Math 507 Linear Algebra and Matrix Theory Fall 2008 Exam 3 Feingold

1. (30 Pts) Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

   (a) Find the **characteristic polynomial** of A, $p_A(t) = \det(tI_4 - A)$, and find the **eigenvalues** of A, and their **algebraic multiplicities**.

   (b) Can A can be diagonalized? **If not, give reasons why**. If it can, **find an invertible matrix** $P \in \mathbf{R}^4_4$ and a **diagonal matrix** D such that $D = P^{-1}AP$.

   (c) Find the minimal polynomial $m_A(t)$ of A and verify that $m_A(A) = 0^n_n$.

2. (50 points, 10 points each) Answer each of the following questions separately.

   (a) Let $L : \mathbf{C}^n \to \mathbf{C}^n$ and suppose that L satisfies the polynomial $t^m - 1$ for some integer $m > 1$. What are the possible eigenvalues for L?

   (b) Suppose $A \in \mathbf{C}^7_7$ has characteristic and minimal polynomials

   $$p_A(t) = (t-3)^4(t-5)^3 \qquad \text{and} \qquad m_A(t) = (t-3)^2(t-5)^2.$$

   Find all the possible Jordan Canonical Form matrices that could be similar to A, but not similar to each other. In each case, give the geometric multiplicities for each eigenvalue, g_3 and g_5.

   (c) Suppose $L : V \to V$ has characteristic and minimal polynomials

   $$p_L(t) = \prod_{i=1}^{r}(t-\lambda_i)^{k_i} \qquad \text{and} \qquad m_L(t) = \prod_{i=1}^{r}(t-\lambda_i)^{m_i}$$

   where $\lambda_1, \cdots, \lambda_r$ are the distinct eigenvalues of L. Let $g_i = \dim(L_{\lambda_i})$, $1 \le i \le r$, be the geometric multiplicities of L. (1) In general, what is the most you can say about the relation between g_i and k_i? (2) When L is **diagonalizable** what can you say about the numbers g_i, k_i and m_i?

   (d) Suppose $A \in \mathbf{R}^8_8$ has characteristic and minimal polynomials

   $$p_A(t) = (t^2+3)^2(t^2+2t+3)^2 \qquad \text{and} \qquad m_A(t) = (t^2+3)^2(t^2+2t+3).$$

   Find all the possible Rational Canonical Form matrices that could be similar to A, but not similar to each other.

   (e) Let $A, B \in \mathbf{F}^n_n$ be invertible matrices satisfying the relation $ABA^{-1} = B^{-1}$. What is the most you can say about $\det(A)$? What is the most you can say about $\det(B)$?

3. (20 points) Let $L : \mathbf{R}_2^2 \to \mathbf{R}_2^2$ be $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} (a+b) & (a-c) \\ (b+d) & (a+c) \end{bmatrix}$,

let $v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and let $T = \{v, L(v), L^2(v), L^3(v)\}$.

(a) Find T and show it is independent, so it is a basis of $\mathbf{R}_2^2$.

(b) Find $L^4(v)$ and express it as a linear combination of the vectors in T.

(c) Using the answers to parts (a) and (b) find the companion matrix $C = [L]_T^T$ that represents L from T to T and find its characteristic polynomial.

1. (30 Points)

(a) (10 Points) The characteristic polynomial is $p_A(t) = \det(tI_4 - A) =$

$$\det\begin{bmatrix} t-1 & -2 & -3 & -4 \\ -1 & t-2 & -3 & -4 \\ -1 & -2 & t-3 & -4 \\ -1 & -2 & -3 & t-4 \end{bmatrix} = \det\begin{bmatrix} t & -t & 0 & 0 \\ 0 & t & -t & 0 \\ 0 & 0 & t & -t \\ -1 & -2 & -3 & t-4 \end{bmatrix}$$

$$= t^3 \det\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & -2 & -3 & t-4 \end{bmatrix} = t^3 \det\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & t-10 \end{bmatrix} = t^3\ (t-10).$$

So the eigenvalues are $\lambda_1 = 0$ with algebraic multiplicity $k_1 = 3$ and $\lambda_2 = 10$ with algebraic multiplicity $k_2 = 1$.

