

(1) (20 Pts) Let $L : \mathbf{R}_2^2 \rightarrow \mathbf{R}^2$ be given by

$$L \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a - b + 2c - 3d \\ -a + 3b + 5c + d \end{bmatrix}.$$

Let S be the standard basis of \mathbf{R}_2^2 and let T be the standard basis of \mathbf{R}^2 . Let other ordered bases be

$$S' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\} \text{ and } T' = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$$

- (a) (5 pts) Find the matrix $[L]_S^T$ representing L from S to T .
- (b) (5 pts) Find the matrix $[L]_{S'}^{T'}$ representing L from S' to T' **without using transition matrices**. (Do it directly.)
- (c) (10 pts) Find the transition matrices ${}_S P_{S'}$ and ${}_T Q_{T'}$ and show that $[L]_{S'}^{T'} = ({}_T Q_{T'})^{-1} [L]_S^T ({}_S P_{S'})$.
- (2) (30 Points) Answer (with brief justification) each question **separately**.
- (a) Let $S = \{[1 \ 2 \ 3], [1 \ 1 \ 1], [0 \ 1 \ 1]\}$ be a basis of \mathbf{R}_3 and let $v = [3 \ 8 \ 7] \in \mathbf{R}_3$. Find the coordinate vector $[v]_S$.
- (b) If $L : V \rightarrow W$ where $\dim(V) = n > m = \dim(W)$ then what are all the possibilities for $\dim(\text{Ker}(L))$?
- (c) If $L : V \rightarrow W$ where $\dim(V) = n < m = \dim(W)$ then what are all the possibilities for $\dim(\text{Range}(L))$?
- (d) If $S = \{v_1, \dots, v_n\}$ is a **basis** of V and $L : V \rightarrow W$, what is the **most** you can say about $L(S) = \{L(v_1), \dots, L(v_n)\}$?
- (e) What is $\dim(\mathcal{L}(\mathbf{F}_2^4, \mathbf{F}_3^5))$? What isomorphism justifies this?
- (f) Suppose $L : V \rightarrow W$ with $\dim(V) = \dim(W)$ finite, and L is injective. What else can you say must be true about L ? Give as many implied properties of L as you can.

(3) (30 Points) Let $L : \mathbf{R}_2^2 \rightarrow \mathbf{R}^4$ be the linear transformation defined by

$$L \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (a + b - 2c + 2d) \\ (a + 2b + d) \\ (-2a + 3b + 14c - 9d) \\ (-a + b + 6c - 4d) \end{bmatrix}$$

- (a) (8 points) Find the set of all vectors in $\text{Ker}(L)$.
 - (b) (4 points) Find a basis for $\text{Ker}(L)$.
 - (c) (1 points) Find $\dim(\text{Ker}(L))$.
 - (d) (2 points) Is L one-to one? **Explain why!**
 - (e) (8 points) Find a basis for $\text{Range}(L)$.
 - (f) (2 points) Find $\dim(\text{Range}(L))$.
 - (g) (2 points) Is L onto? **Explain why!**
 - (h) (3 points) Is L invertible? **Explain why!**
- (4) (20 pts) Let $L : \mathbf{R}_2^2 \rightarrow \mathbf{R}^3$ and $K : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ be the linear transformations

$$L \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a - b + c - d \\ 2a + b + d \\ -a + 2c + 3d \end{bmatrix} \quad \text{and} \quad K \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + z \\ x - y - z \end{bmatrix}.$$

Let R , S and T be the standard bases of \mathbf{R}_2^2 , \mathbf{R}^3 and \mathbf{R}^2 , respectively.

- (a) (5 pts) Find the matrix $[L]_R^S$ representing L from R to S .
- (b) (5 pts) Find the matrix $[K]_S^T$ representing K from S to T .
- (c) (5 pts) Use composition of functions to find the formula for $(K \circ L) \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ from the formulas for K and L .
- (d) (5 pts) Find matrix $[K \circ L]_R^T$ representing $K \circ L$ from R to T . What should be the relationship between the matrices $[L]_R^S$, $[K]_S^T$ and $[K \circ L]_R^T$? Check that relationship by direct calculation.

