Math 507 Linear Algebra and Matrix Theory Fall 2008 Exam 1 Feingold

(1) (30 Points) Let

$$W = \{p(t) = \sum_{i=0}^{3} a_i t^i \in Poly_3(\mathbf{R}) \mid p(0)+p(1) = 0 \text{ and } 2p(0)+3p(1)+p(2) = 0\}.$$

(a) Prove that $W$ is a subspace of $Poly_3(\mathbf{R})$.
(b) Find a basis for $W$ and find $\dim(W)$.

(2) (35 Points) Answer (with brief justification) each question separately. Assume that $U$, $V$ and $W$ are vector spaces.

(a) If a set of $n$ vectors spans $V$, what is the relationship between $n$ and $\dim(V)$?
(b) If a set of $m$ vectors is independent in $W$, what is the relationship between $m$ and $\dim(W)$?
(c) If $T = \{w_1, w_2, \ldots, w_m\}$ is a subset of $W$, and $w \in W$ is not in the span of $T$, what is the most you can say about $T \cup \{w\} = \{w_1, \ldots, w_m, w\}$?
(d) If $S = \{v_1, \ldots, v_n\}$ is an independent set in $V$ and $v \in V$ is in the span of $S$, then what is the most you can you say about $S \cup \{v\} = \{v_1, \ldots, v_n, v\}$?
(e) If $S = \{v_1, v_2, \ldots, v_m\}$ is an independent set in vector space $V$, and $T = \{w_1, w_2, \ldots, w_n\}$ spans the same vector space $V$, what is the relationship between $m$ and $n$ and $\dim(V)$?
(f) Let $A \in \mathbf{F}^m_n$ with $m < n$. What is the most you can say about the dimension of the subspace $W = \{X \in \mathbf{F}^n \mid AX = 0\}$?
(g) If $U$ and $W$ are subspaces of $V$ with $\dim(U) = 5$, $\dim(W) = 7$ and $\dim(V) = 10$, what is the most you can say about $\dim(U \cap W)$? Hint: $(U + W) \leq V$ and $(U \cap W) \leq W$.

(3) (35 Points) Define the following two subspaces of $\mathbf{R}^2_2$:

$$U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{R}^2_2 \mid a + b + c + d = 0 \right\},$$

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{R}^2_2 \mid a + 2b + 3c - d = 0 \text{ and } 2a + 3b - c + 5d = 0 \right\}.$$

(a) Find a basis for $U$.
(b) Find a basis for $W$.
(c) Find a basis for the intersection $U \cap W$.
(d) Knowing the dimensions of $U$, $W$ and $U \cap W$, what can you say about the dimension of the sum $U + W$? What does that say about the subspace $U + W$? Is it a direct sum? Explain your answer.

1. (30 Points) Let

$$W = \{p(t) = \sum_{i=0}^{3} a_i t^i \in Poly_3(\mathbf{R}) \mid p(0)+p(1) = 0 \text{ and } 2p(0)+3p(1)+p(2) = 0\}.$$

(a) (15 Points) Prove that $W$ is a subspace of $Poly_3(\mathbf{R})$.

**SOLUTION:** One way to show $W$ is a subspace is to show the zero polynomial is in $W$ and that $W$ is closed under addition and scalar multiplication. The zero polynomial $\theta(t) = 0 + 0t + 0t^2 + 0t^3$ is in $W$ because $\theta(0) = 0$, $\theta(1) = 0$ and $\theta(2) = 0$, so $\theta(0) + \theta(1) = 0$ and $2\theta(0) + 3\theta(1) + \theta(2) = 0$. If $p(t), q(t) \in W$, then $(p+q)(t) \in W$ because $(p+q)(t) = p(t) + q(t)$ so $(p+q)(0) + (p+q)(1) = (p(0) + q(0)) + (p(1) + q(1)) = (p(0) + p(1)) + (q(0) + q(1)) = 0 + 0 = 0$ and $2(p+q)(0)+3(p+q)(1)+(p+q)(2) = 2(p(0)+q(0))+3(p(1)+q(1))+(p(2)+q(2)) = (2p(0)+3p(1)+p(2))+(2q(0)+3q(1)+q(2)) = 0+0 = 0$. If $p(t) \in W$ and $b \in \mathbf{R}$ then $(bp)(t) \in W$ since $(bp)(0) + (bp)(1) = b(p(0)) + b(p(1)) = b(p(0) + p(1)) = b(0) = 0$ and $2(bp)(0) + 3(bp)(1) + (bp)(2) = b(2p(0) + 3p(1) + p(2)) = b(0) = 0$.

(b) (15 Points) Find a basis for $W$ and find $\dim(W)$.

**SOLUTION:** There is a way of answering both parts (a) and (b) at the same time, as follows. The condition $p(0) + p(1) = 0$ means $2a_0 + a_1 + a_2 + a_3 = 0$, and the condition $2p(0) + 3p(1) + p(2) = 0$ means $6a_0 + 5a_1 + 7a_2 + 11a_3 = 0$. To solve these two equations in four variables, row reduce

$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 0 \\ 6 & 5 & 7 & 11 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & 2 & 4 & 0 \end{array}\right] \text{ so } \begin{array}{c} a_0 = \frac{1}{2}r + \frac{3}{2}s \\ a_1 = -2r - 4s \\ a_2 = r \in \mathbf{R} \\ a_3 = s \in \mathbf{R} \end{array}$$

Then $W = \{(\frac{1}{2}r + \frac{3}{2}s) + (-2r - 4s)t + rt^2 + st^3 \in Poly_3(\mathbf{R}) \mid r, s \in \mathbf{R}\}$ is the span of the set of two polynomials $\{\frac{1}{2} - 2t + t^2, \frac{3}{2} - 4t + t^3\}$, making $W$ a subspace, and this set is independent since the only way to get $\theta(t) = (\frac{1}{2}r + \frac{3}{2}s) + (-2r - 4s)t + rt^2 + st^3$ is when $r = 0$ and $s = 0$. So that set is a basis of $W$. Since a basis for $W$ has two vectors, we get $\dim(W) = 2$.

