Math 304 Linear Algebra Fall 2006 Practice Exam 2 Feingold

(1) (20 Points) Let S be the following subset of $\mathbf{R}_2^2$,

$$S = \left\{ \begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ 1 & -4 \end{bmatrix} \right\}$$

and let $W = \langle S \rangle$ be the span of S.

(a) Is S linearly independent or dependent? If S is dependent, find the non-trivial dependence relations which allow redundant vectors to be removed.

(b) Does $W = \mathbf{R}_2^2$? If not, give conditions on a, b, c, d such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \langle S \rangle$.

(c) Find a basis for W.

(d) Find $\dim(W)$.

(2) (20 Points) Let $L : \mathbf{R}_2^2 \to \mathbf{R}_4$ be the linear transformation given by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = [\,(a+b-c-d) \quad (a+2b+d) \quad (-a+b+3c+5d) \quad (2a+3b-c)\,]$$

(a) (5 points) Find the set of all vectors in $\mathrm{Ker}(L)$.

(b) (3 points) Find a basis for $\mathrm{Ker}(L)$.

(c) (1 points) Find $\dim(\mathrm{Ker}(L))$.

(d) (1 points) Is L one-to one? **Explain why!**

(e) (5 points) Find a basis for $\mathrm{Range}(L)$.

(f) (2 points) Find $\dim(\mathrm{Range}(L))$.

(g) (2 points) Is L onto? **Explain why!**

(h) (1 points) Is L invertible? **Explain why!**

(3) (20 Points, 4 Points Each) Answer each question separately

(a) What elementary matrix corresponds to the elementary row operation $R_3 + 7R_2 \to R_3$ done to a matrix in $\mathbf{R}_n^3$?

(b) If $S = \{v_1, v_2, \ldots, v_n\}$ is an independent set in vector space V, what is the relationship between n and $\dim(V)$?

(c) If $T = \{w_1, w_2, \ldots, w_m\}$ spans vector space W, what is the relationship between m and $\dim(W)$?

(d) If $T = \{w_1, w_2, \ldots, w_m\}$ spans vector space W, and $w \in W$, what is the most you can say about $T \cup \{w\} = \{w_1, \ldots, w_m, w\}$?

(e) If $S = \{v_1, \ldots, v_n\}$ is an independent set in a vector space V and $v \in V$ is not in the span of S, then what is the most you can you say about $S \cup \{v\} = \{v_1, \ldots, v_n, v\}$?

(4) (20 points, 4 points each) Answer each question separately.

(a) If $L : V \to W$ is a linear transformation, $\dim(V) = 15$ and $\dim(\text{Ker}(L)) = 6$ then what is $\dim(\text{Range}(L))$?

(b) If $L : \mathbf{R}_5^6 \to \mathbf{R}^{18}$ what are all the possibilities for $\dim(\text{Ker}(L))$?

(c) If $L : \mathbf{R}_6 \to \mathbf{R}^{11}$ what are all the possibilities for $\dim(\text{Range}(L))$?

(d) If $S = \{v_1, \ldots, v_n\}$ is a basis of a vector space V, and $v \in V$ has coordinates

$$[v]_S = \begin{bmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ n \end{bmatrix} \in \mathbf{R}^n, \text{ then write } v \text{ as a linear combination from } S.$$

(e) Let $S = \{[1\ 2\ 3], [0\ 1\ 2], [-1\ -1\ -2]\}$ be a basis of $\mathbf{R}_3$ and let $v = [8\ 1\ 9] \in \mathbf{R}_3$. Find the coordinate vector $[v]_S$.

(5) (20 Pts) Let $L : \mathbb{R}_4 \to \mathbb{R}^3$ be given by

$$L\left(\begin{bmatrix} a & b & c & d \end{bmatrix}\right) = \begin{bmatrix} a - b + c - d \\ 2a + b - 3c + d \\ -a + 2b + 2c + 3d \end{bmatrix}.$$

Let S be the standard basis of $\mathbb{R}_4$ and let T be the standard basis of $\mathbb{R}^3$. Let other ordered bases be

$$S' = \{[1\ 2\ 3\ 4], [1\ 1\ 1\ 1], [0\ 1\ 1\ 2], [0\ 0\ 1\ -1\} \text{ and } T' = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

(a) (4 pts) Find the matrix $[L]_S^T$ representing L from S to T.

