Math 507 Linear Algebra and Matrix Theory Fall 2008 Final Exam Feingold

(1) (30 Points) For the matrix $A = \begin{bmatrix} 3 & -4 & 0 & 0 \\ 1 & 7 & 0 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & -1 & 4 \end{bmatrix}$ do the following.

(a) Find the **characteristic polynomial** $p_A(t)$, the **eigenvalues** of A, and their **algebraic multiplicities**.

(b) Find the **minimal polynomial** $m_A(t)$.

(c) Determine whether or not A is **diagonalizable** and **justify your answer**.

(d) If A can be diagonalized, find the diagonal matrix D and the matrix P such that $D = P^{-1}AP$. If A cannot be diagonalized, just find the **Jordan Canonical Form** matrix J similar to A.

(2) (20 Points) Suppose that a matrix A has **characteristic polynomial** $p_A(t) = (t-2)^6(t-7)^5$ and **minimal polynomial** $m_A(t) = (t-2)^3(t-7)^4$. Find all possible Jordan canonical form matrices J to which A might be similar, and for each one give the geometric multiplicities g_2 and g_7 of the two eigenvalues.

(3) (20 Points, 5 Points Each) Answer each question separately.

(a) Let S and T be bases for a vector space V. What is the relationship between the transition matrices ${}_TP_S$ and ${}_SP_T$?

(b) If $W_1, \cdots, W_m$ are subspaces of V, the subspace $W = W_1 + \cdots + W_m = \{w_1 + \cdots + w_m \in V \mid w_i \in W_i, 1 \leq i \leq m\}$ is called a **direct sum** when each element $w \in W$ has a **unique** expression of the form $w = w_1 + \cdots + w_m$ for $w_i \in W_i$. Under what conditions can you say that the sum W is a direct sum?

(c) Suppose $A, B \in \mathbf{F}_n^m$ are matrices such that $AX = BX$ for any $X \in \mathbf{F}^n$. Prove that $A = B$.

(d) If $L : \mathbf{R}_4^3 \to \mathbf{R}_5^2$, what is the most you can say about $\dim(Ker(L))$?

(4) (25 Points) Let $L : \mathbf{R}_2^2 \to \mathbf{R}^2$ be the linear transformation given by

$$L \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-b \\ c-d \end{bmatrix},$$

let $K = Ker(L)$ and let $U = \mathbf{R}_2^2/K = \{v + K \mid v \in \mathbf{R}_2^2\}$ be the quotient space.

(a) (4 points) Find the set of **all vectors** in $\text{Ker}(L)$.

(b) (4 points) Find a **basis** $\{v_1, \cdots, v_k\}$ for $\text{Ker}(L)$ and find $k = \dim(Ker(L))$.

(c) (2 points) Is L one-to one? **Explain why!**

(d) (2 points) Extend your basis for K to a basis $S = \{v_1, \cdots, v_k, \cdots, v_4\}$ for all of $\mathbf{R}_2^2$.

(e) (2 points) Use your answer to (d) to give a **basis** S' for U and find $\dim(U)$.

(f) (2 points) Let T be the standard basis of $\mathbf{R}^2$. Find $[L]_S^T$, the matrix representing L from S to T.

(g) (6 points) Try to define $\bar{L} : U \to \mathbf{R}^2$ by $\bar{L}(v + K) = L(v)$. Show that $\bar{L}$ is **well-defined** and **injective**.

(h) (3 points) Find $[\bar{L}]_{S'}^T$, the matrix representing $\bar{L}$ from S' to T.

(5) (25 Points) Answer each question separately.

(a) (5 Points) Let $A \in \mathbf{R}^5_{10}$ be a matrix whose rows are **linearly independent**. What is the **most** you can say about the dimension of the span of the **columns** of A?

(b) (10 Points) Find $\det \begin{bmatrix} 4 & 6 & 6 & 8 \\ 3 & -9 & 6 & 3 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

(c) (5 Points) If $\dim(V) = \dim(W)$ is finite and $L : V \to W$ is **onto**, what is the most you can say about L?

(d) (5 Points) If V and W are finite dimensional and $L : V \to W$ is **bijective**, what is the most you can say about the relationship between $\dim(V)$ and $\dim(W)$?

(6) (30 Points) Let V be a vector space with $\dim(V) = n$ and basis $S = \{v_1, \cdots, v_n\}$. For any $v, w \in V$, we have coordinates

$$[v]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \qquad [w]_S = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

in $\mathbf{F}^n$. Let $M = [m_{ij}] \in \mathbf{F}^n_n$ and use it do define a function $B : V \times V \to \mathbf{F}$ by $B(v, w) = [v]^{Tr}_S M [w]_S$ where $[v]^{Tr}_S = [\, a_1 \quad a_2 \quad \cdots \quad a_n \,] \in \mathbf{F}_n$ is the transpose of $[v]_S$. In that equation we understand the left side is the single entry of the 1×1 matrix on the right side.

(a) (5 pts) Prove that the function B is **bilinear**, that is, a linear function of each input separately, that is,

$$B(c_1 v_1 + c_2 v_2, w) = c_1 B(v_1, w) + c_2 B(v_2, w) \quad \text{and}$$
$$B(v, c_1 w_1 + c_2 w_2) = c_1 B(v, w_1) + c_2 B(v, w_2).$$

(b) (5 pts) Prove that $B(v_i, v_j) = m_{ij}$ for all $1 \leq i, j \leq n$.

(c) (5 Points) Write out the formula for $B(v, w)$ as a summation of terms involving the coordinates a_i, b_i and the matrix entries m_{ij} of M.

(d) (5 pts) Prove that $B(v, w) = B(w, v)$ iff $M = M^{Tr}$ is symmetric.

(e) (5 pts) In the special case when $V = \mathbf{R}^n$, S is the standard basis and $M = I_n$ is the identity matrix, what is the formula for $B(v, w)$?

(f) (5 pts) In the special case of (e), prove that $B(v, v) \geq 0$ for any $v \in V$, and $B(v, v) = 0$ implies $v = \theta$.