2. (35 points, 5 points each)
   (a) $\dim(V) \leq n$ (A spanning set can be cut down to a basis.)

   (b) $m \leq \dim(W)$ (An independent set can be extended to a basis.)

   (c) The span of $T \cup \{w\}$ is strictly larger than the span of $T$. (It contains all of the span of $T$ as well as $w$.) $\dim(\langle T \cup \{w\}\rangle) = \dim(\langle T\rangle) + 1$. If $T$ were independent, then $T \cup \{w\}$ would also be independent, since no vector would be a combination of previous vectors on the list. If $T$ were dependent, then $T \cup \{w\}$ would also be dependent, since any set containing a dependent set is also dependent.

   (d) $S \cup \{v\}$ is dependent: the last vector is a linear combo of previous vectors. Also, $\langle S \cup \{v\}\rangle = \langle S\rangle$ since $v \in \langle S\rangle$.

   (e) $m \leq \dim(V) \leq n$. (combining results of parts (a) and (b).)

   (f) Since $m < n$ the linear system has more variables than equations, so it must have nontrivial solutions. When the augmented matrix is row reduced, it can have at most $m$ leading ones, so there are at least $n - m$ columns without leading ones, giving at least $n - m$ free variables in the solution space. Therefore, $\dim(W) \geq n - m$.

   (g) If $U$ and $W$ are of $V$ with $\dim(U) = 5$, $\dim(W) = 7$ and $\dim(V) = 10$, what is the most you can say about $\dim(U \cap W)$? Hint: $(U + W) \leq V$ and $(U \cap W) \leq W$. From the formula $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$ and the fact that $\dim(U + W) \leq \dim(V)$, we can say that $5 + 7 - \dim(U \cap W) \leq 10$, so $2 = 5 + 7 - 10 \leq \dim(U \cap W)$ is a lower bound. Also, since $(U \cap W) \leq W$, we have the upper bound $\dim(U \cap W) \leq 5$, so the most we can say is $2 \leq \dim(U \cap W) \leq 5$.

3. (35 points) (a) (5 Points) Since $U$ is given by only one equation, and the matrix $[1 \quad 1 \quad 1 \quad 1 \quad | \quad 0]$ is already in RREF, the interpretation is $a = -b - c - d$, with $b, c, d \in \mathbf{R}$ free variables. Then

$$U = \left\{ \begin{bmatrix} -b-c-d & b \\ c & d \end{bmatrix} = b\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; b, c, d \in \mathbf{R} \right\}$$

$$\text{showing that a basis for } U \text{ is } \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(b) (10 Points) Since $W$ is given by two equations, solve it by reducing

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 0 \\ 2 & 3 & -1 & 5 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & -11 & 13 & 0 \\ 0 & 1 & 7 & -7 & 0 \end{array}\right] \text{ so } \begin{array}{c} a = 11c - 13d \\ b = -7c + 7d \\ c \in \mathbf{R} \\ d \in \mathbf{R} \end{array}$$

so that

$$W = \left\{ \begin{bmatrix} 11c - 13d & -7c + 7d \\ c & d \end{bmatrix} = c\begin{bmatrix} 11 & -7 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} -13 & 7 \\ 0 & 1 \end{bmatrix} \;\middle|\; c, d \in \mathbf{R} \right\}$$

$$\text{showing that a basis for } W \text{ is } \left\{ \begin{bmatrix} 11 & -7 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -13 & 7 \\ 0 & 1 \end{bmatrix} \right\}.$$

(c) (10 Points) The intersection $U \cap W$ is given by the three equations, one giving conditions to be in $U$ and the other two to be in $W$. Since we already row reduced the two equations for $W$, we can just add on the one more row for $U$ and reduce

$$\left[\begin{array}{cccc|c} 1 & 0 & -11 & 13 & 0 \\ 0 & 1 & 7 & -7 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \text{ so } \begin{array}{c} a = -2d \\ b = 0 \\ c = d \\ d \in \mathbf{R} \end{array}$$

$$\text{so that } U \cap W = \left\{ \begin{bmatrix} -2d & 0 \\ d & d \end{bmatrix} = d\begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \;\middle|\; d \in \mathbf{R} \right\}$$

$$\text{showing that a basis for } U \cap W \text{ is } \left\{ \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

(d) (10 Points) Since $\dim(U) = 3$, $\dim(W) = 2$ and $\dim(U \cap W) = 1$, from the formula $\dim(U+W) = \dim(U)+\dim(W)-\dim(U\cap W) = 3+2-1 = 4 = \dim(\mathbf{R}^2_2)$ we get that $U + W = \mathbf{R}^2_2$. The sum $U + W$ is not direct because the intersection $U \cap W$ is non-trivial.