(b) (4 pts) Find the matrix $[L]_{S'}^{T'}$ representing L from S' to T' **without using transition matrices**. (Do it directly.)

(c) (12 pts) Find the transition matrices ${}_SP_{S'}$ and ${}_TQ_{T'}$ and show that $[L]_{S'}^{T'} = ({}_TQ_{T'})^{-1}\ [L]_S^T\ ({}_SP_{S'})$.

Math 304 Linear Algebra Fall 2006 Practice Exam 2 Solutions Feingold

1. (a) Let the 5 vectors of S be denoted $v_1, \cdots, v_5$. Determine if $\sum_{i=1}^{5} a_i v_i = \theta$ has nontrivial solutions. Reduce

$$\left[\begin{array}{ccccc|c} -1 & 1 & 1 & 1 & 0 & 0 \\ 3 & -2 & -1 & 1 & 5 & 0 \\ 2 & -2 & -2 & -1 & 1 & 0 \\ -4 & 3 & 2 & 1 & -4 & 0 \end{array}\right] \text{ to } \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = -r - 2s \\ x_2 = -2r - s \\ x_3 = r \in \mathbf{R} \\ x_4 = -s \\ x_5 = s \in \mathbf{R} \end{array}$$

which has nontrivial solutions so S is dependent. Each free variable gives a dependence relation. When $r = 1$ and $s = 0$, we get $-v_1 - 2v_2 + v_3 = \theta$ and when $r = 0$ and $s = 1$ we get $-2v_1 - v_2 - v_4 + v_5 = \theta$. These allow the redundant vectors v_3 and v_5 to be expressed as linear combinations of v_1, v_2 and v_4, so S is spanned by just those three vectors.

(b) Since S is spanned by just 3 vectors and $\dim(\mathbf{R}^2_2) = 4$, S does not span $\mathbf{R}^2_2$. Since $\langle S \rangle = \langle \{v_1, v_2, v_4\} \rangle$, a matrix is in this span when the following system is consistent:

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & a \\ 3 & -2 & 1 & b \\ 2 & -2 & -1 & c \\ -4 & 3 & 1 & d \end{array}\right] \text{ reduces to } \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4a + b - 3c \\ 0 & 1 & 0 & -5a + b - 4c \\ 0 & 0 & 1 & 2a + c \\ 0 & 0 & 0 & -3a + b - c + d \end{array}\right]$$

which is consistent iff $0 = -3a + b - c + d$. This is the condition we wanted to find.

(c) From part (a), after removing the redundant vectors from S, we have $\{v_1, v_2, v_4\}$ is a basis for W. Another possible answer is obtained by using the condition from part (b), which expresses any element of W as

$$\begin{bmatrix} a & b \\ c & 3a - b + c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

so a basis of W is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$.

(d) $\dim(W) = 3$ since we have three vectors in a basis for W.

2. (20 points) To find Ker(L), row reduce

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ -1 & 1 & 3 & 5 & 0 \\ 2 & 3 & -1 & 0 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & -2 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} a = 2r+3s \\ b = -r-2s \\ c = r \in \mathbf{R} \\ d = s \in \mathbf{R} \end{array}.$$

(a) (5 points) $\text{Ker}(L) = \left\{ \begin{bmatrix} 2r+3s & -r-2s \\ r & s \end{bmatrix} \in \mathbf{R}_2^2 \;\middle|\; r, s \in \mathbf{R} \right\}$.

(b) (3 points) $\left\{ \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis of Ker(L).

(c) (1 points) dim(Ker(L))= 2.

(d) (1 points) L is not one-to-one since Ker(L) is nontrivial.

(e) (5 points) *Range*(L) =

$$\left\{ [(a+b-c-d)\ (a+2b+d)\ (-a+b+3c+5d)\ (2a+3b-c)] \in \mathbf{R}_4 \;\middle|\; a,b,c,d \in \mathbf{R} \right\}$$

is spanned by the set

$$\{[1 \quad 1 \quad -1 \quad 2], [1 \quad 2 \quad 1 \quad 3], [-1 \quad 0 \quad 3 \quad -1], [-1 \quad 1 \quad 5 \quad 0]\}.$$

The two free variables in Ker(L) mean the last two matrices are redundant vectors, so a basis for Range(L) is $\{[1 \quad 1 \quad -1 \quad 2], [1 \quad 2 \quad 1 \quad 3]\}$. Using the row space method, another answer is $\{[1 \quad 0 \quad -3 \quad 1], [0 \quad 1 \quad 2 \quad 1]\}$.

