Math 404 Advanced Linear Algebra Spring 2006 Practice Exam 1 Feingold

(1) (20 Points) In each part use *row reduction* to find the set of solutions W.

(a) $\begin{aligned} x_1 + 2x_2 &= 2 \\ 3x_1 + 4x_2 &= 5 \end{aligned}$ (b) $\begin{aligned} x_1 - 2x_2 + x_3 &= 2 \\ 2x_1 - x_2 + x_3 &= 3 \\ x_1 + x_2 &= 4 \end{aligned}$ (c) $\begin{aligned} x_1 + 2x_2 - 3x_3 + 2x_4 &= 0 \\ -2x_1 - 3x_2 + 7x_3 - x_4 &= 0 \\ 2x_1 + 4x_2 - 8x_3 &= 0 \\ x_1 + 3x_2 - x_3 + 7x_4 &= 0 \end{aligned}$

(2) (15 Points) For each of the following matrices find the inverse if it exists.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -3 \\ -2 & -4 & -5 \end{bmatrix}$

(3) (25 Points) Let $L : \mathbf{R}^5 \to \mathbf{R}^4$ be the linear transformation given by $L(X) = AX$ for $A = \begin{bmatrix} 1 & 2 & -3 & 2 & 1 \\ -2 & -3 & 7 & -1 & 1 \\ 2 & 4 & -8 & 0 & 0 \\ 1 & 3 & -1 & 7 & 5 \end{bmatrix}$.

(a) Find all the vectors in $Ker(L) = \{X \in \mathbf{R}^5 \mid L(X) = 0_{4\times1}\}$ in terms of some free variables.

(b) Find all the vectors in $Range(L) = \{B = L(X) \in \mathbf{R}^4 \mid X \in \mathbf{R}^5\}$ in terms of a consistency condition on the entries of B.

(c) Determine whether L is injective and whether L is surjective.

(4) (20 Points) Answer each question separately

(a) What elementary row operation is the inverse of operation $R_3 + 7R_5 \to R_3$?

(b) If $A \in \mathbf{R}^8_5$ has two rows of zeros, what is the most that rank(A) could be?

(c) If the homogeneous linear system $A_{m\times n}X_{n\times1} = 0_{m\times1}$ has only the trivial solution, then what is the most you can say about the relation between m and n?

(d) What is the most you can say about the solutions of a homogeneous linear system $A_{n\times n}X_{n\times1} = 0_{n\times1}$ when the square matrix A is row equivalent to a matrix with a row of zeros?

(e) For any two anti-symmetric $n \times n$ matrices A and B, show that $AB - BA$ is also anti-symmetric.

(5) (20 Points) Let $L : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation given by $L(X) = AX$ for an $m \times n$ matrix A. Let $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ be the standard vectors in $\mathbf{R}^n$.

(a) What can you say about $L(\mathbf{e}_1)$, $L(\mathbf{e}_2)$,...,$L(\mathbf{e}_n)$?

(b) What relation between m and n would guarantee that L is not injective?

(c) What relation between m and n would guarantee that L is not surjective?

(d) What relation between rank(A) and the size of A would mean that L is injective?

(e) What relation between rank(A) and the size of A would mean that L is surjective?

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1. (a) $\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 3 & 4 & 5 \end{array}\right]$ row reduces to $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \end{array}\right]$ so $W = \left\{ \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \right\}$. (5 points)

(b) $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 2 & -1 & 1 & 3 \\ 1 & 1 & 0 & 4 \end{array}\right]$ row reduces to $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & -3 \end{array}\right]$ inconsistent, so W is empty. (5 points)

(c) $\left[\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 0 \\ -2 & -3 & 7 & -1 & 0 \\ 2 & 4 & -8 & 0 & 0 \\ 1 & 3 & -1 & 7 & 0 \end{array}\right]$ row reduces to $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 6 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ so

$W = \left\{ \begin{bmatrix} -6r \\ -r \\ -2r \\ r \end{bmatrix} \in \mathbf{R}^4 \middle| r \in \mathbf{R} \right\}$. (10 points)

2. (a) $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$ row reduces to $\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \end{array}\right]$ so $A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$. This is the coefficient matrix from problem 1(a). (5 points)

