Math 507 Linear Algebra and Matrix Theory Fall 2008 Final Exam Feingold

(1) (30 Points) For the matrix $A = \begin{bmatrix} 3 & -4 & 0 & 0 \\ 1 & 7 & 0 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & -1 & 4 \end{bmatrix}$ do the following.

(a) Find the **characteristic polynomial** $p_A(t)$, the **eigenvalues** of A, and their **algebraic multiplicities**.

(b) Find the **minimal polynomial** $m_A(t)$.

(c) Determine whether or not A is **diagonalizable** and **justify your answer**.

(d) If A can be diagonalized, find the diagonal matrix D and the matrix P such that $D = P^{-1}AP$. If A cannot be diagonalized, just find the **Jordan Canonical Form** matrix J similar to A.

(2) (20 Points) Suppose that a matrix A has **characteristic polynomial** $p_A(t) = (t-2)^6(t-7)^5$ and **minimal polynomial** $m_A(t) = (t-2)^3(t-7)^4$. Find all possible Jordan canonical form matrices J to which A might be similar, and for each one give the geometric multiplicities g_2 and g_7 of the two eigenvalues.

(3) (20 Points, 5 Points Each) Answer each question separately.

(a) Let S and T be bases for a vector space V. What is the relationship between the transition matrices ${}_TP_S$ and ${}_SP_T$?

(b) If $W_1, \cdots, W_m$ are subspaces of V, the subspace $W = W_1 + \cdots + W_m = \{w_1 + \cdots + w_m \in V \mid w_i \in W_i, 1 \leq i \leq m\}$ is called a **direct sum** when each element $w \in W$ has a **unique** expression of the form $w = w_1 + \cdots + w_m$ for $w_i \in W_i$. Under what conditions can you say that the sum W is a direct sum?

(c) Suppose $A, B \in \mathbf{F}^m_n$ are matrices such that $AX = BX$ for any $X \in \mathbf{F}^n$. Prove that $A = B$.

(d) If $L : \mathbf{R}^3_4 \to \mathbf{R}^2_5$, what is the most you can say about $\dim(Ker(L))$?

(4) (25 Points) Let $L : \mathbf{R}^2_2 \to \mathbf{R}^2$ be the linear transformation given by

$$L \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-b \\ c-d \end{bmatrix},$$

let $K = Ker(L)$ and let $U = \mathbf{R}^2_2/K = \{v + K \mid v \in \mathbf{R}^2_2\}$ be the quotient space.

(a) (4 points) Find the set of **all vectors** in $\text{Ker}(L)$.

(b) (4 points) Find a **basis** $\{v_1, \cdots, v_k\}$ for $\text{Ker}(L)$ and find $k = \dim(Ker(L))$.

(c) (2 points) Is L one-to one? **Explain why!**

(d) (2 points) Extend your basis for K to a basis $S = \{v_1, \cdots, v_k, \cdots, v_4\}$ for all of $\mathbf{R}^2_2$.

(e) (2 points) Use your answer to (d) to give a **basis** S' for U and find $\dim(U)$.

(f) (2 points) Let T be the standard basis of $\mathbf{R}^2$. Find $[L]^T_S$, the matrix representing L from S to T.

(g) (6 points) Try to define $\bar{L} : U \to \mathbf{R}^2$ by $\bar{L}(v + K) = L(v)$. Show that $\bar{L}$ is **well-defined** and **injective**.

(h) (3 points) Find $[\bar{L}]^T_{S'}$, the matrix representing $\bar{L}$ from S' to T.

(5) (25 Points) Answer each question separately.

(a) (5 Points) Let $A \in \mathbf{R}^5_{10}$ be a matrix whose rows are **linearly independent**. What is the **most** you can say about the dimension of the span of the **columns** of A?

(b) (10 Points) Find $\det \begin{bmatrix} 4 & 6 & 6 & 8 \\ 3 & -9 & 6 & 3 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

(c) (5 Points) If $\dim(V) = \dim(W)$ is finite and $L : V \to W$ is **onto**, what is the most you can say about L?

(d) (5 Points) If V and W are finite dimensional and $L : V \to W$ is **bijective**, what is the most you can say about the relationship between $\dim(V)$ and $\dim(W)$?