(1) (20 Pts)

(a) (5 Pts) $[L]_S^T = \begin{bmatrix} 1 & -1 & 2 & -3 \\ -1 & 3 & 5 & 1 \end{bmatrix}$ is easy to get since S and T are standard.

$$L \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad L \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad L \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad L \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

$$[T \mid L(S)] = \left[\begin{array}{cc|cccc} 1 & 0 & 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 3 & 5 & 1 \\ \hline & T & & & L(S) & \end{array} \right] \text{ is in RREF, so the right side is } [L]_S^T.$$

(b) (5 Pts)

$$L \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \quad L \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}, \quad L \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}, \quad L \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}.$$

$$\text{Row reduce } \left[\begin{array}{cc|cccc} 3 & 5 & -2 & 1 & -1 & -3 \\ 1 & 2 & 0 & 8 & 6 & 3 \\ \hline & T' & & & L(S') & \end{array} \right] \text{ to } \left[\begin{array}{cc|cccc} 1 & 0 & -4 & -38 & -32 & -21 \\ 0 & 1 & 2 & 23 & 19 & 12 \\ \hline & I_2 & & & [L]_{S'}^{T'} & \end{array} \right]$$

(c) (10 Pts) ${}_S P_{S'} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ and ${}_T Q_{T'} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ since S and T are the

standard bases.

To get ${}_{T'} Q_T = ({}_T Q_{T'})^{-1}$, reduce

$$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ \hline & T' & & T \end{array} \right] \text{ to } \left[\begin{array}{cc|cc} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & 3 \\ \hline & I_2 & & {}_{T'} Q_T \end{array} \right]$$

$$({}_T Q_{T'})^{-1} [L]_S^T ({}_S P_{S'}) = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & -3 \\ -1 & 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 7 & -17 & -21 & -11 \\ -4 & 10 & 13 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -38 & -32 & -21 \\ 2 & 23 & 19 & 12 \end{bmatrix} = [L]_{S'}^{T'}$$

checks.

(2) (30 Points) Answer (with brief justification) each question separately.

$$(a) [v]_S = \begin{bmatrix} -1 \\ 4 \\ 6 \end{bmatrix} \text{ since } \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 2 & 1 & 1 & 8 \\ 3 & 1 & 1 & 7 \end{array} \right] \text{ reduces to } \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 6 \end{array} \right]$$

$$\text{Check that: } (-1)[1 \ 2 \ 3] + 4[1 \ 1 \ 1] + 6[0 \ 1 \ 1] = [3 \ 8 \ 7].$$

(b) $L : V \rightarrow W$ where $\dim(V) = n > m = \dim(W)$. The dimension formula says

$$n = \dim(V) = \dim(\text{Ker}(L)) + \dim(\text{Range}(L))$$

and $\dim(\text{Range}(L)) \leq \dim(W) = m$, so

$$\dim(\text{Ker}(L)) = n - \dim(\text{Range}(L)) \geq n - m.$$

Also, $\dim(\text{Ker}(L)) \leq n$, so

$$n - m \leq \dim(\text{Ker}(L)) \leq n.$$

(c) $L : V \rightarrow W$ where $\dim(V) = n < m = \dim(W)$. Then from the dimension formula

$$\dim(\text{Range}(L)) = n - \dim(\text{Ker}(L)) \leq n$$

since $\dim(\text{Ker}(L)) \geq 0$. All values between the extreme cases, $\dim(\text{Ker}(L)) = n$ and $\dim(\text{Ker}(L)) = 0$, can occur, so

$$0 \leq \dim(\text{Range}(L)) \leq n.$$

(d) $S = \{v_1, \dots, v_n\}$ is a **basis** of V and $L : V \rightarrow W$, then $\text{Range}(L) = \left\{ \sum_{i=1}^n a_i L(v_i) \mid a_i \in \mathbf{F} \right\}$ is the span of $L(S)$. If L is injective then $L(S)$ is independent, so it is a basis of $\text{Range}(L)$, but if L is not injective then $L(S)$ is dependent. In general, the most we can say is that $L(S)$ spans $\text{Range}(L)$.