(b) $\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ row reduces to $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -3 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right]$ so A is not invertible. This is the coefficient matrix from problem 1(b). (5 points)

(c) Row reduce $[A|I_3]$ to $[I_3|A^{-1}]$ so $A^{-1} = \begin{bmatrix} -7 & -2 & -3 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. (5 points)

3. (a) $\left[\begin{array}{ccccc|c} 1 & 2 & -3 & 2 & 1 & 0 \\ -2 & -3 & 7 & -1 & 1 & 0 \\ 2 & 4 & -8 & 0 & 0 & 0 \\ 1 & 3 & -1 & 7 & 5 & 0 \end{array}\right]$ row reduces to $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 6 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ so

$Ker(L) = \left\{ \begin{bmatrix} -6r \\ -r-2s \\ -2r-s \\ r \\ s \end{bmatrix} \in \mathbf{R}^5 \middle| r, s \in \mathbf{R} \right\}$ is nontrivial so (c) L is not injective.

(b) $\left[\begin{array}{ccccc|c} 1 & 2 & -3 & 2 & 1 & b_1 \\ -2 & -3 & 7 & -1 & 1 & b_2 \\ 2 & 4 & -8 & 0 & 0 & b_3 \\ 1 & 3 & -1 & 7 & 5 & b_4 \end{array}\right]$ row reduces to $\left[\begin{array}{ccccc|c} 1 & 2 & -3 & 2 & 1 & b_1 \\ 0 & 1 & -1 & -1 & 1 & b_2+b_3 \\ 0 & 0 & 1 & 2 & 1 & \frac{-1}{2}b_3+b_1 \\ 0 & 0 & 0 & 0 & 0 & -4b_1-b_2+\frac{1}{2}b_3+b_4 \end{array}\right]$

so the consistency condition describing $Range(L)$ is $-4b_1 - b_2 + \frac{1}{2}b_3 + b_4 = 0$ and

$Range(L) = \left\{ B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \in \mathbf{R}^4 \middle| -4b_1 - b_2 + \frac{1}{2}b_3 + b_4 = 0 \right\}$ is not all of $\mathbf{R}^4$ so (c) $L$is not surjective.

4. (20 Points, 4 points each)
   (a) The inverse of operation $R_3 + 7R_5 \to R_3$ is $R_3 - 7R_5 \to R_3$.
   (b) If $A \in \mathbf{R}_5^8$ has two rows of zeros, then $\text{rank}(A) \le 5$.
   (c) If the homogeneous linear system $A_{m\times n}X_{n\times 1} = 0_{m\times 1}$ has only the trivial solution, then $n \le m$ because otherwise $n > m$ guarantees nontrivial solutions.
   (d) There are infinitely many nontrivial solutions of a homogeneous linear system $A_{n\times n}X_{n\times 1} = 0_{n\times 1}$ when the square matrix A is row equivalent to a matrix with a row of zeros since there must be at least one free variable in the solution.
   (e) Given that $A^T = -A$ and $B^T = -B$, we have

$$\begin{aligned}(AB - BA)^T &= (AB)^T - (BA)^T = B^T A^T - A^T B^T \\ &= (-B)(-A) + (-A)(-B) = BA - AB = -(AB - BA)\end{aligned}$$

   which means $AB - BA$ is anti-symmetric.

5. (20 Points, 4 points each)
   (a) $L(\mathbf{e}_1)$, $L(\mathbf{e}_2)$,...,$L(\mathbf{e}_n)$ are the column vectors of matrix A since $L(\mathbf{e}_i) = A\mathbf{e}_i$ is the i^{th} column of A.
   (b) The relation $m < n$ would guarantee that L is not injective since more variables than equations guarantees there are free variables.
   (c) The relation $m > n$ would guarantee that L is not surjective since at most n leading ones leaves at least one row of zeros in the RREF, giving a consistency condition for $AX = B$.
   (d) The relation $\text{rank}(A) = n$ would mean that L is injective since n leading ones means no free variables in $AX = 0_{m\times 1}$.
   (e) The relation $\text{rank}(A) = m$ would mean that L is surjective since m leading ones give a leading one in each row and there are no consistency conditions for $AX = B$.