Math 222 Calculus 2 Fall 2008 Exam 3 Solutions

(1) (10 Points) In each part test the series for convergence or divergence. Write all steps of the test you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{7n^3+n^2+1}$

(5 Points) Since $0 \le \frac{1}{7n^3+n^2+1} \le \frac{1}{7n^3}$ for all $n \ge 1$, and $\sum_{n=1}^{\infty} \frac{1}{7n^3} = \frac{1}{7}\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a multiple of a convergent p-series with $p = 3$, the series converges by the Comparison test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3-n^2+1}}$

(5 Points) Apply the Limit Comparison test where the series $\sum b_n$ being compared to is the divergent p-series $\sum \frac{1}{n^{3/4}}$. We have

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt[4]{n^3-n^2+1}}}{\frac{1}{\sqrt[4]{n^3}}} = \lim_{n\to\infty} \sqrt[4]{\frac{n^3}{n^3-n^2+1}} = \lim_{n\to\infty} \sqrt[4]{\frac{1}{(1-\frac{1}{n}+\frac{1}{n^3})}} = 1 > 0$$

so the Limit Comparison test says both series have the same behavior, they both diverge.

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(2) (10 Points) Test the following series for absolute convergence, conditional convergence, or divergence. Explain what tests you are applying and how you apply them.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

(5 Points) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ is an alternating series, and for $n \ge 1$ we have $(n+1)^2+1 > n^2+1$ so $\frac{1}{(n+1)^2+1} < \frac{1}{n^2+1}$. Also, $lim_{n\to\infty} \frac{1}{n^2+1} = 0$, so this series converges by the Alternating Series test. Compare the absolute values of the terms of this series with the terms of the convergent p-series with $p = 2 > 1$, $\frac{1}{n^2+1} < \frac{1}{n^2}$. So the given series converges absolutely.

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$

(5 Points) Applying the root test to the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$ gives $L = \displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1$, which gives no information. But $\left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+(1/n)}\right)^n \to \frac{1}{e} \ne 0$ as $n \to \infty$ so the n^{th} term of the series does not converge to zero. Therefore, the series diverges by the Test for Divergence.

(3) (12 Points) Find all values of x where the following power series converge (**interval of convergence**).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-2)^n}{3^n}$

(8 Points) The Ratio Test gives $\displaystyle\left|\frac{(n+1)(x-2)^{n+1}}{3^{n+1}} \frac{3^n}{n(x-2)^n}\right| = \frac{n+1}{n}\frac{|x-2|}{3}$. As $n \to \infty$ this ratio goes to $L = \frac{|x-2|}{3}$ so that $L < 1$ iff $|x-2| < 3$ iff $-1 < x < 5$. We must check for convergence at the endpoints separately. At $x = -1$ the series is $\sum_{n=1}^{\infty} \frac{n(-3)^n}{3^n} = \sum_{n=1}^{\infty} n(-1)^n$ which diverges. At $x = 5$ the series is $\sum_{n=1}^{\infty} \frac{n3^n}{3^n} = \sum_{n=1}^{\infty} n$ which diverges. So the complete domain of convergence is $(-1, 5)$, that is, $-1 < x < 5$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n\ n!}$

(4 Points) The Ratio Test gives $\displaystyle\left|\frac{x^{n+1}}{2^{n+1}\ (n+1)!} \frac{2^n\ n!}{x^n}\right| = \frac{|x|}{2(n+1)}$. As $n \to \infty$ for any fixed x, this ratio goes to zero, so the series converges (absolutely) for all real x, that is, on $(-\infty, \infty)$.

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(4) (10 Points) Write a power series representation for each function and give its radius of convergence. Use summation notation to express all terms.

(a) $\ln(1+x)$

(5 Points) The series $\displaystyle\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$ converges absolutely for $|x| < 1$. Taking the integral of it, we get $\displaystyle\ln(1+x) + C = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$ and at $x = 0$ we get $\ln(1) + C = 0$ so $C = 0$. So for $|x| < 1$ we have

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{m=1}^{\infty}(-1)^{m-1}\frac{x^m}{m}.$$

(b) $\tan^{-1}(x^2)$

(5 Points) The series $\displaystyle\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ converges absolutely for $|x| < 1$. So

$$\tan^{-1}(x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{2n+1}$$ converges absolutely for $|x| < 1$.

(5) (10 Points) Find the second degree Taylor polynomial $T_2(x)$ which approximates the function $f(x) = \dfrac{1}{\sqrt[3]{x}} = x^{-1/3}$ at $a = 27$.

(10 Points) We have

$$f^{(1)}(x) = \frac{-1}{3}x^{-4/3} \qquad \text{and} \qquad f^{(2)}(x) = \frac{4}{9}x^{-7/3}.$$

Then

$$\begin{aligned} f(27) &= (27)^{-1/3} = \frac{1}{3} \\ f^{(1)}(27) &= \frac{-1}{3}(27)^{-4/3} = \frac{-1}{3^5} = \frac{-1}{243} \\ f^{(2)}(27) &= \frac{4}{9}(27)^{-7/3} = \frac{4}{3^9} = \frac{4}{19683} \end{aligned}$$

so

$$\begin{aligned} T_2(x) &= f(27) + \frac{f^{(1)}(27)}{1!}(x-27) + \frac{f^{(2)}(27)}{2!}(x-27)^2 \\ &= \frac{1}{3} + \frac{-1}{243}(x-27) + \frac{2}{19683}(x-27)^2. \end{aligned}$$

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(6) (10 Points) The alternating series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k}}$ converges to S and has N^{th} partial sum $S_N = \displaystyle\sum_{k=1}^{N} \frac{(-1)^k}{\sqrt[3]{k}}$. Find the smallest N such that you can be sure $|S - S_N| < \frac{1}{100}$.

