Math 507 Linear Algebra Fall 2002 Final Exam Feingold

YOU MUST SHOW WORK TO GET CREDIT FOR SOLUTIONS.

1. (40 points, 5 points each) Answer each of the following questions separately.

(a) If $S = \{v_1, \cdots, v_n\}$ is an orthonormal subset of an inner product space V with $dim(V) = n$, what is the most you can say about S?

(b) If $L \in \mathcal{L}(V)$ and $S = \{v_1, \cdots, v_n\}$ is a basis of V, what is the most you can say about $L(S) = \{L(v_1), \ldots, L(v_n)\}$?

(c) If $\{v_1, \ldots, v_t\}$ spans V and $\{w_1, \ldots, w_r\}$ is an independent subset of V, what is the most you can say about $dim(V)$?

(d) Let $\mathcal{P}_n(\mathbf{R})$ be the inner product space with inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. Write out the statement of the Cauchy-Schwarz inequality in this space as a relationship between integrals.

(e) If $L \in \mathcal{L}(\mathbf{R}^n)$ is represented with respect to the standard basis S by a symmetric matrix A, what is the most you can say about the diagonalization of L?

(f) Suppose that $P \in \mathbf{C}_n^n$ satisfies $P\bar{P}^t = I_n$. What is the most you can say about the columns of P?

(g) Let $L \in \mathcal{L}(\mathbf{C}^n)$ be represented with respect to the standard basis S by a matrix A. Suppose there is an invertible matrix $P \in \mathbf{C}_n^n$ such that $P^{-1} = \bar{P}^t$ and $P^{-1}AP = D$ is diagonal. What is the most you can say about L?

(h) If $L : \mathbf{R}_4^6 \to \mathbf{R}_{14}$, give all possible values of $dim(Ker(L))$.

2. (40 pts) Let $V = \mathcal{P}_3(\mathbf{R})$ with the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. Let U be the subspace with basis $\{1, x^2\}$. Find a basis for $U^\perp$, the orthogonal complement of U in V.

3. (40 Pts) Let S be the standard basis of $\mathbf{R}^4$ and let $L : \mathbf{R}^4 \to \mathbf{R}^4$ be given by $L(X) = AX$ where

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = {}_S\mathcal{M}_S(L).$$

Find a basis T, a transition matrix $P = {}_SP_T$, and a diagonal matrix $D = {}_T\mathcal{M}_T(L)$ such that $P^{-1}AP = D$.

4. (30 Pts) Let V be an inner product space and suppose $N \in \mathcal{L}(V)$ is self-adjoint ($N^* = N$) and nilpotent ($N^k = 0$ for some $k \geq 1$). Prove that $N = 0$.

1. (40 points, 5 points each)

(a) S is an independent set of n vectors in an n-dimensional space, so S is an orthonormal basis of V.

(b) $L(S)$ spans the range of L.

(c) $r \leq dim(V) \leq t$.

(d) Cauchy-Schwarz says $\left(\int_0^1 f(x)g(x)dx\right)^2 \leq \left(\int_0^1 f(x)^2dx\right)\left(\int_0^1 g(x)^2dx\right)$

(e) In the real inner product space $\mathbf{R}^n$ the adjoint of L is represented with respect to the o.n. standard basis by the transpose $A^t = A$ since A is symmetric, so $L = L^*$ is self-adjoint. The real spectral theorem then says that L is orthogonally diagonalizable.

(f) The columns of P form an orthonormal basis in $\mathbf{C}^n$ with respect to the standard inner product.

(g) There is an orthonormal basis of $\mathbf{C}^n$ (the columns of P) with respect to which L is represented by the diagonal matrix D, so L is orthogonally diagonalizable, its eigenvalues are the diagonal entries of D, and L is normal by the complex spectral theorem.

(h) $10 \leq dim(Ker(L)) \leq 24$.

