Manual for SOA Exam FM/CAS Exam 2.

Chapter 1. Basic Interest Theory.

Section 1.1. Amount and accumulation functions.

Extract from:
"Arcones' Manual for the SOA Exam FM/CAS Exam 2, Financial Mathematics. Fall 2009 Edition",
available at http://www.actexmadriver.com/

Interest

- When money is invested or loaned the amount of money returned is different from the initial one.

Interest

- When money is invested or loaned the amount of money returned is different from the initial one.
- The amount of money invested (or loaned) is called the **principal** or **principle**.

Interest

- When money is invested or loaned the amount of money returned is different from the initial one.
- The amount of money invested (or loaned) is called the **principal** or **principle**.
- The **amount of interest** earned during a period of time is

$$I = \text{final balance} - \text{invested amount}.$$

Interest

- When money is invested or loaned the amount of money returned is different from the initial one.
- The amount of money invested (or loaned) is called the **principal** or **principle**.
- The **amount of interest** earned during a period of time is

$$I = \text{final balance} - \text{invested amount}.$$

- The **effective rate of interest** earned in the period $[s, t]$ is

$$\frac{\text{final balance} - \text{invested amount}}{\text{invested amount}}.$$

Example 1

Simon invests \$1000 in a bank account. Six months later, the amount in his bank account is \$1049.23.
(i) Find the amount of interest earned by Simon in those 6 months.
(ii) Find the (semiannual) effective rate of interest earned in those 6 months.

Example 1

Simon invests \$1000 in a bank account. Six months later, the amount in his bank account is \$1049.23.
(i) Find the amount of interest earned by Simon in those 6 months.
(ii) Find the (semiannual) effective rate of interest earned in those 6 months.

Solution: (i) The amount of interest earned by Simon in those 6 months is $I = 1049.23 - 1000 = 49.23$.

Example 1

Simon invests \$1000 in a bank account. Six months later, the amount in his bank account is \$1049.23.
(i) Find the amount of interest earned by Simon in those 6 months.
(ii) Find the (semiannual) effective rate of interest earned in those 6 months.

Solution: (i) The amount of interest earned by Simon in those 6 months is $I = 1049.23 - 1000 = 49.23$.
(ii) The (semiannual) effective rate of interest earned is

$$\frac{1049.23 - 1000}{1000} = 0.004923 = 0.4923\%.$$

Amount function

Suppose that an amount $A(0)$ of money is invested at time 0. $A(0)$ is the principal. Let $A(t)$ denote the value at time t of the initial investment $A(0)$. The function $A(t)$, $t \geq 0$, is called the **amount function**. Usually, we assume that the amount function satisfies the following properties:

(i) For each $t \geq 0$, $A(t) > 0$.

(ii) A is nondecreasing.

Amount function

Suppose that an amount $A(0)$ of money is invested at time 0. $A(0)$ is the principal. Let $A(t)$ denote the value at time t of the initial investment $A(0)$. The function $A(t)$, $t \geq 0$, is called the **amount function**. Usually, we assume that the amount function satisfies the following properties:

(i) For each $t \geq 0$, $A(t) > 0$.

(ii) A is nondecreasing.

In this situation,

- The amount of interest earned over the period $[s, t]$ is

$$A(t) - A(s).$$

Amount function

Suppose that an amount $A(0)$ of money is invested at time 0. $A(0)$ is the principal. Let $A(t)$ denote the value at time t of the initial investment $A(0)$. The function $A(t)$, $t \geq 0$, is called the **amount function**. Usually, we assume that the amount function satisfies the following properties:
(i) For each $t \geq 0$, $A(t) > 0$.
(ii) A is nondecreasing.
In this situation,

- The amount of interest earned over the period $[s, t]$ is
$$A(t) - A(s).$$
- The effective rate of interest earned in the period $[s, t]$ is
$$\frac{A(t) - A(s)}{A(s)}.$$

Example 2

Jessica invests \$5000 on March 1, 2008, in a fund which follows the accumulation function $A(t) = (5000)\left(1 + \frac{t}{40}\right)$, where t is the number of years after March 1, 2008.

