(1) (25 Points, 5 Points each) Determine whether each sequence \( \{a_n\}_{n=1}^{\infty} \) converges or diverges, and if it converges, then find its limit.

\[(a) \quad a_n = \sqrt{\frac{3n}{n+1}} + \frac{2n}{n-1} \quad \quad \quad (b) \quad a_n = \frac{6n^3 - 3n^2 + n - 1}{7n^3 + 4n^2 + 3n + 2} \]

\[(c) \quad a_n = \frac{1 + \ln(n)}{1 + \sqrt{n}} \quad \quad \quad (d) \quad a_n = \arcsin \left( \frac{1 + n}{1 + 2n} \right) \]

\[(e) \quad a_n = \frac{n!}{n^n} = \frac{(1)(2)(3)\cdots(n)}{(n)(n)(n)\cdots(n)} \quad \text{(Hint: Use the Squeeze Theorem.)} \]

(2) (10 Points) A convergent sequence of positive numbers satisfies the recursive relation \( a_na_{n+1} = a_n + 6 \). Find \( \lim_{n \to \infty} a_n \).

(3) (20 Points) In each part test the series for convergence or divergence. Write all steps of the test you use.

\[(a) \quad \sum_{n=1}^{\infty} \frac{1}{n^4 + 2n^2 + 1} \quad \quad \quad (b) \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^7 - n^3 + 2}} \]

(4) (20 Points) Test the following series for absolute convergence, conditional convergence, or divergence. Explain what tests you are applying and how you apply them.

\[(a) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n^3 + 1}} \quad \quad \quad (b) \quad \sum_{n=1}^{\infty} \left( \frac{2n}{3n+1} \right)^{2n} \]

(5) (20 Points) Answer these questions about the polar curve \( r = \theta^2 \) for \( 0 \leq \theta \leq 2\pi \).

(a) Find the slope of the tangent line to the curve at \( \theta = \frac{\pi}{3} \).
(b) Find the area of the polar region determined by the curve.
(c) Set up the integral for the arclength of that polar curve.
(d) Compute the integral in part (c) to find the arclength of that polar curve.

(6) (20 Points) The alternating series \( \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[4]{k}} \) converges to \( S \) and has \( N^{th} \) partial sum \( S_N = \sum_{k=1}^{N} \frac{(-1)^k}{\sqrt[4]{k}} \). Find the smallest \( N \) such that you can be sure \( |S - S_N| < \frac{1}{100} \).
(1) (25 Points, 5 Points Each) Determine whether each sequence \( \{a_n\}_{n=1}^{\infty} \) converges or diverges, and if it converges, then find its limit.

(a) \( a_n = \sqrt{\frac{3n}{n+1}} + \frac{2n}{n-1} = \sqrt{\frac{5n^2 - n}{n^2 - 1}} = \sqrt{\frac{5 - \frac{1}{n}}{1 - \frac{1}{n^2}}} \to \sqrt{5} \) as \( n \to \infty \) converges since \( \frac{1}{n^2} \to 0 \) and \( \frac{1}{n} \to 0 \).

(b) \( a_n = \frac{6n^3 - 3n^2 + n - 1}{7n^3 + 4n^2 + 3n + 2} = \frac{6 - \frac{3}{n} + \frac{1}{n^2} - \frac{1}{n^3}}{7 + \frac{4}{n} + \frac{3}{n^2} + \frac{2}{n^3}} \to \frac{6}{7} \) converges as \( n \to \infty \).

(c) \( a_n = \frac{1 + \ln(n)}{1 + \sqrt{n}} = f(n) \) where \( f(x) = \frac{1 + \ln(x)}{1 + \sqrt{x}} \). Using L’Hospital’s Rule (\( \infty / \infty \)-type):

\[
\lim_{x \to \infty} \frac{1 + \ln(x)}{1 + \sqrt{x}} = \lim_{x \to \infty} \frac{x^{-1}}{2\sqrt{x}} = \lim_{x \to \infty} \frac{2\sqrt{x}}{x} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} \to 0 \text{ converges.}
\]

(d) \( a_n = \arcsin \left( \frac{1 + n}{1 + 2n} \right) \to \arcsin(1/2) = \frac{\pi}{6} \) converges since \( \frac{1 + n}{1 + 2n} \to \frac{1}{2} \) as \( n \to \infty \), and \( \arcsin(x) \) is continuous at \( 1/2 \).

