Math 404 Advanced Linear Algebra Spring 2006 Practice Exam 2 Feingold

(1) (30 Pts) Let $L : \mathbf{R}_4 \to \mathbf{R}^2$ be given by

$$L([a \quad b \quad c \quad d]) = \begin{bmatrix} 2a - b + c - 3d \\ -a + 4b - 2c - d \end{bmatrix}.$$

Let S be the standard basis of $\mathbf{R}_4$ and let T be the standard basis of $\mathbf{R}^2$. Let other ordered bases be

$$S' = \{[1\ 1\ 0\ 1], [1\ 0\ 0\ 1], [0\ 0\ 1\ 1], [0\ 1\ 1\ 0]\} \quad \text{and} \quad T' = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}.$$

(a) (2 pts) Find the matrix $[L]_S^T$ representing L from S to T.

(b) (8 pts) Find the matrix $[L]_{S'}^{T'}$ representing L from S' to T' **without using transition matrices**. (Do it directly.)

(c) (10 pts) Find the transition matrices ${}_SP_{S'}$ and ${}_TQ_{T'}$ and show that $[L]_{S'}^{T'} = ({}_TQ_{T'})^{-1}\ [L]_S^T\ ({}_SP_{S'})$.

(2) (30 Points) Answer (with brief justification) each question separately.

(a) What elementary matrix corresponds to the elementary row operation $R_3 + 7R_2 \to R_3$ done to a matrix in $\mathbf{R}_m^3$?

(b) If $S = \{v_1, \ldots, v_n\}$ is a basis of a vector space V, and $v \in V$ has coordinates $[v]_S = [1\ 2\ 3\ \cdots\ n]^T \in \mathbf{R}^n$, then write v as a linear combination from S.

(c) If $A = [a_{ij}] \in \mathbb{R}_4^4$ and $a_{ij} = i + j$, what is the most you can say about $\det(A)$?

(d) Suppose $A \in \mathbb{R}_n^n$ and $B = P^T AP$ for a matrix P with $\det(P) = 5$. (P^T is the transpose of P.) What can you say about the relation between $\det(A)$ and $\det(B)$?

(e) If $\det(A) = 7$, $\det(B) = 4$ and $\det(C) = 3$, find $\det(A^{-1}BC)$.

(3) (20 pts) Let $V = P_2(\mathbf{R}) = \{a_2t^2 + a_1t + a_0 \mid a_i \in \mathbf{R}\}$ with basis

$$S = \{p_1 = 3t^2 + 2t + 1, \quad p_2 = t^2 + t, \quad p_3 = -1\}$$

and let V^* be the dual space of V. Find the dual basis $S^* = \{f_1, f_2, f_3\}$ by giving formulas for each $f_i(a_2t^2 + a_1t + a_0)$, $1 \le i \le 3$.

(4) (20 pts) Let $L : \mathbf{C}^2 \to \mathbf{C}^2$ and $K : \mathbf{C}^2 \to \mathbf{C}^2$ be the linear transformations

$$L\left(\begin{bmatrix} z \\ w \end{bmatrix}\right) = \begin{bmatrix} \mathbf{i}z - w \\ z - \mathbf{i}w \end{bmatrix} \quad \text{and} \quad K\left(\begin{bmatrix} z \\ w \end{bmatrix}\right) = \begin{bmatrix} \mathbf{i}w \\ -\mathbf{i}z \end{bmatrix}.$$

Let S be the standard basis of $\mathbf{C}^2$.

(a) (2 pts) Find the matrix $[L]_S^S$ representing L from S to S.

(b) (2 pts) Find the matrix $[K]_S^S$ representing K from S to S.

(c) (4 pts) Use composition of functions to find the formula for $(K \circ L)\left(\begin{bmatrix} z \\ w \end{bmatrix}\right)$ from the formulas for K and L.

(d) (2 pts) Find matrix $[K \circ L]_S^S$ representing $K \circ L$ from S to S. What should be the relationship between the matrices $[L]_S^S$, $[K]_S^S$ and $[K \circ L]_S^S$? Check that relationship by direct calculation.

