Math 503 Graduate Algebra Spring 2009 Final Exam Feingold

SHOW WORK TO JUSTIFY ALL ANSWERS.

1. (30 Points) Let $|G| = 45$.
   (a) Determine the possibilities for all the Sylow subgroups of G.
   (b) What is the most you can say about the structure of G?

2. (10 Points) Let $|G| = p^n$ for $n \geq 1$ and p prime. Prove that $|Z(G)| \neq p^{n-1}$.

3. (20 Points) Determine whether each assertion is true or false. If it is true, prove it. If it is false, give a counterexample.
   (a) If $P_1 \neq P_2$ are Sylow p-subgroups of G, then $P_2 \not\subseteq N_G(P_1)$.
   (b) A direct sum $G = \bigoplus_{j \in J} G_j$ is abelian if and only if G_j is abelian for each $j \in J$.
   (c) If $\phi : G \to H$ is any group homomorphism, $g \in G$ and $|\phi(g)|$ is finite, then $|g|$ must be finite and $|\phi(g)|$ divides $|g|$.
   (d) If $N \trianglelefteq G$ with $|N| = 2$ then $N \leq Z(G)$, that is, N is a subgroup of the center of G.

4. (30 Points) Let $C_7 = \langle x \mid x^7 = 1\rangle$ and $C_3 = \langle y \mid y^3 = 1\rangle$ be cyclic groups of orders 7 and 3, respectively. Let $f \in Aut(C_7)$ be the automorphism determined by $f(x) = x^2$, and define $\phi : C_3 \to Aut(C_7)$ by $\phi(y^i) = f^i$ for $i \in \mathbf{Z}$. We have seen on Exam 1 that ϕ is a well-defined function which is a group homomorphism, so ϕ determines the semidirect product $G = C_7 \rtimes_\phi C_3$. Then, **as a set**, $G = C_7 \times C_3$, but with the product $(a, b)(c, d) = (a(b \cdot c), bd)$ where $b \cdot c = \phi(b)(c)$.
   (a) Find the order of the element $(x, y) \in G$.
   (b) Prove that $|Z(G)| = 1$.

5. (30 Points) Let $G = \mathbf{Z}_{12} \times \mathbf{Z}_{15} = \{(a, b) \mid a \in \mathbf{Z}_{12}, b \in \mathbf{Z}_{15}\}$. Write the elements in G as ordered pairs of integers with the understanding that the first number is taken modulo 12 and the second number is taken modulo 15. Let $\langle(3, 5)\rangle = H \leq G$ be the cyclic subgroup generated by the element $(3, 5) \in G$.
   (a) Find the order of $(3, 5) \in G$ from the orders of $3 \in \mathbf{Z}_{12}$ and $5 \in \mathbf{Z}_{15}$.
   (b) Find all the elements in H. Does the number you found match $|(3, 5)|$?
   (c) G is Abelian so $H \trianglelefteq G$. What is the order $|G/H|$?
   (d) Find the order of the coset $(9, 8) + H$ in the quotient group G/H.
   (e) Is G/H a cyclic group? If so, find a coset which is a generator. If not, what can you say about it's structure?

Math 503 Graduate Algebra Spring 2009 Final Exam Solutions Feingold

1. (30 Points)
   (a) Since $|G| = 45 = 3^2\,5$, there are n_3 Sylow 3-subgroups, and n_5 Sylow 5-subgroups. We know $n_3 \equiv 1\ (mod\ 3)$ and n_3 divides 5, so $n_3 = 1$ making the Sylow 3-subgroup unique and normal. Call it P. We know $n_5 \equiv 1\ (mod\ 5)$ and n_5 divides 9, so out of the divisors $\{1, 3, 9\}$ only $n_5 = 1$ satisfies the congruence, making the Sylow 5-subgroup unique and normal. Call it Q. Those all all the possibilities.

   (b) Since $|P| = 9$ and $|Q| = 5$, $|P \cap Q| = 1$, and both are normal in G, so $PQ \leq G$ and $|PQ| = (9)(5) = |G|$, so $PQ = G$. In fact, both normal implies (as done in class) that for any $x \in P$ and any $y \in Q$, $xy = yx$. $|Q| = 5$ prime means Q is cyclic, and $|P| = 3^2$ means P is abelian by a theorem proved in the book. From the commutation of elements between P and Q we get that G is abelian, and it must be a direct sum (or product) of P and Q. The only possibilities are $C_9 \times C_5$ or $C_3 \times C_3 \times C_5$. In additive notation for abelian groups, that would be $\mathbf{Z}_9 \oplus \mathbf{Z}_5$ or $\mathbf{Z}_3 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_5$.

---

2. (10 Points) Let $Z = Z(G)$. If $|Z| = p^{n-1}$ then $|G/Z| = p$ so it would be a cyclic group. But we proved the theorem that if G/Z is cyclic, then G is Abelian, which means that $|Z| = |G| = p^n$, a contradiction.

Another proof: If $|Z| = p^{n-1}$ then let $a \in G$ with $a \notin Z$. Then we have $a \in C_G(a)$ and $Z \leq C_G(a) \leq G$ but $Z \neq C_G(a) \neq G$. This means $p^{n-1} = |Z| < |C_G(a)| < |G| = p^n$, which is impossible since there are no divisors of p^n strictly between p^{n-1} and p^n.

---

3. (20 Points) True or false. If true, prove it. If false, give a counterexample.

   (a) If $P_1 \neq P_2$ are Sylow p-subgroups of G, then $P_2 \not\subseteq N_G(P_1)$.