2. (40 points) $U^\perp = \{f \in V \mid \int_0^1 f(x)1dx = 0 \text{ and } \int_0^1 f(x)x^2dx = 0\}$. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Then the conditions for $f(x) \in U^\perp$ are that

$$0 = \int_0^1 f(x)dx = \left(a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4\right)\Big|_{x=0}^{x=1} = a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4}$$

and

$$0 = \int_0^1 f(x)x^2dx = \left(\frac{a_0}{3}x^3 + \frac{a_1}{4}x^4 + \frac{a_2}{5}x^5 + \frac{a_3}{6}x^6\right)\Big|_{x=0}^{x=1} = \frac{a_0}{3} + \frac{a_1}{4} + \frac{a_2}{5} + \frac{a_3}{6}$$

Solve these two linear equations by reducing

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & \frac{-1}{5} & \frac{-1}{4} & 0 \\ 0 & 1 & \frac{16}{15} & 1 & 0 \end{array}\right] \text{ so } \begin{array}{c} a_0 = \frac{1}{5}r + \frac{1}{4}s \\ a_1 = \frac{-16}{15}r - s \\ a_2 = r \in \mathbf{R} \\ a_3 = s \in \mathbf{R} \end{array}.$$

so $U^\perp = \{\frac{1}{5}r + \frac{1}{4}s + (\frac{-16}{15}r - s)x + rx^2 + sx^3 \in \mathcal{P}_3(\mathbf{R}) \mid r, s \in \mathbf{R}\}$ which has basis $\{\frac{1}{4} - x + x^3,\ \frac{1}{5} - \frac{16}{15}x + x^2\}$.

3. (40 Pts) First find eigenvalues by finding the λ such that $[A - \lambda I_4 | 0_4]$ has non-trivial solutions.

$$\left[\begin{array}{cccc|c} -\lambda & 1 & 1 & 1 & 0 \\ 1 & -\lambda & 1 & 1 & 0 \\ 1 & 1 & -\lambda & 1 & 0 \\ 1 & 1 & 1 & -\lambda & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 0 & 1-\lambda^2 & \lambda+1 & \lambda+1 & 0 \\ 1 & -\lambda & 1 & 1 & 0 \\ 0 & \lambda+1 & -\lambda-1 & 0 & 0 \\ 0 & \lambda+1 & 0 & -\lambda-1 & 0 \end{array}\right]$$

So $\lambda = -1$ is clearly one eigenvalue, making three of the rows all zeros, and the remaining row becomes $[1\ 1\ 1\ 1 \mid 0]$ whose solutions are $x_1 = -r - s - t$, $x_2 = r$, $x_3 = s$, $x_4 = t$, giving a three dimensional eigenspace L_{-1} with basis in $\mathbf{R}^4$ $\{[-1\ 1\ 0\ 0]^t, [-1\ 0\ 1\ 0]^t, [-1\ 0\ 0\ 1]^t\}$. If $\lambda \neq -1$ we can further reduce the matrix above to

$$\left[\begin{array}{cccc|c} 0 & 1-\lambda & 1 & 1 & 0 \\ 1 & -\lambda & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right] \text{ and then to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \lambda-3 & 0 \end{array}\right]$$

So the only other eigenvalue is $\lambda = 3$ making the last row all zeros, and giving the solutions $x_1 = r$, $x_2 = r$, $x_3 = r$, $x_4 = r$, so L_3 is one-dimensional with basis in $\mathbf{R}^4$ $\{[1\ 1\ 1\ 1]^t\}$. This shows L is diagonalizable since we got the basis of eigenvectors

$$T = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ so that } P = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

4. (30 Pts) Let V be an inner product space and suppose $N \in \mathcal{L}(V)$ is self-adjoint ($N^* = N$) and nilpotent ($N^k = 0$ for some $k \geq 1$). Prove that $N = 0$.

Proof: Let k be the smallest positive integer for which $N^k = 0$. Suppose that $k \geq 2$. Then $N^{k-1}v \neq \theta$ for some $\theta \neq v \in V$. So $0 \neq \langle N^{k-1}v, N^{k-1}v \rangle = \langle N^{k-2}v, N^k v \rangle = \langle N^{k-2}v, \theta \rangle = 0$ because $k - 1 \geq 1$ so there is at least one factor of N on the left side which can be moved to the right side as $N^* = N$. But this contradiction shows that $k = 1$ so $N = 0$.