(b) (15 Points) Check the $\lambda_1 = 0$ eigenspace first since the algebraic multiplicity $k_1 = 3$. Solve the homogeneous linear system whose coefficient matrix is obtained by plugging in $t = 0$ to $tI_4 - A$. Row reduce

$$\left[\begin{array}{cccc|c} -1 & -2 & -3 & -4 & 0 \\ -1 & -2 & -3 & -4 & 0 \\ -1 & -2 & -3 & -4 & 0 \\ -1 & -2 & -3 & -4 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = -2r - 3s - 4t \\ x_2 = r \in \mathbf{R} \\ x_3 = s \in \mathbf{R} \\ x_4 = t \in \mathbf{R} \end{array}, \text{ then}$$

$$A_{\lambda_1} = \left\{ \begin{bmatrix} -2r - 3s - 4t \\ r \\ s \\ t \end{bmatrix} \in \mathbf{R}^4 \;\middle|\; r, s, t \in \mathbf{R} \right\} \text{ has basis } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

with three vectors. Since there will be one more independent eigenvector from the other eigenvalue, we will have the necessary three eigenvectors to form a basis for $\mathbf{R}^4$, so this A is diagonalizable.

Now find the $\lambda_2 = 10$ eigenspace. Solve the homogeneous linear system whose coefficient matrix is obtained by plugging in $t = 10$ to $tI_4 - A$. Row reduce

$$\left[\begin{array}{cccc|c} 9 & -2 & -3 & -4 & 0 \\ -1 & 8 & -3 & -4 & 0 \\ -1 & -2 & 7 & -4 & 0 \\ -1 & -2 & -3 & 6 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{r} x_1 = r \\ x_2 = r \\ x_3 = r \\ x_4 = r \in \mathbf{R} \end{array}, \text{ then}$$

$$A_{\lambda_2} = \left\{ \begin{bmatrix} r \\ r \\ r \\ r \end{bmatrix} \in \mathbf{R}^4 \;\middle|\; r \in \mathbf{R} \right\} \qquad \text{has basis} \qquad \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Therefore, $T = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, $D = {}_TD_T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$

and $P = {}_SP_T = \begin{bmatrix} -2 & -3 & -4 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $P^{-1} = {}_TP_S = \frac{1}{10} \begin{bmatrix} -1 & 8 & -3 & -4 \\ -1 & -2 & 7 & -4 \\ -1 & -2 & -3 & 6 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

We check:

$$P^{-1}AP = \frac{1}{10} \begin{bmatrix} -1 & 8 & -3 & -4 \\ -1 & -2 & 7 & -4 \\ -1 & -2 & -3 & 6 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & -3 & -4 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & -3 & -4 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} = D.$$

(c) (5 Points) Since the characteristic polynomial is $p_A(t) = t^3(t - 10)$ the only possibilities for $m_A(t)$ are the divisors sharing the same irreducible factors, namely, $t(t - 10)$, $t^2(t - 10)$ and $t^3(t - 10)$. Checking the first one, we find

$$A(A - 10I) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -9 & 2 & 3 & 4 \\ 1 & -8 & 3 & 4 \\ 1 & 2 & -7 & 4 \\ 1 & 2 & 3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $t(t - 10)$ is the minimal polynomial of that matrix A.

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2. (50 points, 10 points each) Answer each of the following questions separately.

(a) Let $L : \mathbf{C}^n \to \mathbf{C}^n$ and suppose that L satisfies the polynomial $t^m - 1$ for some integer $m > 1$. What are the possible eigenvalues for L?

Solution: The minimal polynomial $m_A(t)$ must divide the polynomial $t^m - 1$ satisfied by L, and so the roots of $m_A(t)$, which are the eigenvalues of L, must be roots of $t^m - 1$. The roots of $t^m - 1$ are the complex numbers such that $t^m = 1$, and these consist of the m distinct powers of $e^{2\pi i/m} = \cos(2\pi/m) + i \sin(2\pi/m)$.

(b) Suppose $A \in \mathbf{C}^7_7$ has characteristic and minimal polynomials

$$p_A(t) = (t-3)^4(t-5)^3 \qquad \text{and} \qquad m_A(t) = (t-3)^2(t-5)^2.$$

Find all the possible Jordan Canonical Form matrices that could be similar to A, but not similar to each other. In each case, give the geometric multiplicities for each eigenvalue, g_3 and g_5.