(6) (30 Points) Let V be a vector space with $\dim(V) = n$ and basis $S = \{v_1, \cdots, v_n\}$. For any $v, w \in V$, we have coordinates

$$[v]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \qquad [w]_S = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

in $\mathbf{F}^n$. Let $M = [m_{ij}] \in \mathbf{F}^n_n$ and use it do define a function $B : V \times V \to \mathbf{F}$ by $B(v, w) = [v]_S^{Tr} M [w]_S$ where $[v]_S^{Tr} = [\, a_1 \quad a_2 \quad \cdots \quad a_n \,] \in \mathbf{F}_n$ is the transpose of $[v]_S$. In that equation we understand the left side is the single entry of the 1×1 matrix on the right side.

(a) (5 pts) Prove that the function B is **bilinear**, that is, a linear function of each input separately, that is,

$$B(c_1 v_1 + c_2 v_2, w) = c_1 B(v_1, w) + c_2 B(v_2, w) \quad \text{and}$$
$$B(v, c_1 w_1 + c_2 w_2) = c_1 B(v, w_1) + c_2 B(v, w_2).$$

(b) (5 pts) Prove that $B(v_i, v_j) = m_{ij}$ for all $1 \le i, j \le n$.

(c) (5 Points) Write out the formula for $B(v, w)$ as a summation of terms involving the coordinates a_i, b_i and the matrix entries m_{ij} of M.

(d) (5 pts) Prove that $B(v, w) = B(w, v)$ iff $M = M^{Tr}$ is symmetric.

(e) (5 pts) In the special case when $V = \mathbf{R}^n$, S is the standard basis and $M = I_n$ is the identity matrix, what is the formula for $B(v, w)$?

(f) (5 pts) In the special case of (e), prove that $B(v, v) \ge 0$ for any $v \in V$, and $B(v, v) = 0$ implies $v = \theta$.

(1) (30 Points) (a) The characteristic polynomial $p_A(t)$ is

$$\det(tI_4 - A) = \det \begin{bmatrix} t-3 & 4 & 0 & 0 \\ -1 & t-7 & 0 & 0 \\ 0 & 0 & t-6 & -1 \\ 0 & 0 & 1 & t-4 \end{bmatrix} = [(t-3)(t-7)+4][(t-6)(t-4)+1]$$

$$= [t^2 - 10t + 25][t^2 - 10t + 25] = (t-5)^4.$$

The only eigenvalue is $\lambda_1 = 5$ with algebraic multiplicity $k_1 = 4$.

(b) The minimal polynomial divides the characteristic polynomial, and has the same irreducible factors. Since

$$(5I_4 - A)^2 = \begin{bmatrix} 2 & 4 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0_4^4$$

we have $m_A(t) = (t-5)^2$ so $m_1 = 2$.

(c) A is diagonalizable iff $g_1 = k_1 = 4$. To find the λ_1-eigenspace row reduce

$$\left[\begin{array}{cccc|c} 2 & 4 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \text{ to } \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{array}{c} x_1 = -2r \\ x_2 = r \in \mathbf{R} \\ x_3 = -s \\ x_4 = s \in \mathbf{R} \end{array}, \text{ then}$$

$$A_{\lambda_1} = \left\{ \begin{bmatrix} -2r \\ r \\ -s \\ s \end{bmatrix} \in \mathbf{R}^4 \;\middle|\; r, s \in \mathbf{R} \right\} \text{ has dimension 2 so A is not diagonalizable.}$$

(d) Since $g_1 = 2 < 4 = k_1$, A cannot be diagonalized, but it does have two independent eigenvectors with eigenvalue 5. So there must be two Jordan blocks for $\lambda_1 = 5$, with the maximum size being $m_1 = 2$. This implies that there are two Jordan blocks, each of size 2×2. The Jordan canonical form matrix similar to A must be $J = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$.

(2) (20 Points) The **characteristic polynomial** is $p_A(t) = (t-2)^6(t-7)^5$ and the **minimal polynomial** is $m_A(t) = (t-2)^3(t-7)^4$ so there are two eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = 7$ with algebraic multiplicities $k_1 = 6$ and $k_2 = 5$. The powers in the minimal polynomial $m_1 = 3$ and $m_2 = 4$ tell the sizes of the largest Jordan blocks of each type. Let

$$B = J_{2,3} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad C = J_{2,2} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad D = J_{7,4} = \begin{bmatrix} 7 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}.$$

Then there are only three possible Jordan canonical form matrices similar to A,

$$\begin{bmatrix} B & & & \\ & B & & \\ & & D & \\ & & & 7 \end{bmatrix}, \text{ or } \begin{bmatrix} B & & & & \\ & C & & & \\ & & 2 & & \\ & & & D & \\ & & & & 7 \end{bmatrix}, \text{ or } \begin{bmatrix} B & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & 2 & & \\ & & & & D & \\ & & & & & 7 \end{bmatrix}.$$

In the first case, there are two Jordan blocks for eigenvalue 2 so $g_2 = 2$. In the second case, there are three Jordan blocks for eigenvalue 2 so $g_2 = 3$. In the third case, there are four Jordan blocks for eigenvalue 2 so $g_2 = 4$. In all cases, there are two Jordan blocks for eigenvalue 7, so $g_7 = 2$.

