (t+2)^3 \ \det\begin{bmatrix} -1 & 1 \\ -5 & -t+8 \end{bmatrix} = (t+2)^3 \ (t-3).$$

So the eigenvalues are $\lambda_1 = -2$ with algebraic multiplicity $k_1 = 3$ and $\lambda_2 = 3$ with algebraic multiplicity $k_2 = 1$.

(2) Check the $\lambda_1 = -2$ eigenspace first since the algebraic multiplicity $k_1 = 3$. Solve the homogeneous linear system whose coefficient matrix is obtained by plugging in $t = -2$ to $A - tI_4$. Row reduce

$$\left[\begin{array}{cccc|c} 20 & -20 & -20 & -20 & 0 \\ 5 & -5 & -5 & -5 & 0 \\ 5 & -5 & -5 & -5 & 0 \\ 5 & -5 & -5 & -5 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = r+s+t \\ x_2 = r \in \mathbf{R} \\ x_3 = s \in \mathbf{R} \\ x_4 = t \in \mathbf{R} \end{array}, \text{ then}$$

$$A_{\lambda_1} = \left\{ \begin{bmatrix} r+s+t \\ r \\ s \\ t \end{bmatrix} \in \mathbf{R}^4 \;\middle|\; r, s, t \in \mathbf{R} \right\} \text{ has basis } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

with three vectors. Since there will be one more independent eigenvector from the other eigenvalue, we will have the necessary three eigenvectors to form a basis for $\mathbf{R}^4$, so this L is diagonalizable.

Now find the $\lambda_2 = 3$ eigenspace. Solve the homogeneous linear system whose coefficient matrix is obtained by plugging in $t = 3$ to $A - tI_4$. Row reduce

$$\left[\begin{array}{cccc|c} 15 & -20 & -20 & -20 & 0 \\ 5 & -10 & -5 & -5 & 0 \\ 5 & -5 & -10 & -5 & 0 \\ 5 & -5 & -5 & -10 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = 4r \\ x_2 = r \\ x_3 = r \\ x_4 = r \in \mathbf{R} \end{array}, \text{ then}$$

$$A_{\lambda_2} = \left\{ \begin{bmatrix} 4r \\ r \\ r \\ r \end{bmatrix} \in \mathbf{R}^4 \;\middle|\; r \in \mathbf{R} \right\} \qquad \text{has basis} \qquad \left\{ \begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Therefore, $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, ${}_TD_T = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ and $P =$ ${}_SP_T = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $P^{-1} = {}_TP_S = \begin{bmatrix} -1 & 2 & 1 & 1 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & 1 & 2 \\ 1 & -1 & -1 & -1 \end{bmatrix}$. We check:

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} -1 & 2 & 1 & 1 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & 1 & 2 \\ 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 18 & -20 & -20 & -20 \\ 5 & -7 & -5 & -5 \\ 5 & -5 & -7 & -5 \\ 5 & -5 & -5 & -7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -4 & -2 & -2 \\ 2 & -2 & -4 & -2 \\ 2 & -2 & -2 & -4 \\ 3 & -3 & -3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = D. \end{aligned}$$

1(b) (10 Points) (1) The characteristic polynomial is $\Delta_L(t) = \det(A - tI_2) =$

$$\det \begin{bmatrix} -7-t & -4 \\ 1 & -3-t \end{bmatrix} = (t+7)(t+3) - (-4) = t^2 + 10t + 21 + 4 = (t+5)^2$$

So the only eigenvalue is $\lambda_1 = -5$ with algebraic mulitplicity $k_1 = 2$.

(2) To get the $\lambda_1 = -5$ eigenspace solve the homogeneous linear system whose coefficient matrix is obtained by plugging $t = -5$ into $A - tI_2$. Row reduce

$$\left[\begin{array}{cc|c} -2 & -4 & 0 \\ 1 & 2 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{r} x_1 = -2r \\ x_2 = r \in \mathbf{R} \end{array}$$

so the $\lambda_1 = -5$ eigenspace

$$A_{\lambda_1} = \left\{ \begin{bmatrix} -2r \\ r \end{bmatrix} \in \mathbf{R}^2 \;\middle|\; r \in \mathbf{R} \right\} \qquad \text{has basis} \qquad \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

which means we could not find enough independent eigenvectors to make a basis of $\mathbf{R}^2$, so this L is not diagonalizable.