Solution: There are only two eigenvalues, $\lambda_1 = 3$ and $\lambda_2 = 5$, with $k_1 = 4$ and $k_2 = 3$, so there are two kinds of Jordan blocks. The largest 3-Jordan block $J_{3,2}$ is size 2×2 since $m_1 = 2$, and there are a total of $k_1 = 4$ diagonal entries with $\lambda_1 = 3$. The largest 5-Jordan block $J_{5,2}$ is size 2×2 since $m_2 = 2$, and there are a total of $k_2 = 3$ diagonal entries with $\lambda_2 = 5$. So there are either two $J_{3,2}$ blocks or there is one $J_{3,2}$ and two 1×1 blocks $J_{3,1}$. Also, there can only be one $J_{5,2}$ block and one $J_{5,1}$ block. The two options look like:

$$diag[J_{3,2}, J_{3,2}, J_{5,2}, J_{5,1}] \quad \text{or} \quad diag[J_{3,2}, J_{3,1}, J_{3,1}, J_{5,2}, J_{5,1}].$$

In the first case, $g_3 = 2$ and $g_5 = 2$, but in the second case $g_3 = 3$ and $g_5 = 2$.

(c) Suppose $L : V \to V$ has characteristic and minimal polynomials

$$p_L(t) = \prod_{i=1}^{r} (t - \lambda_i)^{k_i} \qquad \text{and} \qquad m_L(t) = \prod_{i=1}^{r} (t - \lambda_i)^{m_i}$$

where $\lambda_1, \cdots, \lambda_r$ are the distinct eigenvalues of L. Let $g_i = \dim(L_{\lambda_i})$, $1 \le i \le r$, be the geometric multiplicities of L. (1) In general, what is the most you can say about the relation between g_i and k_i? (2) When L is **diagonalizable** what can you say about the numbers g_i, k_i and m_i?

Solution: (1) In general you can only say that $1 \le g_i \le k_i$ for each i. (2) When L is diagonalizable you must have $g_i = k_i$ and $m_i = 1$ for each i.

(d) Suppose $A \in \mathbf{R}^8_8$ has characteristic and minimal polynomials

$$p_A(t) = (t^2+3)^2(t^2+2t+3)^2 \qquad \text{and} \qquad m_A(t) = (t^2+3)^2(t^2+2t+3).$$

Find all the possible Rational Canonical Form matrices that could be similar to A, but not similar to each other.

Solution: Since there are two irreducible factors in $p_A(t)$, there are two kinds of blocks in the RCF similar to A. The blocks coming from $(t^2+3)^2$ have a total size of 4, the degree of this factor, and the blocks coming from $(t^2+2t+3)^2$ also have a total size of 4 for the same reason. But because in $m_A(t)$ we have $(t^2+3)^2$, there must be only one 4×4 companion matrix $C = \begin{bmatrix} 0 & 0 & 0 & -9 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ associated with $(t^2+3)^2 = t^4+6t^2+9$. Since in $m_A(t)$ we have (t^2+2t+3), there must be two 2×2 companion matrices $D = \begin{bmatrix} 0 & -3 \\ 1 & -2 \end{bmatrix}$ associated with (t^2+2t+3). So the only possible RCF is $diag[C, D, D]$.

(e) Let $A, B \in \mathbf{F}^n_n$ be invertible matrices satisfying the relation $ABA^{-1} = B^{-1}$. What is the most you can say about $\det(A)$? What is the most you can say about $\det(B)$?

Solution: Since $\det(ABA^{-1}) = \det(B^{-1})$ we get from the multiplicative property of det that

$$\det(B) = \det(A)\det(B)\det(A)^{-1} = \det(A)\det(B)\det(A^{-1}) = \det(B^{-1}) = \det(B)^{-1}$$

so $\det(B)^2 = 1$ which gives $\det(B) = \pm 1$. Since A is invertible, all we can say is that $\det(A) \neq 0$.

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3. (20 points) We have $L : \mathbf{R}_2^2 \to \mathbf{R}_2^2$ by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} (a+b) & (a-c) \\ (b+d) & (a+c) \end{bmatrix}$, and $v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $T = \{v, L(v), L^2(v), L^3(v)\}$.

(a) $T = \left\{ v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, L(v) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, L^2(v) = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, L^3(v) = \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix} \right\}$.
To show it is independent, we row reduce

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 4 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \end{array}$$

(b) $L^4(v) = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$. To express it as a linear combination of the vectors in T we row reduce

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 1 & 1 & 4 & 5 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \text{ so } \begin{array}{c} x_1 = -2 \\ x_2 = 0 \\ x_3 = 1 \\ x_4 = 1 \end{array}$$

This means $L^4(v) = -2v + 0L(v) + 1L^2(v) + 1L^3(v)$.

(c) From the answers to parts (a) and (b), the companion matrix that represents L from T to T is

$$C = [L]_T^T = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and the characteristic polynomial is read off from the right column, or from the dependence relation found in part (b). It is

$$t^4 - t^3 - t^2 + 2.$$