(3) (20 Points, 5 Points Each)

(a) ${}_TP_S$ and ${}_SP_T$ are inverses of each other.

(b) There are several possible answers to this question. One answer is that W is a direct sum when for each i, $1 \le i \le m$, we have $W_i \cap (\sum_{j \ne i} W_j) = \{\theta\}$. Another possible answer is that $\dim(W) = \sum_{i=1}^{m} \dim(W_i)$. Another possible answer is that the union of bases for each W_i is a basis for W.

(c) If $AX = BX$ for any $X \in \mathbf{F}^n$ then using for X the standard basis vectors e_j, $1 \le j \le n$, we get $Col_j(A) = Ae_j = Be_j = Col_j(B)$, so that A and B have identical corresponding columns, so $A = B$.

(d) Since $L : \mathbf{R}^3_4 \to \mathbf{R}^2_5$, we know $12 = \dim(\mathbf{R}^3_4) = \dim(Ker(L)) + \dim(Range(L))$. But $0 \le \dim(Range(L)) \le 10$, so $2 \le \dim(Ker(L)) \le 12$.

(4) (25 Points) Let $L \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-b \\ c-d \end{bmatrix}$, $K = Ker(L)$ and

$U = \mathbf{R}_2^2/K = \{v + K \mid v \in \mathbf{R}_2^2\}$.

(a) (4 points) $Ker(L) = \left\{ \begin{bmatrix} a & a \\ c & c \end{bmatrix} \;\middle|\; a, c \in \mathbf{R} \right\}$.

(b) (4 points) $\left\{ v_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis for $Ker(L)$ and $\dim(Ker(L)) = k = 2$.

(c) (2 points) L is not one-to one because $\dim(Ker(L)) > 0$.

(d) (2 points) There are many ways to extend the basis for K to a basis of $\mathbf{R}_2^2$.

$$S = \left\{ v_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

(e) (2 points) A basis for U could be $S' = \{v_3 + K, v_4 + K\}$ so $\dim(U) = 2$.

(f) (2 points) If T is the standard basis of $\mathbf{R}^2$ then $[L]_S^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(g) (6 points) Try to define $\bar{L} : U \to \mathbf{R}^2$ by $\bar{L}(v + K) = L(v)$. To show that $\bar{L}$ is well-defined suppose that $v + K = v' + K$, so that $v - v' \in K$. Then we have $L(v - v') = \theta$ so $L(v) = L(v')$ as required. To show $\bar{L}$ is injective suppose that $\bar{L}(v + K) = \theta$. It means $L(v) = \theta$ so $v \in K$ so $v + K = K$ is the trivial vector in the quotient space U. So $Ker(\bar{L})$ is trivial, and $\bar{L}$ is injective.

(h) (3 points) $[\bar{L}]_S^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(5) (25 Points)

(a) (5 Points) We are given that $row - rank(A) = 5$, but $column - rank(A) = rank(A) = row - rank(A)$, so the dimension of the span of the columns of A equals $column - rank(A) = 5$.

(b) (10 Points)

$$\det \begin{bmatrix} 4 & 6 & 6 & 8 \\ 3 & -9 & 6 & 3 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} = (2)(3) \det \begin{bmatrix} 2 & 3 & 3 & 4 \\ 1 & -3 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$= (6) \det \begin{bmatrix} 0 & -1 & -3 & -4 \\ 0 & -5 & -1 & -3 \\ 0 & -3 & -5 & -8 \\ 1 & 2 & 3 & 4 \end{bmatrix} = -(6) \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & -1 & -3 \\ 0 & -3 & -5 & -8 \\ 0 & -1 & -3 & -4 \end{bmatrix}$$

$$= (6) \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -4 \\ 0 & -3 & -5 & -8 \\ 0 & -5 & -1 & -3 \end{bmatrix} = (6)(-1)(-1) \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -4 \\ 0 & 3 & 5 & 8 \\ 0 & 5 & 1 & 3 \end{bmatrix}$$

$$= (6) \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -4 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & -14 & -17 \end{bmatrix} = (6)(-4) \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$=(6)(-4)(1)(-1)(1)(-3) = -72.$$

(c) (5 Points) L must also be injective, so it is bijective, invertible and an isomorphism.