(i) Find the balance in Jessica's account on October 1, 2008.

(ii) Find the amount of interest earned in those 7 months.

(iii) Find the effective rate of interest earned in that period.

Example 2

Jessica invests \$5000 on March 1, 2008, in a fund which follows the accumulation function $A(t) = (5000)\left(1 + \frac{t}{40}\right)$, where t is the number of years after March 1, 2008.

(i) Find the balance in Jessica's account on October 1, 2008.

(ii) Find the amount of interest earned in those 7 months.

(iii) Find the effective rate of interest earned in that period.

Solution: (i) The balance of Jessica's account on 10–1–2008 is

$$A(7/12) = (5000)\left(1 + \frac{7/12}{40}\right) = 5072.917.$$

Example 2

Jessica invests \$5000 on March 1, 2008, in a fund which follows the accumulation function $A(t) = (5000)\left(1 + \frac{t}{40}\right)$, where t is the number of years after March 1, 2008.

(i) Find the balance in Jessica's account on October 1, 2008.

(ii) Find the amount of interest earned in those 7 months.

(iii) Find the effective rate of interest earned in that period.

Solution: (i) The balance of Jessica's account on 10–1–2008 is

$$A(7/12) = (5000)\left(1 + \frac{7/12}{40}\right) = 5072.917.$$

(ii) The amount of interest earned in those 7 months is

$$A(7/12) - A(0) = 5072.917 - 5000 = 72.917.$$

Example 2

Jessica invests \$5000 on March 1, 2008, in a fund which follows the accumulation function $A(t) = (5000)\left(1 + \frac{t}{40}\right)$, where t is the number of years after March 1, 2008.

(i) Find the balance in Jessica's account on October 1, 2008.

(ii) Find the amount of interest earned in those 7 months.

(iii) Find the effective rate of interest earned in that period.

Solution: (i) The balance of Jessica's account on 10–1–2008 is

$$A(7/12) = (5000)\left(1 + \frac{7/12}{40}\right) = 5072.917.$$

(ii) The amount of interest earned in those 7 months is

$$A(7/12) - A(0) = 5072.917 - 5000 = 72.917.$$

(iii) The effective rate of interest earned in that period is

$$\frac{A(7/12) - A(0)}{A(0)} = \frac{72.917}{5000} = 0.0145834 = 1.45834\%.$$

Cashflows

Often, we consider the case when several deposits/withdrawals are made into an account following certain amount function. A series of (deposits/withdrawals) payments made at different times is called a **cashflow**. The payments can be either made by the individual or to the individual. An **inflow** is payment to the individual. An **outflow** is a payment by the individual. We represent inflows by positive numbers and outflows by negative numbers. In a cashflow, we have a contribution of C_j at time t_j, for each $j = 1, \dots, n$. C_j can be either positive or negative. We can represent a cashflow in a table:

Investments	C_1	C_2	$\cdots$	C_n
Time (in years)	t_1	t_2	$\cdots$	t_n

Cashflow rules

Rule 1: Proportionality. *If an investment strategy follows the amount function* $A(t)$, $t > 0$, *an investment of \$k made at time 0 with the previous investment strategy, has a value of* $\$\frac{kA(t)}{A(0)}$ *at time* t.

Cashflow rules

Rule 1: Proportionality. *If an investment strategy follows the amount function* $A(t)$, $t > 0$, *an investment of \$k made at time 0 with the previous investment strategy, has a value of* $\$\frac{kA(t)}{A(0)}$ *at time* t.

Using the amount function $A(\cdot)$ and proportionality:

Cashflow rules

Rule 1: Proportionality. *If an investment strategy follows the amount function* $A(t)$, $t > 0$, *an investment of \$k made at time 0 with the previous investment strategy, has a value of* $\$\frac{kA(t)}{A(0)}$ *at time* t.

