Math 222 Calculus 2 Fall 2008 Exam 1 Solutions

(1) (10 Points)

(a) (2 points) Write down the formula relating the derivatives $f'(x)$ and $(f^{-1})'(y)$. (No work needs to be shown.)

The relationship between them is $(f^{-1})'(y) = \dfrac{1}{f'(x)}$.

(b) (4 points) For the specific function $f(x) = x^5 + x^3 + x$ explain why $f(x)$ has an inverse function.

Since $f(x) = x^5+x^3+x$, $f'(x) = 5x^4+3x^2+1 \geq 1 > 0$ so $f(x)$ is an increasing function, which is then one-to-one, so it has an inverse.

(c) (4 points) For $y = f(x) = x^5+x^3+x$, use your answer to part (a) to find $(f^{-1})'(3)$. Note that $f(1) = 3$.

From part (a) we know that $(f^{-1})'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{5(1)^4 + 3(1)^2 + 1} = \dfrac{1}{9}$.

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(2) (10 Points) Simplify each of the following expressions.

(a) $\cos^{-1}(\sqrt{3}/2) = \pi/6$ since $\cos(\pi/6) = \sqrt{3}/2$.

(b) $\ln(5^{x\sqrt{x}}) = \ln(e^{x\sqrt{x}\ln(5)}) = x\sqrt{x}\ln(5)$.

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(3) (5 Points) Use the fact that $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ to find $\displaystyle\lim_{n\to\infty}\left(1+\frac{2}{3n}\right)^n$.

Let $x = 3n/2$. Since $x \to \infty$ as $n \to \infty$, the limit equals

$$\lim_{n\to\infty}\left(1+\frac{1}{3n/2}\right)^n = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x/3} = \left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right)^{2/3} = e^{2/3}.$$

(4) (15 Points) Find each of these derivatives.

(a) $\dfrac{d}{dx}\ \sin^{-1}(x^3) = \dfrac{3x^2}{\sqrt{1-(x^3)^2}} = \dfrac{3x^2}{\sqrt{1-x^6}}.$

(b) $\dfrac{d}{dx}\ 3^{\tan^{-1}(x)} = 3^{\tan^{-1}(x)}\dfrac{1}{1+x^2}\ln(3)$

(c) $\dfrac{d}{dx}\ \log_2(x^4+x^2+1) = \dfrac{4x^3+2x}{(x^4+x^2+1)\ln(2)}$

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(5) (15 Points) Find each of these integrals.

(a) $\displaystyle\int \frac{dx}{1+9x^2}$

Using the substitution $u = 3x$ so $du = 3dx$, we get

$$\int \frac{du/3}{1+u^2} = \frac{1}{3}\tan^{-1}(u) + C = \frac{1}{3}\ \tan^{-1}(3x) + C.$$

(b) $\displaystyle\int_e^{e^2} \frac{dx}{x(\ln(x))^4}$

Using substitution $u = \ln(x)$ we get $du = \frac{dx}{x}$ and the bounds of the integral are from $u = \ln(e) = 1$ to $u = \ln(e^2) = 2$, so

$$\int_e^{e^2} \frac{dx}{x(\ln(x))^4} = \int_1^2 u^{-4}\ du = \frac{u^{-3}}{-3}\Big]_1^2 = \frac{-1}{3}\left(\frac{1}{2^3} - \frac{1}{1^3}\right) = \frac{7}{24}.$$

(c) $\displaystyle\int_0^{\pi/4} \sin(x)\ e^{\cos(x)}\ dx$

Using substitution $u = \cos(x)$ we get $du = -\sin(x)dx$ and the bounds of the integral are from $u = \cos(0) = 1$ to $u = \cos(\pi/4) = 1/\sqrt{2}$, so $\displaystyle\int_0^{\pi/4} \sin(x)\ e^{\cos(x)}dx = \int_1^{1/\sqrt{2}} -e^u du = -e^u\Big]_{u=1}^{u=1/\sqrt{2}} = -e^{1/\sqrt{2}} - (-e^1) = e - e^{1/\sqrt{2}}.$

(6) (20 Points) Evaluate the following limits. If you use L'Hospital's Rule, show where you use it and explain what type of limit you are using it on.

(a) $\lim_{x\to 0} \frac{\sin(2x^3)}{\sin(3x^3)}$

(7 points) By L'Hospital's Rule for a $\frac{0}{0}$-type indeterminate form,

$$\lim_{x\to 0} \frac{\sin(2x^3)}{\sin(3x^3)} \overset{L'H}{=} \lim_{x\to 0} \frac{6x^2\cos(2x^3)}{9x^2\cos(3x^3)} = \lim_{x\to 0} \frac{(6)(1)}{(9)(1)} = \frac{2}{3}.$$

(b) $\lim_{x\to 0^+} \sqrt{x}\ln(x)$

(6 points) $\lim_{x\to 0^+} \sqrt{x}\ln(x)$ is an indeterminate form of the type $(0)(-\infty)$, which we make into a $\frac{\infty}{\infty}$-type by writing it as $\lim_{x\to 0^+} \frac{\ln(x)}{x^{-1/2}} \overset{L'H}{=} \lim_{x\to 0^+} \frac{1/x}{\frac{-1}{2}x^{-3/2}} = \lim_{x\to 0^+} \frac{-2x^{3/2}}{x} = \lim_{x\to 0^+} -2x^{1/2} = 0$

(c) $\lim_{x\to 0} \left(1+x+2x^2\right)^{1/x}$.

(7 points) Let $y = \left(1+x+2x^2\right)^{1/x}$ so $\ln(y) = \frac{\ln(1+x+2x^2)}{x}$. Then

$$\lim_{x\to 0} \ln(y) = \lim_{x\to 0} \frac{\ln(1+x+2x^2)}{x}$$

is a type $\frac{0}{0}$ indeterminate form. L'Hospital's Rule gives

$$\lim_{x\to 0} \frac{\ln(1+x+2x^2)}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1+4x}{1+x+2x^2}}{1} = 1$$

so $y \to e^1 = e$ is the limit we seek.

(7) (10 Points) In the year 1980 your parents invested $\$10,000$ in a special bank account which earned interest **compounded continuously**. In the year 2000 that account was worth $\$160,000$. Assuming no withdrawals, and the same interest rate, what will that account be worth in the year 2020?

For continuous compounding, the value of the account is $P(t) = 10000e^{It}$ where I is the annual interest rate. Then we know $160000 = 10000e^{20I}$ so $16 = e^{20I}$ gives $I = \frac{\ln(16)}{20}$. Then in the year 2020, when $t = 40$, the value of the account would be

$$P(40) = 10000e^{40I} = 10000e^{2\ln(16)} = 10000(16)^2 = 10000(256) = 2,560,000 \textit{ dollars}.$$