(f) (2 points) dim(Range(L)) = 2.

(g) (2 points) L is not onto, Range(L) $\neq \mathbf{R}_4$.

(h) (1 points) L is not invertible since it is not bijective.

3. (20 Points, 4 points each)

(a) The elementary matrix that corresponds to that row operation is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$ obtained by doing the operator to I_3.

(b) $\dim(V) \geq n$ (An independent set can be extended to a basis.)

(c) $\dim(W) \leq m$ (A spanning set can be cut down to a basis.)

(d) $T \cup \{w\}$ is dependent.

(e) $S \cup \{v\}$ is independent.

4. (20 Points, 4 points each)

(a) $\dim(Range(L)) = 9$.

(b) $12 \leq dim(Ker(L)) \leq 30$.

(c) $0 \leq dim(Range(L)) \leq 6$.

(d) $v = 1v_1 + 2v_2 + \cdots + nv_n$.

(e) $[v]_S = \begin{bmatrix} -7 \\ 0 \\ -15 \end{bmatrix}$.

5(a) (4 Pts) $[L]_S^T = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & 1 & -3 & 1 \\ -1 & 2 & 2 & 3 \end{bmatrix}$ is easy to get since S and T are standard.

$L([1\,0\,0\,0]) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $L([0\,1\,0\,0]) = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $L([0\,0\,1\,0]) = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $L([0\,0\,0\,1]) =$ $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$. $[T \mid L(S)] = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 2 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 & 2 & 2 & 3 \\ & T & & & L(S) & & \end{array}\right]$ is already reduced, so the right side is $[L]_S^T$.

5(b) (4 Pts) $L([1\ 2\ 3\ 4]) = \begin{bmatrix} -2 \\ -1 \\ 21 \end{bmatrix}$, $L([1\ 1\ 1\ 1]) = \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$, $L([0\ 1\ 1\ 2]) = \begin{bmatrix} -2 \\ 0 \\ 10 \end{bmatrix}$, $L([0\ 0\ 1\ -1]) = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}$.

Row reduce $\left[\begin{array}{ccc|cccc} 3 & 1 & 1 & -2 & 0 & -2 & 2 \\ 2 & -1 & 2 & -1 & 1 & 0 & -4 \\ 1 & 1 & 0 & 21 & 6 & 10 & -1 \\ & T' & & & L(S') & & \end{array}\right]$ to $\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & -66 & -19 & -34 & 11 \\ 0 & 1 & 0 & 87 & 25 & 44 & -12 \\ 0 & 0 & 1 & 109 & 32 & 56 & -19 \\ & I_3 & & & {}_{T'}A_{S'} & & \end{array}\right]$

5(c) (12 Pts) ${}_SP_{S'} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 \\ 4 & 1 & 2 & -1 \end{bmatrix}$ and ${}_TQ_{T'} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ since S and T are the standard bases.
To get ${}_{T'}Q_T = ({}_TQ_{T'})^{-1}$, reduce

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ & T' & & & T & \end{array}\right] \text{ to } \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -3 \\ 0 & 1 & 0 & -2 & 1 & 4 \\ 0 & 0 & 1 & -3 & 2 & 5 \\ & I_3 & & & {}_{T'}Q_T & \end{array}\right]$$

$$({}_TQ_{T'})^{-1}[L]_S^T({}_SP_{S'}) = \begin{bmatrix} 2 & -1 & -3 \\ -2 & 1 & 4 \\ -3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & 1 & -3 & 1 \\ -1 & 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 \\ 4 & 1 & 2 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} 3 & -9 & -1 & -12 \\ -4 & 11 & 3 & 15 \\ -4 & 15 & 1 & 20 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 \\ 4 & 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -66 & -19 & -34 & 11 \\ 87 & 25 & 44 & -12 \\ 109 & 32 & 56 & -19 \end{bmatrix} = [L]_{S'}^{T'}$$

checks.