Using the amount function $A(\cdot)$ and proportionality:

- Investing $A(0)$ at time zero, we get $A(t)$ at time t.

Cashflow rules

Rule 1: Proportionality. *If an investment strategy follows the amount function* $A(t)$, $t > 0$, *an investment of \$k made at time 0 with the previous investment strategy, has a value of* $\$\frac{kA(t)}{A(0)}$ *at time* t.

Using the amount function $A(\cdot)$ and proportionality:

- Investing $A(0)$ at time zero, we get $A(t)$ at time t.
- Investing 1 at time zero, we get $\frac{A(t)}{A(0)}$ at time t.

Cashflow rules

Rule 1: Proportionality. *If an investment strategy follows the amount function* $A(t)$, $t > 0$, *an investment of $k made at time 0 with the previous investment strategy, has a value of* $\$\frac{kA(t)}{A(0)}$ *at time* t.

Using the amount function $A(\cdot)$ and proportionality:

- Investing $A(0)$ at time zero, we get $A(t)$ at time t.
- Investing 1 at time zero, we get $\frac{A(t)}{A(0)}$ at time t.
- Investing k at time zero, we get $\frac{kA(t)}{A(0)}$ at time t.

Present value

Definition 1

The **present value** *at certain time of a cashflow is the amount of the money which need to invest at certain time in other to get the same balance as that obtained from a cashflow.*

Present value

Definition 1

The **present value** *at certain time of a cashflow is the amount of the money which need to invest at certain time in other to get the same balance as that obtained from a cashflow.*

Since investing k at time zero, we get $\frac{kA(t)}{A(0)}$ at time t, we have that: the present value at time t of a deposit of k made at time zero is $\frac{kA(t)}{A(0)}$.

Present value

Definition 1

The **present value** *at certain time of a cashflow is the amount of the money which need to invest at certain time in other to get the same balance as that obtained from a cashflow.*

Since investing k at time zero, we get $\frac{kA(t)}{A(0)}$ at time t, we have that: the present value at time t of a deposit of k made at time zero is $\frac{kA(t)}{A(0)}$.

Let x be the amount which need to invest at time zero to get a balance of k at time t. We have that $k = \frac{xA(t)}{A(0)}$. So, $x = \frac{kA(0)}{A(t)}$. Hence, the present value at time 0 of a balance of k had at time t is $\frac{kA(0)}{A(t)}$.

To know how the value of money changes over time we need to see how the value of \$1 varies over time. The **accumulation function** $a(t)$, $t \geq 0$, is defined as the value at time t of \$1 invested at time 0.

To know how the value of money changes over time we need to see how the value of \$1 varies over time. The **accumulation function** $a(t)$, $t \geq 0$, is defined as the value at time t of \$1 invested at time 0.
By proportionality, $a(t) = \frac{A(t)}{A(0)}$. Observe that $a(0) = 1$.

To know how the value of money changes over time we need to see how the value of \$1 varies over time. The **accumulation function** $a(t)$, $t \geq 0$, is defined as the value at time t of \$1 invested at time 0.

By proportionality, $a(t) = \frac{A(t)}{A(0)}$. Observe that $a(0) = 1$.

Knowing the value function $a(t)$ and the principal $A(0)$, we can find the amount function $A(t)$ using the formula $A(t) = A(0)a(t)$.

To know how the value of money changes over time we need to see how the value of \$1 varies over time. The **accumulation function** $a(t)$, $t \geq 0$, is defined as the value at time t of \$1 invested at time 0.
By proportionality, $a(t) = \frac{A(t)}{A(0)}$. Observe that $a(0) = 1$.
Knowing the value function $a(t)$ and the principal $A(0)$, we can find the amount function $A(t)$ using the formula $A(t) = A(0)a(t)$.
Using the accumulation function $a(t)$, $t \geq 0$, we have:

- The present value at time t of a deposit of k made at time zero is $ka(t)$ $(= \frac{kA(t)}{A(0)})$.