2. (15 Points) Since the characteristic polynomial of the first matrix in problem 3 is $\Delta_A(t) = (t{+}2)^3(t{-}3)$ the only possibilities for $m_A(t)$ are the divisors sharing the same irreducible factors, namely, $(t+2)(t-3)$, $(t+2)^2(t-4)$ and $(t+2)^3(t-4)$. Checking the first one, we find

$$(A{+}2I)(A{-}3I) = \begin{bmatrix} 20 & -20 & -20 & -20 \\ 5 & -5 & -5 & -5 \\ 5 & -5 & -5 & -5 \\ 5 & -5 & -5 & -5 \end{bmatrix} \begin{bmatrix} 15 & -20 & -20 & -20 \\ 5 & -10 & -5 & -5 \\ 5 & -5 & -10 & -5 \\ 5 & -5 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $(t+2)(t-3)$ is the minimal polynomial of that matrix A.

The characteristic polynomial of the second matrix in problem 1 is $\Delta_A(t) = (t+5)^2$, so the only possible minimal polynomials are $(t+5)$ and $(t+5)^2$. Since $A \neq -5I_2$, the first one is clearly not satisfied by A, so $m_A(t) = (t+5)^2$ and we verify

$$(A + 5I_2)^2 = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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3. (30 points, 6 points each)
   (a) If $L(v) = \lambda v$ for $v \neq \theta$ then $L^n = 0$ gives $\theta = L^n(v) = \lambda^n v$, so $\lambda^n = 0$ which means $\lambda = 0$ is the only possible eigenvalue of L.

   (b) We have $[\,x \quad y \quad z\,] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2x^2 - 2xy + 2y^2 - 2yz + 2z^2 =$ $(x-y)^2 + (y-z)^2 + x^2 + z^2 \geq 0$ since it is a positive linear combination of squares of reals, and if it equals zero then $x - y = 0$, $y - z = 0$, $x = 0$ and $z = 0$ implies $x = y = z = 0$.

   (c) $S^{\perp} = \{[x_1\ x_2\ x_3\ x_4] \in \mathbf{R}_4 \mid 1x_1{+}2x_2{+}4x_3{+}5x_4 = 0, 2x_1{+}3x_2{+}5x_3{+}6x_4 = 0\}$. A simple row reduction finds the solutions to be $x_1 = 2r{+}3s$, $x_2 = -3r{-}4s$, $x_3 = r$, $x_4 = s$ so a basis could be $\{[2\ \ -3\ 1\ 0], [3\ \ -4\ 0\ 1]\}$.

   (d) W has basis $\{[-1\ 1\ 0], [-1\ 0\ 1]\}$ but this is not orthogonal. Applying Gram-Schmidt, we get the orthogonal basis of W $\{w_1 = [-1\ 1\ 0], w_2 = [-1\ -1\ 2]\}$. Then

$$\begin{aligned} \mathit{Proj}_W(v) &= \frac{v \cdot w_1}{w_1 \cdot w_1}\, w_1 + \frac{v \cdot w_2}{w_2 \cdot w_2}\, w_2 \\ &= \frac{\mathbf{i} - 1}{2}\, w_1 + \frac{\mathbf{i} + 1}{6}\, w_2 = \frac{1}{3}[(1 - 2\mathbf{i})\ (-2 + \mathbf{i})\ (1 + \mathbf{i})]. \end{aligned}$$

   (e) If $A \in \mathbf{R}_n^n$ is an orthogonal matrix, that is, $A^T = A^{-1}$, then $\det(A^T) = \det(A^{-1})$, but $\det(A^T) = \det(A)$ and $\det(A^{-1}) = (\det(A))^{-1}$, so $\det(A) = (\det(A))^{-1}$ which means $(\det(A))^2 = 1$, so $\det(A) = \pm 1$.

4. (15 points) We have $L : \mathbf{R}^2_2 \to \mathbf{R}^2_2$ by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & a+b \\ a+b+c & a+b+c+d \end{bmatrix}$, and $v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $S = \{v, L(v), L^2(v), L^3(v)\}$.

(a) $S = \left\{ v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, L(v) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, L^2(v) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, L^3(v) = \begin{bmatrix} 1 & 3 \\ 6 & 10 \end{bmatrix} \right\}$.
To show it is independent, we row reduce

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 3 & 6 & 0 \\ 0 & 1 & 4 & 10 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \end{array}$$

(b) $L^4(v) = \begin{bmatrix} 1 & 4 \\ 10 & 20 \end{bmatrix}$. To express it as a linear combination of the vectors in S we row reduce

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 6 & 10 \\ 0 & 1 & 4 & 10 & 20 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \text{ so } \begin{array}{r} x_1 = -1 \\ x_2 = 4 \\ x_3 = -6 \\ x_4 = 4 \end{array}$$

This means $L^4(v) = -1v + 4L(v) - 6L^2(v) + 4L^3(v)$.

(c) From the answers to parts (a) and (b), the companion matrix that represents L from S to S is

$$C = [L]_S^S = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

and the characteristic polynomial is read off from the right column, or from the dependence relation found in part (b). It is

$$t^4 - 4t^3 + 6t^2 - 4t + 1.$$