(e) (4 pts) Show that both L and K are invertible transformations.

(f) (4 pts) Find formulas for L^{-1} and K^{-1}.

(g) (2 pts) What matrix represents $(K \circ L)^{-1}$ from S to S?

(1) (a) (2 Pts) ${}_TA_S = \begin{bmatrix} 2 & -1 & 1 & -3 \\ -1 & 4 & -2 & -1 \end{bmatrix}$ is easy to get since S and T are standard.

$L([1\ 0\ 0\ 0]) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $L([0\ 1\ 0\ 0]) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $L([0\ 0\ 1\ 0]) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$,

$L([0\ 0\ 0\ 1]) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$. $[T \mid L(S)] = \left[\begin{array}{cc|cccc} 1 & 0 & 2 & -1 & 1 & -3 \\ 0 & 1 & -1 & 4 & -2 & -1 \\ \multicolumn{2}{c|}{T} & \multicolumn{4}{c}{L(S)} \end{array}\right]$ is already reduced, so the right side is ${}_TA_S$.

(b) (8 Pts) $L([1\ 1\ 0\ 1]) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, $L([1\ 0\ 0\ 1]) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, $L([0\ 0\ 1\ 1]) = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$, $L([0\ 1\ 1\ 0]) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Row reduce $\left[\begin{array}{cc|cccc} 2 & 3 & -2 & -1 & -2 & 0 \\ 1 & 2 & 2 & -2 & -3 & 2 \\ \multicolumn{2}{c|}{T'} & \multicolumn{4}{c}{L(S')} \end{array}\right]$ to $\left[\begin{array}{cc|cccc} 1 & 0 & -10 & 4 & 5 & -6 \\ 0 & 1 & 6 & -3 & -4 & 4 \\ \multicolumn{2}{c|}{I_2} & \multicolumn{4}{c}{{}_{T'}A_{S'}} \end{array}\right]$

(c) (10 Pts) ${}_SP_{S'} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ and ${}_TQ_{T'} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ since S and T are the standard bases.

To get ${}_{T'}Q_T = ({}_TQ_{T'})^{-1}$, reduce

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ \multicolumn{2}{c|}{T'} & \multicolumn{2}{c}{T} \end{array}\right] \text{ to } \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ \multicolumn{2}{c|}{I_2} & \multicolumn{2}{c}{{}_{T'}Q_T} \end{array}\right]$$

$$({}_TQ_{T'})^{-1}\,{}_TA_S({}_SP_{S'}) = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 & -3 \\ -1 & 4 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 7 & -14 & 8 & -3 \\ -4 & 9 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -10 & 4 & 5 & -6 \\ 6 & -3 & -4 & 4 \end{bmatrix} = {}_{T'}A_{S'}$$

checks.

(2) (30 Points) Answer (with brief justification) each question separately.

(a) The elementary matrix that corresponds to that row operation is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$ obtained by doing the operator to I_3.

(b) $v = 1v_1 + 2v_2 + \cdots + nv_n$.

(c) $\det(A) = \det \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix} = \det \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 0$ since two rows are identical.

(d) $\det(B) = \det(P^T AP) = \det(P^T)\det(A)\det(P) = \det(P)\det(A)\det(P) = (5)\det(A)(5) = (25)\det(A)$.

(e) $\det(A^{-1}BC) = \det(A)^{-1}\det(B)\det(C) = 7^{-1}(4)(3) = \frac{12}{7}$.

(3) (20 Pts) The dual basis vectors are uniquely determined by the conditions $f_i(p_j) = \delta_{ij}$ for $1 \leq i, j \leq 3$. To make use of this it easiest to first express the polynomial $p = a_2t^2 + a_1t + a_0$ as a linear combination of the basis vectors in S.