(e) \( a_n = \frac{n!}{n^n} = \frac{(1)(2)(3)\cdots(n)}{(n)(n)(n)\cdots(n)} \) (Hint: Use the Squeeze Theorem.)

Since \( \frac{i}{n} \leq 1 \) for \( 2 \leq i \leq n \), we get

\[
0 \leq a_n = \frac{1 2 3 \cdots n - 1}{n n n \cdots n} < \frac{1}{n}
\]

This means that \( 0 \leq a_n \leq \frac{1}{n} \), and since \( \lim_{n \to \infty} \frac{1}{n} = 0 \), the Squeeze Theorem says the sequence converges to 0.

(2) (10 Points) A **convergent** sequence of **positive** numbers satisfies the recursive relation \( a_n a_{n+1} = a_n + 6 \). Find \( \lim_{n \to \infty} a_n \).

Taking the limit as \( n \to \infty \) of the recursive relation, and using that \( \lim_{n \to \infty} a_n = L = \lim_{n \to \infty} a_{n+1} \), get

\[
(L)(L) = L + 6 \quad \text{so} \quad L^2 - L - 6 = (L-3)(L+2) = 0 \quad \text{so} \quad L = 3 \quad \text{is the only positive solution.}
\]
(3) (20 Points) In each part test the series for convergence or divergence. Write all steps of the test you use.

(a) \[\sum_{n=1}^{\infty} \frac{1}{n^4 + 2n^2 + 1}\]

(10 Points) Since \(0 \leq \frac{1}{n^4 + 2n^2 + 1} \leq \frac{1}{n^4}\) for all \(n \geq 1\), and \(\sum_{n=1}^{\infty} \frac{1}{n^4}\) is a convergent \(p\)-series with \(p = 4 > 1\), the series converges by the Comparison test.

(b) \[\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n^3 - n^3 + 2}}\]

(10 Points) Apply the Limit Comparison test where the series \(\sum b_n\) being compared to is the convergent \(p\)-series \(\sum \frac{1}{n^{7/4}}\). We have

\[
\lim_{n \to \infty} \frac{\frac{1}{\sqrt[n]{n^3 - n^3 + 2}}}{\frac{1}{n^{7/4}}} = \lim_{n \to \infty} 4^{\frac{n^3}{n^3 - n^3 + 2}} = \lim_{n \to \infty} 4^{\frac{1}{(1 - \frac{1}{n^4} + \frac{2}{n^7})}} = 1 > 0
\]

so the Limit Comparison test says both series have the same behavior, they both converge.

(4) (20 Points) Test the following series for absolute convergence, conditional convergence, or divergence. Explain what tests you are applying and how you apply them.

(a) \[\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n^3 + 1}}\]

(10 Points) The series \(\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n^3 + 1}}\) is an alternating series, and for \(n \geq 1\) we have \(\sqrt[4]{(n+1)^3 + 1} > \sqrt[4]{n^3 + 1}\) so \(\frac{(-1)^n}{\sqrt[4]{(n+1)^3 + 1}} < \frac{1}{\sqrt[4]{n^3 + 1}}\). Also, \(\lim_{n \to \infty} \frac{1}{\sqrt[4]{n^3 + 1}} = 0\), so this series converges by the Alternating Series test. Do the Limit Comparison Test of the absolute values of the terms of this series with the terms of the divergent \(p\)-series with \(p = \frac{3}{4} < 1\),

\[
\lim_{n \to \infty} \frac{\frac{1}{\sqrt[4]{n^3 + 1}}}{\frac{1}{\sqrt[4]{n^3 + 1}}} = \lim_{n \to \infty} \sqrt[4]{\frac{n^3}{n^3 + 1}} = \lim_{n \to \infty} \sqrt[4]{\frac{1}{(1 + \frac{1}{n^3})}} = 1 > 0
\]

so the Limit Comparison test says both series have the same behavior, they both diverge. So the given series converges conditionally.