(e) $\dim(\mathcal{L}(\mathbf{F}_2^4, \mathbf{F}_3^5)) = (8)(15) = 120$. This is justified by the isomorphism $\mathcal{L}(\mathbf{F}_2^4, \mathbf{F}_3^5) \cong \mathbf{F}_8^{15}$ sending a linear transformation $L : \mathbf{F}_2^4 \rightarrow \mathbf{F}_3^5$ to the 15×8 matrix $[L]_S^T$ representing L from S to T .

(f) Since $L : V \rightarrow W$ with $\dim(V) = \dim(W)$ finite and L is injective, $\dim(\text{Ker}(L)) = 0$, so the dimension formula $\dim(V) = \dim(\text{Ker}(L)) + \dim(\text{Range}(L))$ says $\dim(V) = \dim(\text{Range}(L))$ so $\dim(W) = \dim(\text{Range}(L))$ says L is surjective. Therefore, L is bijective, invertible, and an isomorphism.

(3) (30 points) To find $\text{Ker}(L)$, row reduce

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 2 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ -2 & 3 & 14 & -9 & 0 \\ -1 & 1 & 6 & -4 & 0 \end{array} \right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & -4 & 3 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \text{ so } \begin{array}{l} a = 4r - 3s \\ b = -2r + s \\ c = r \in \mathbf{R} \\ d = s \in \mathbf{R} \end{array}.$$

(a) (8 points) $\text{Ker}(L) = \left\{ \begin{bmatrix} 4r - 3s & -2r + s \\ r & s \end{bmatrix} \in \mathbf{R}_2^2 \mid r, s \in \mathbf{R} \right\}.$

(b) (4 points) $\left\{ \begin{bmatrix} 4 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis of $\text{Ker}(L)$.

(c) (1 points) $\dim(\text{Ker}(L)) = 2$.

(d) (2 points) L is not one-to-one since $\text{Ker}(L)$ is nontrivial.

(e) (8 points) $\text{Range}(L) =$

$$\left\{ \begin{bmatrix} (a + b - 2c + 2d) \\ (a + 2b + d) \\ (-2a + 3b + 14c - 9d) \\ (-a + b + 6c - 4d) \end{bmatrix} \in \mathbf{R}^4 \mid a, b, c, d \in \mathbf{R} \right\}$$

is spanned by the set

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 14 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -9 \\ -4 \end{bmatrix} \right\}.$$

The two free variables in $\text{Ker}(L)$ mean the last two matrices are redundant vectors, so a basis for $\text{Range}(L)$ is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

Using the row space method, another answer is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -7 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \\ 2 \end{bmatrix} \right\}.$$

(f) (2 points) $\dim(\text{Range}(L)) = 2$.

(g) (2 points) L is not onto, $\text{Range}(L) \neq \mathbf{R}_4$.

(h) (3 points) L is not invertible since it is not bijective.

(4) (20 Pts)

(a) (5 Pts)
$$\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 2 & 3 \end{array} \right]$$
 is reduced so $[L]_R^S = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & 1 & 0 & 1 \\ -1 & 0 & 2 & 3 \end{bmatrix}$

(b) (5 Pts)
$$\left[\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 & -1 \end{array} \right]$$
 is reduced so $[K]_S^T = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

(c) (5 Pts)
$$(K \circ L) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = K \begin{bmatrix} a - b + c - d \\ 2a + b + d \\ -a + 2c + 3d \end{bmatrix}$$

$$= \begin{bmatrix} (a - b + c - d) + 2(2a + b + d) + (-a + 2c + 3d) \\ (a - b + c - d) - (2a + b + d) - (-a + 2c + 3d) \end{bmatrix} = \begin{bmatrix} 4a + b + 3c + 4d \\ -2b - c - 5d \end{bmatrix}$$

(d) (5 Pts)
$$\left[\begin{array}{ccc|cccc} 1 & 0 & 4 & 1 & 3 & 4 \\ 0 & 1 & 0 & -2 & -1 & -5 \end{array} \right]$$
 is reduced so $[K \circ L]_R^T = \begin{bmatrix} 4 & 1 & 3 & 4 \\ 0 & -2 & -1 & -5 \end{bmatrix}$.

The relationship should be $[K]_S^T [L]_R^S = [K \circ L]_R^T$.

Check:
$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & 1 & 0 & 1 \\ -1 & 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 3 & 4 \\ 0 & -2 & -1 & -5 \end{bmatrix}.$$