(10 Points) We have $|S - S_N| \le \dfrac{1}{\sqrt[3]{N+1}}$ from the Alternating Series Estimation theorem, so to guarantee the accuracy given, we need $\dfrac{1}{\sqrt[3]{N+1}} < \dfrac{1}{100}$ which means $100 < \sqrt[3]{N+1}$ which means $1,000,000 < N+1$. The smallest N such that this is true is $N = 1,000,000$.

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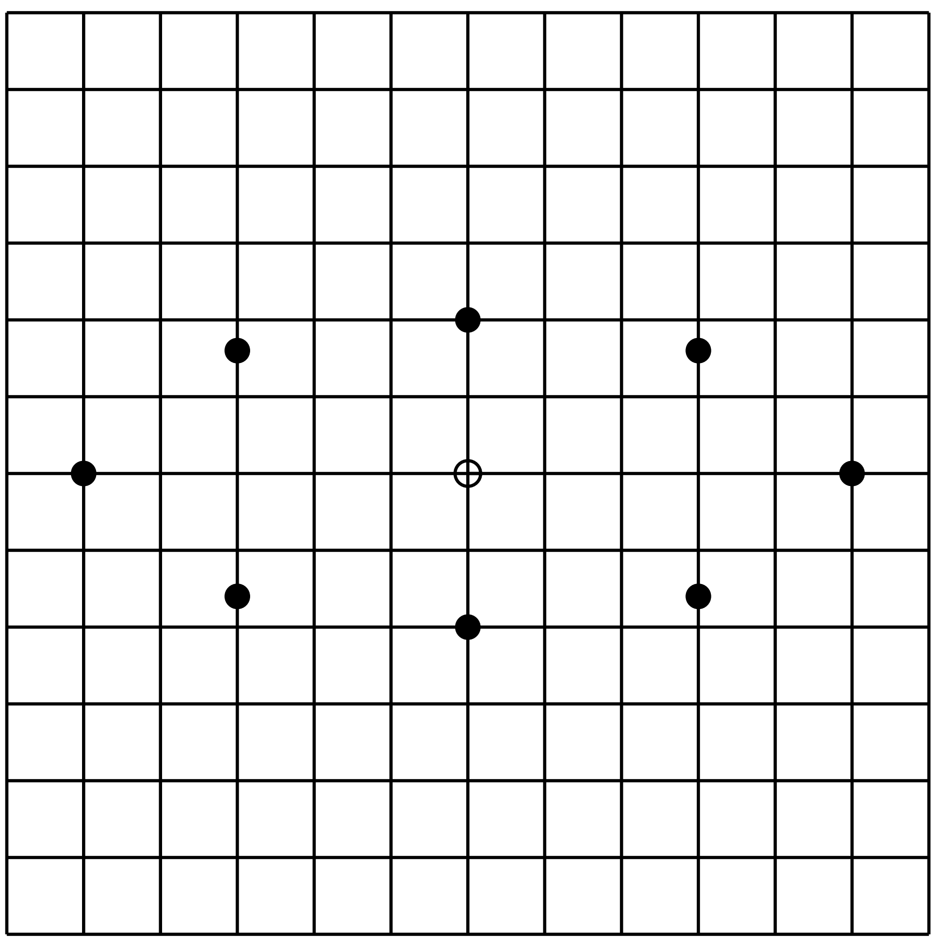
(7) (8 Points) The motion of a particle is given by the parametric equations $x = 5\cos(t)$ and $y = 2\sin(t)$ for $0 \le t \le 2\pi$.

(a) Find the equation relating x and y without parameter t.

(4 Points) Since $x^2 = 25\cos^2(t)$ and $y^2 = 4\sin^2(t)$ we see that $\frac{x^2}{25} + \frac{y^2}{4} = \cos^2(t) + \sin^2(t) = 1$. This is an ellipse with x-radius 5 and y-radius 2 centered at the origin.

(b) Sketch the curve and indicate the direction of motion of the particle.

(4 Points) The motion of a particle given by the parametric equations $x = 5\cos(t)$ and $y = 2\sin(t)$ for $0 \le t \le 2\pi$ is along an ellipse. As t hits the values 0, $\pi/2$, π, $3\pi/2$, 2π the points (x, y) hit are $(5, 0)$, $(0, 2)$, $(-5, 0)$, $(0, -2)$ and $(5, 0)$. Four other points which are easy to find from the ellipse equation are $(x, y) = (\pm 3, \pm 1.6)$. (See Sketch).



(8) (15 Points) Parametric equations for a curve are $x = t^2 + e^t$ and $y = t^3 + e^{-t}$ for $0 \le t \le 2$.

(a) Set up the integral for the length of that curve, but DO NOT EVALUATE OR SIMPLIFY THAT INTEGRAL. (5 Points)

$$L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^2 \sqrt{(2t + e^t)^2 + (3t^2 - e^{-t})^2}\, dt$$

(b) Set up the integral for the surface area obtained by rotating that curve around the x-axis, but DO NOT EVALUATE OR SIMPLIFY THAT INTEGRAL. (5 Points)

$$L = \int_0^2 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^2 2\pi (t^3 + e^{-t}) \sqrt{(2t + e^t)^2 + (3t^2 - e^{-t})^2}\, dt$$

(c) Find the equation of the tangent line to that curve at $t = 0$. (5 Points)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - e^{-t}}{2t + e^t}$$

and at $t = 0$ this slope is -1. At $t = 0$ we also have $x = 1$ and $y = 1$, so the equation of the tangent line is $y - 1 = -1(x - 1)$, which can be written as $x + y = 2$ or as $y = -x + 2$.