(d) (5 Points) Since L is bijective, we have $\dim(Ker(L)) = 0$ and $\dim(Range(L)) = \dim(W)$, so

$$\dim(V) = \dim(Ker(L)) + \dim(Range(L)) = 0 + \dim(W) = \dim(W)$$

and we get the relation is

$$\dim(V) = \dim(W).$$

(6) (30 Points) $M = [m_{ij}] \in \mathbf{F}_n^n$ defines a function $B : V \times V \to \mathbf{F}$ by $B(v, w) = [v]_S^{Tr} M [w]_S$ where $[v]_S^{Tr} = [\, a_1 \quad a_2 \quad \cdots \quad a_n \,] \in \mathbf{F}_n$ is the transpose of $[v]_S$.

(a) (5 pts) To prove that the function B is bilinear, we have

$$\begin{aligned}
&B(c_1v_1 + c_2v_2, w) = [c_1v_1 + c_2v_2]_S^{Tr} M[w]_S \\
&= (c_1[v_1]_S^{Tr} + c_2[v_2]_S^{Tr})M[w]_S = c_1[v_1]_S^{Tr} M[w]_S + c_2[v_2]_S^{Tr} M[w]_S \\
&= c_1B(v_1, w) + c_2B(v_2, w)
\end{aligned}$$

and

$$\begin{aligned}
&B(v, c_1w_1 + c_2w_2) = [v]_S^{Tr} M[c_1w_1 + c_2w_2]_S \\
&= [v]_S^{Tr} M(c_1[w_1]_S + c_2[w_2]_S) = c_1[v]_S^{Tr} M[w_1]_S + c_2[v]_S^{Tr} M[w_2]_S \\
&= c_1B(v, w_1) + c_2B(v, w_2).
\end{aligned}$$

(b) (5 pts) To prove that $B(v_i, v_j) = m_{ij}$ for all $1 \le i, j \le n$, use that $[v_i]_S = e_i$ are the standard basis vectors of $\mathbf{F}^n$, so

$$B(v_i, v_j) = [v_i]_S^{Tr} M[v_j]_S = e_i^{Tr} M e_j = m_{ij}$$

since $e_i^{Tr} M = Row_i(M)$ and then $e_i^{Tr} M e_j = Col_j(Row_i(M)) = m_{ij}$.

(c) (5 Points) To get the formula for $B(v, w)$ we use parts (a) and (b):

$$B(v, w) = B(\sum_{i=1}^{n} a_i v_i, \sum_{j=1}^{n} b_j v_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j B(v_i, v_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j m_{ij}.$$

(d) (5 pts) Prove that $B(v, w) = B(w, v)$ iff $M = M^{Tr}$ is symmetric. By the definition, $B(v, w) = [v]_S^{Tr} M[w]_S$ and $B(w, v) = [w]_S^{Tr} M[v]_S$. These 1×1 matrices are equal iff

$$[v]_S^{Tr} M[w]_S = ([w]_S^{Tr} M[v]_S)^{Tr} = [v]_S^{Tr} M^{Tr}[w]_S$$

using the fact that $(AB)^{Tr} = B^{Tr} A^{Tr}$. But this is true for all $v, w \in V$ iff it is true for all $v = v_i$ and $w = v_j$, $1 \le i, j \le n$. As we have seen in part (b), this means $e_i^{Tr} M e_j = e_i^{Tr} M^{Tr} e_j$ which says $m_{ij} = m_{ji}$, that is, $M = M^{Tr}$.

(e) (5 pts) When $V = \mathbf{R}^n$, S is the standard basis and $M = I_n = [\delta_{ij}]$, from part (c) the formula for $B(v, w)$ is

$$B(v,w) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \delta_{ij} = \sum_{i=1}^{n} a_i b_i$$

which we recognize as a generalization of the famous “dot product” from $\mathbf{R}^3$ to $\mathbf{R}^n$.

(f) (5 pts) In the special case of (e), we have for any $v \in V$,

$$B(v,v) = \sum_{i=1}^{n} a_i^2 \geq 0$$

and since each $a_i^2 \geq 0$, $B(v, v) = 0$ implies each $a_i = 0$, so $v = \theta$.