

- (1) (25 Points) Let $L_A : \mathbf{R}^4 \rightarrow \mathbf{R}^4$ be the linear function $L_A(X) = AX$ associated with the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 3 & 5 \\ 2 & 3 & -1 & 0 \end{bmatrix}.$$

- (a) Find all the vectors in $\text{Ker}(L_A) = \{X \in \mathbf{R}^4 \mid L_A(X) = 0\}$ in terms of some free variables.
- (b) Find all the vectors in $\text{Range}(L_A) = \{Y = L_A(X) \in \mathbf{R}^4 \mid X \in \mathbf{R}^4\}$ in terms of a consistency condition on the entries of $Y = [y_i]$.
- (c) Determine whether L is injective and whether L is surjective.
- (2) (15 Points) Let $L : \mathbf{R}_2^2 \rightarrow \mathbf{R}_2^2$ be the function $L \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & c \end{bmatrix}$
- (a) Show that L is a linear transformation.
- (b) Find formulas for the compositions $L^2 = L \circ L$, $L^3 = L \circ L^2$, and $L^4 = L \circ L^3$. What does this tell you about L^k for $k \geq 5$?
- (3) (20 Points) Answer each question separately

- (a) If $A \in \mathbf{R}_n^m$, $B \in \mathbf{R}_p^n$, and $C \in \mathbf{R}_p^m$ are matrices such that the associated functions $L_A : \mathbf{R}^n \rightarrow \mathbf{R}^m$, $L_B : \mathbf{R}^p \rightarrow \mathbf{R}^n$, and $L_C : \mathbf{R}^p \rightarrow \mathbf{R}^m$ satisfy $L_A \circ L_B = L_C$, then what is the relationship between the matrices A , B and C ?
- (b) If a nonzero matrix $A \in \mathbf{R}_4^7$ row reduces to B in RREF having r leading ones, what are the possible values of r ?
- (c) For $A \in \mathbf{R}_n^m$ what relation between m and n would guarantee that the homogeneous linear system $AX = 0$ has nontrivial solutions?
- (d) What condition on the rank of $A \in \mathbf{R}_n^m$ is equivalent to the inhomogeneous linear system $AX = B$ being consistent for any choice of B ?
- (e) For $A, B, C \in \mathbf{R}_n^n$ write a formula for $(ABC)^T$ in terms of A^T , B^T and C^T .
- (4) (20 Points) For $A \in \mathbf{R}_n^m$, let $L_A : \mathbf{R}^n \rightarrow \mathbf{R}^m$ be the linear function $L_A(X) = AX$. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be the standard basis vectors in \mathbf{R}^n and let $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m$ be the standard basis vectors in \mathbf{R}^m .
- (a) Write a list of vectors guaranteed to span $\text{Range}(L_A)$.
- (b) If L_A is injective then what is the most you can say about the relation between m and n ?
- (c) If L_A is surjective then what is the most you can say about the relation between m and n ?
- (d) If $\text{rank}(A) = m$ what does that tell you about L_A ?
- (e) If $\text{rank}(A) = n$ what does that tell you about L_A ?

- (5) (20 Points) Let $L : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be linear with $L(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $L(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. Find $A \in \mathbf{R}_2^2$ such that $L = L_A$, and check that $L(X) = L_A(X) = AX$ for all $X \in \mathbf{R}^2$.

1. (a) (10 Points) To find $\text{Ker}(L)$ we must solve a linear system by row reducing

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ -1 & 1 & 3 & 5 & 0 \\ 2 & 3 & -1 & 0 & 0 \end{array} \right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & -2 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \text{ so } \begin{cases} x_1 = 2r + 3s \\ x_2 = -r - 2s \\ x_3 = r \in \mathbf{R} \\ x_4 = s \in \mathbf{R} \end{cases}.$$

$$\text{Ker}(L) = \left\{ \begin{bmatrix} 2r + 3s \\ -r - 2s \\ r \\ s \end{bmatrix} \in \mathbf{R}^4 \mid r, s \in \mathbf{R} \right\}.$$