Solution: True. If $P_2 \leq N_G(P_1)$ then there would be two distinct Sylow p-subgroups in $N_G(P_1)$ because both P_1 and P_2 would be in it and have order the maximal power of p possible. But $P_1 \trianglelefteq N_G(P_1)$ means it is the unique Sylow p-subgroup in $N_G(P_1)$.

   (b) A direct sum $G = \bigoplus_{j \in J} G_j$ is abelian if and only if G_j is abelian for each $j \in J$.

Solution: True. For any $f, g \in G$, $fg = gf$ iff $f(j)g(j) = g(j)f(j)$ for all $j \in J$. Since $f(j), g(j) \in G_j$ can be arbitrary, G is abelian iff each G_j is abelian.

   (c) If $\phi : G \to H$ is any group homomorphism, $g \in G$ and $|\phi(g)|$ is finite, then $|g|$ must be finite and $|\phi(g)|$ divides $|g|$.

Solution: False. Let $\phi : \mathbf{Z} \to \mathbf{Z}_n$ be the projection $\phi(i) = [i]_n$. Then $\phi(1) = [1]_n$ has finite order n, but $1 \in \mathbf{Z}$ has infinite order.

   (d) If $N \trianglelefteq G$ with $|N| = 2$ then $N \leq Z(G)$, that is, N is a subgroup of the center of G.

Solution: If $N \trianglelefteq G$ with $|N| = 2$ then let $N = \{e, n\}$, so for any $x \in G$, $xnx^{-1} \in N$. If $xnx^{-1} = e$ then $n = x^{-1}ex = e$ contradicts $|N| = 2$, so $xnx^{-1} = n$ giving $xn = nx$. This true for all $x \in G$ says $n \in Z(G)$. Since $e \in Z(G)$, we get $N \leq Z(G)$.

---

4. (30 Points)

(a) $(x,y)^2 = (x,y)(x,y) = (x(y\cdot x), y^2) = (x\phi(y)(x), y^2) = (xf(x), y^2) = (xx^2, y^2) = (x^3, y^2)$ and $(x,y)^3 = (x,y)(x^3,y^2) = (x(y\cdot x^3), y^3) = (x\phi(y)(x^3), 1) = (xf(x^3), 1) = (x(f(x))^3, 1) = (x(x^2)^3, 1) = (x^7, 1) = (1,1)$ so the order of (x,y) is 3.

(b) Let $(x^n, y^i) \in Z(G)$. Then $(x^n, y^i)(x^m, y^j) = (x^m, y^j)(x^n, y^i)$ for $0 \le m \le 6$ and $0 \le j \le 2$.. This says $(x^n(y^i \cdot x^m), y^{i+j}) = (x^m(y^j \cdot x^n), y^{j+i})$ which is true iff the first coordinates are equal, that is, $x^n f^i(x^m) = x^m f^j(x^n)$, which means $x^n\ x^{2^i m} = x^m\ x^{2^j n}$. This is true in C_7 iff $n + 2^i m \equiv m + 2^j n\ (mod\ 7)$ for $0 \le m \le 6$, $0 \le j \le 2$. For $m = 0$ it says $n \equiv 2^j n\ (mod\ 7)$ for $j = 0,1,2$, so $n \equiv 2n\ (mod\ 7)$ which gives $n \equiv 0\ (mod\ 7)$ so $x^n = 1$ in C_7. Now we must have $2^i m \equiv m\ (mod\ 7)$ for $m = 0,1,2,...,6$. For $m = 1$ it says $2^i \equiv 1\ (mod\ 7)$ which is not true for $i = 1,2$, but only for $i = 0$. Therefore, $(x^n, y^i) = (1,1)$ in the only element of $Z(G)$.

---

5. (30 Points) Let $G = \mathbf{Z}_{12} \times \mathbf{Z}_{15} = \{(a,b) \mid a \in \mathbf{Z}_{12}, b \in \mathbf{Z}_{15}\}$. Write the elements in G as ordered pairs of integers with the understanding that the first number is taken modulo 12 and the second number is taken modulo 15. Let $\langle(3,5)\rangle = H \le G$ be the cyclic subgroup generated by the element $(3,5) \in G$.

(a) Find the order of $(3,5) \in G$ from the orders of $3 \in \mathbf{Z}_{12}$ and $5 \in \mathbf{Z}_{15}$. $|3| = 4$ in $\mathbf{Z}_{12}$ and $|5| = 3$ in $\mathbf{Z}_{15}$ so $|(3,5)| = lcm(4,3) = 12$ in G.

(b) Find all the elements in H. Does the number you found match $|(3,5)|$? $H = \{(3,5),(6,10),(9,0),(0,5),(3,10),(6,0),(9,5),(0,10),(3,0),(6,5),(9,10),(0,0)\}$. Yes, this matches $|(3,5)| = 12$ as it should.

(c) G is Abelian so $H \trianglelefteq G$. What is the order $|G/H|$? Since $|G| = (12)(15)$ and $|H| = 12$, we have $|G/H| = |G|/|H| = 15$.

(d) Find the order of the coset $(9,8) + H$ in the quotient group G/H. By repeated addition, we find $\langle(9,8) + H\rangle = \{n(9,8) + H \mid n \ge 1\}$

$$= \{(9,8) + H, (6,1) + H, (3,9) + H, (0,2) + H, (9,10) + H = H\}$$

and since $(9,10) \in H$, the last one is the trivial coset, larger n only repeat those listed, so $|(9,8) + H| = 5$.

(e) Is G/H a cyclic group? If so, find a coset which is a generator. If not, what can you say about it's structure? Yes, G/H is cyclic. We have seen in the book that any Abelian group of order 15 is cyclic, but more directly, we can see that $\langle(1,1) + H\rangle = \{(a,a) + H \in G/H \mid 1 \le a \le 15\}$ contains 15 distinct cosets, so it is all of G/H.