(b) \[\sum_{n=1}^{\infty} \left(\frac{2n}{3n + 1}\right)^2\]

(10 Points) Applying the root test to the series gives \(L = \lim_{n \to \infty} \left(\frac{2n}{3n + 1}\right)^2 = \frac{4}{9} < 1\) so the series converges by the Root Test. It is a positive series, so it converges absolutely.
(5) (20 Points) Answer these questions about the polar curve \( r = \theta^2 \) for \( 0 \leq \theta \leq 2\pi \).

(a) Find the slope of the tangent line to the curve at \( \theta = \frac{\pi}{3} \).

The slope of the tangent line to the curve for any \( \theta \) is
\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.
\]

Since \( y = r \sin(\theta) = \theta^2 \sin(\theta) \) and \( x = r \cos(\theta) = \theta^2 \cos(\theta) \), we have
\[
\frac{dy}{dx} = \frac{2\theta \sin(\theta) + \theta^2 \cos(\theta)}{2\theta \cos(\theta) - \theta^2 \sin(\theta)}
\]
so when \( \theta = \frac{\pi}{3} \) this slope equals
\[
\frac{2 \pi \sqrt{3}}{3} + \frac{\pi^2 \sqrt{3}}{9} = \frac{6 \pi \sqrt{3} + \pi}{6 - \pi \sqrt{3}}.
\]

(b) Find the area of the polar region determined by the curve.

The area is
\[
\int_{0}^{2\pi} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_{0}^{2\pi} \theta^4 d\theta = \frac{1}{2} \frac{\theta^5}{5} \bigg|_{\theta=0}^{\theta=2\pi} = \frac{(2\pi)^5}{10} = \frac{32\pi^5}{5} = \frac{16\pi^5}{5}.
\]

(c) Set up the integral for the arclength of that polar curve.

The arclength integral is
\[
\int_{0}^{2\pi} \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_{0}^{2\pi} \sqrt{\theta^4 + (2\theta)^2} d\theta.
\]

(d) Compute the integral in part (c) to find the arclength of that polar curve.

\[
\int_{0}^{2\pi} \sqrt{\theta^4 + (2\theta)^2} d\theta = \int_{0}^{2\pi} \theta \sqrt{\theta^2 + 4} d\theta. \quad \text{Then using the substitution} \quad u = \theta^2 + 4 \quad \text{so} \quad du = 2\theta \, d\theta, \quad \text{where} \quad 4 \leq u \leq (2\pi)^2 + 4 = 4\pi^2 + 4, \quad \text{we get}
\]
\[
\int_{0}^{2\pi} \theta \sqrt{\theta^2 + 4} d\theta = \int_{4}^{4\pi^2 + 4} u^{1/2} du/2 = \frac{1}{3} \left[u^{3/2}\right]_{u=4}^{u=4\pi^2 + 4} = \frac{1}{3} \left[(4\pi^2 + 4)^{3/2} - 4^{3/2}\right].
\]

(6) (20 Points) The alternating series \( \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} \) converges to \( S \) and has \( N^{th} \) partial sum \( S_N = \sum_{k=1}^{N} \frac{(-1)^k}{\sqrt{k}} \). Find the smallest \( N \) such that you can be sure \( |S - S_N| < \frac{1}{100} \).

We have \( |S - S_N| < \frac{1}{\sqrt{N+1}} \) from the Alternating Series Estimation theorem, so to guarantee the accuracy given, we need \( \frac{1}{\sqrt{N+1}} < \frac{1}{100} \) which means \( 10^2 = 100 < \sqrt{N+1} \) which means \( 10^8 < N+1 \). The smallest \( N \) such that this is true is \( N = 10^8 = 100,000,000 \).