$$\text{Row reduce } \left[\begin{array}{ccc|c} 1 & 0 & -1 & a_0 \\ 2 & 1 & 0 & a_1 \\ 3 & 1 & 0 & a_2 \\ & S & & v \end{array}\right] \text{ to } \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_2 - a_1 \\ 0 & 1 & 0 & -2a_2 + 3a_1 \\ 0 & 0 & 1 & a_2 - a_1 - a_0 \\ & I_3 & & [p]_S \end{array}\right] \text{ which means}$$

$$a_2t^2 + a_1t + a_0 = (a_2 - a_1)p_1 + (-2a_2 + 3a_1)p_2 + (a_2 - a_1 - a_0)p_3 \quad \text{so}$$
$$f_i(a_2t^2 + a_1t + a_0) = f_i((a_2 - a_1)p_1 + (-2a_2 + 3a_1)p_2 + (a_2 - a_1 - a_0)p_3).$$

The formulas we get are then just

$$f_1(a_2t^2 + a_1t + a_0) = a_2 - a_1,$$

$$f_2(a_2t^2 + a_1t + a_0) = -2a_2 + 3a_1,$$
$$f_3(a_2t^2 + a_1t + a_0) = a_2 - a_1 - a_0.$$

(4) (20 Pts)

(a) (2 Pts) $L\left(\begin{bmatrix} z \\ w \end{bmatrix}\right) = \begin{bmatrix} \mathbf{i}z - w \\ z - \mathbf{i}w \end{bmatrix} = \begin{bmatrix} \mathbf{i} & -1 \\ 1 & -\mathbf{i} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}$ so $[L]_S^S = \begin{bmatrix} \mathbf{i} & -1 \\ 1 & -\mathbf{i} \end{bmatrix}$.

(b) (2 Pts) $K\left(\begin{bmatrix} z \\ w \end{bmatrix}\right) = \begin{bmatrix} \mathbf{i}w \\ -\mathbf{i}z \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}$ so $[K]_S^S = \begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix}$.

(c) (4 Pts) $(K \circ L)\left(\begin{bmatrix} z \\ w \end{bmatrix}\right) = K\left(\begin{bmatrix} \mathbf{i}z - w \\ z - \mathbf{i}w \end{bmatrix}\right) = \begin{bmatrix} \mathbf{i}z + w \\ z + \mathbf{i}w \end{bmatrix} = \begin{bmatrix} \mathbf{i} & 1 \\ 1 & \mathbf{i} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}$.

(d) (2 Pts) $(K \circ L)\left(\begin{bmatrix} z \\ w \end{bmatrix}\right) = \begin{bmatrix} \mathbf{i} & 1 \\ 1 & \mathbf{i} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}$ so $[K \circ L]_S^S = \begin{bmatrix} \mathbf{i} & 1 \\ 1 & \mathbf{i} \end{bmatrix}$.

The relationship should be $([K]_S^S)([L]_S^S) = [K \circ L]_S^S$.

Check: $\begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{i} & -1 \\ 1 & -\mathbf{i} \end{bmatrix} = \begin{bmatrix} \mathbf{i} & 1 \\ 1 & \mathbf{i} \end{bmatrix}$.

(e) (4 Pts) It is easy to check that the kernels of L and K are trivial, so they are both injective. Since the dimension of the domain and codomain are equal, the transformations are also surjective, so bijective, so invertible. Another way to check it is to see that the matrices $[L]_S^S$ and $[K]_S^S$ are invertible, and that $([L]_S^S)^{-1}$ represents L^{-1}, and $([K]_S^S)^{-1}$ represents K^{-1}.

(f) (4 Pts) $L^{-1}\left(\begin{bmatrix} z \\ w \end{bmatrix}\right) = \frac{1}{2} \begin{bmatrix} -\mathbf{i} & 1 \\ -1 & \mathbf{i} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\mathbf{i}z + w \\ -z + \mathbf{i}w \end{bmatrix}$

Since $([K]_S^S)^{-1} = [K]_S^S$, $K^{-1} = K$

(g) (2 Pts) $(K \circ L)^{-1} = L^{-1} \circ K^{-1}$ is represented by

$$([L]_S^S)^{-1}([K]_S^S)^{-1} = \frac{1}{2} \begin{bmatrix} -\mathbf{i} & 1 \\ -1 & \mathbf{i} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\mathbf{i} & 1 \\ 1 & -\mathbf{i} \end{bmatrix}.$$