(b) (10 Points) $Y = [y_i] \in \text{Range}(L)$ iff the following system is consistent:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & y_1 \\ 1 & 2 & 0 & 1 & y_2 \\ -1 & 1 & 3 & 5 & y_3 \\ 2 & 3 & -1 & 0 & y_4 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & -3 & 2y_1 - y_2 \\ 0 & 1 & 1 & 2 & -y_1 + y_2 \\ 0 & 0 & 0 & 0 & 3y_1 - 2y_2 + y_3 \\ 0 & 0 & 0 & 0 & y_1 + y_2 - y_4 \end{array} \right] \begin{array}{l} \text{is consistent iff} \\ 0 = 3y_1 - 2y_2 + y_3 \\ \text{and} \\ 0 = y_1 + y_2 - y_4 \end{array}.$$

(c) (5 Points) L is not injective since by (a) more than one vector is sent to the zero vector, and L is not surjective since by (b) not all vectors of \mathbf{R}_4 are in $\text{Range}(L)$.

2. (15 Points) $L : \mathbf{R}_2^2 \rightarrow \mathbf{R}_2^2$ is the function $L \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & c \end{bmatrix}$

(a) (10 Points) To show that L is a linear transformation we check both:

$$\begin{aligned} L \left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \right) &= L \left(\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 & a_1 + a_2 \\ b_1 + b_2 & c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 0 & a_1 \\ b_1 & c_1 \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ b_2 & c_2 \end{bmatrix} = L \left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \right) + L \left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \right) \end{aligned}$$

$$L \left(r \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = L \left(\begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix} \right) = \begin{bmatrix} 0 & ra \\ rb & rc \end{bmatrix} = r \begin{bmatrix} 0 & a \\ b & c \end{bmatrix} = rL \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$

(b) (5 Points) We have the formulas

$$L^2 \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = L \left(L \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) = L \left(\begin{bmatrix} 0 & a \\ b & c \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$$

$$L^3 \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = L \left(L^2 \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) = L \left(\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$$

$$L^4 \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = L \left(L^3 \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) = L \left(\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This means $L^k = O$ is the zero transformation for $k \geq 5$.

3. (20 Points, 4 points each)

- (a) The relationship is $AB = C$, so C is the matrix product of A and B .
 - (b) If $A \in \mathbf{R}_4^7$ is not the zero matrix, the number of leading ones in its RREF could only be 1, 2, 3 or 4 since each leading one occupies a column, and there is at least one.
 - (c) The relation $n > m$ guarantees that the homogeneous linear system $AX = 0$ has nontrivial solutions.
 - (d) The inhomogeneous linear system $AX = B$ is consistent iff the rank of A equals m .
 - (e) $(ABC)^T = C^T B^T A^T$.
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4. (20 Points, 4 points each)

- (a) $\text{Range}(L_A)$ consists of all vectors $AX = A(\sum_{j=1}^n x_j e_j) = \sum_{j=1}^n x_j A e_j = \sum_{j=1}^n x_j L_A(e_j)$ for all $x_j \in \mathbf{R}$. So $\text{Range}(L_A)$ is spanned by the n vectors $L_A(e_j) = A e_j$ for $1 \leq j \leq n$, which are the column vectors of matrix A .
 - (b) If L_A is injective then $n \leq m$ since more variables than equations would guarantee free variables.
 - (c) If L_A is surjective then $m \leq n$ since more equations than variables would guarantee a row of zeros in the RREF of A , giving a consistency condition for $AX = B$.
 - (d) If $\text{rank}(A) = m$ then L_A is surjective since m leading ones in the RREF means no zero rows so $AX = B$ is always consistent.
 - (e) If $\text{rank}(A) = n$ then L_A is injective since n leading ones in the RREF means a leading one in each column and there are no free variables.
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5. (20 Points) Let $L : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be linear with $L(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $L(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$. The $A \in \mathbf{R}_2^2$ such that $L(X) = L_A(X) = AX$ for all $X \in \mathbf{R}^2$ would have to satisfy $A\mathbf{e}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $A\mathbf{e}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, but $A\mathbf{e}_1 = \text{Col}_1(A)$ and $A\mathbf{e}_2 = \text{Col}_2(A)$, so $A = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$. This works for all $X \in \mathbf{R}^2$ because L is linear, so

$$\begin{aligned} L(X) &= L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = L(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2) = x_1 L(\mathbf{e}_1) + x_2 L(\mathbf{e}_2) \\ &= x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = AX. \end{aligned}$$