To know how the value of money changes over time we need to see how the value of \$1 varies over time. The **accumulation function** $a(t)$, $t \geq 0$, is defined as the value at time t of \$1 invested at time 0.
By proportionality, $a(t) = \frac{A(t)}{A(0)}$. Observe that $a(0) = 1$.
Knowing the value function $a(t)$ and the principal $A(0)$, we can find the amount function $A(t)$ using the formula $A(t) = A(0)a(t)$.
Using the accumulation function $a(t)$, $t \geq 0$, we have:

- The present value at time t of a deposit of k made at time zero is $ka(t)$ $(= \frac{kA(t)}{A(0)})$.
- The present value at time 0 of a balance of k had at time t is $\frac{k}{a(t)}$ $(= \frac{kA(0)}{A(t)})$.

Example 1

The accumulation function of a fund is $a(t) = (1.03)^{2t}$, $t \geq 0$.
(i) Amanda invests $5000 at time zero in this fund. Find the balance into Amanda's fund at time 2.5 years.
(ii) How much money does Kevin need to invest into the fund at time 0 *to accumulate $10000 at time* 3*?*

Example 1

The accumulation function of a fund is $a(t) = (1.03)^{2t}$, $t \geq 0$.
(i) Amanda invests $5000 at time zero in this fund. Find the balance into Amanda's fund at time 2.5 years.
(ii) How much money does Kevin need to invest into the fund at time 0 *to accumulate $10000 at time* 3*?*

Solution: (i) The balance into Amanda's fund at time 2.5 years is

$$ka(2.5) = (5000)(1.03)^{2(2.5)} = 5796.370371.$$

Example 1

The accumulation function of a fund is $a(t) = (1.03)^{2t}$, $t \geq 0$.
(i) Amanda invests \$5000 at time zero in this fund. Find the balance into Amanda's fund at time 2.5 years.
(ii) How much money does Kevin need to invest into the fund at time 0 *to accumulate \$10000 at time* 3*?*

Solution: (i) The balance into Amanda's fund at time 2.5 years is

$$ka(2.5) = (5000)(1.03)^{2(2.5)} = 5796.370371.$$

(ii) The amount which Kevin needs to invest at time 0 to accumulate \$10000 at time 3 is

$$\frac{10000}{A(3)} = 10000\frac{(1.03)^{2(0)}}{(1.03)^{2(3)}} = (10000)(1.03)^{-6} = 8374.842567.$$

Cashflow rules

Rule 2. Grows–depends–on–balance rule. *If an investment follows the amount function* $A(t)$, $t \geq 0$, *the growth during certain period where no deposits/withdrawals are made depends on the balance on the account at the beginning of the period.*

Cashflow rules

Rule 2. Grows–depends–on–balance rule. *If an investment follows the amount function $A(t)$, $t \geq 0$, the growth during certain period where no deposits/withdrawals are made depends on the balance on the account at the beginning of the period.*

If an account has a balance of k at time t and no deposits/withdrawals are made in the future, then the future balance in this account does not depend on how the balance of k at time t was attained.

Cashflow rules

Rule 2. Grows–depends–on–balance rule. *If an investment follows the amount function $A(t)$, $t \geq 0$, the growth during certain period where no deposits/withdrawals are made depends on the balance on the account at the beginning of the period.*

If an account has a balance of k at time t and no deposits/withdrawals are made in the future, then the future balance in this account does not depend on how the balance of k at time t was attained.

In particular, the following two accounts have the same balance for times bigger than t:

1. An account where a unique deposit of k is made at time t.
2. An account where a unique deposit of $\frac{k}{A(t)}$ is made at time zero.

Theorem 1

If an investment follows the amount function $A(\cdot)$, the present value at time t of a deposit of $\$k$ made at time s is $\$\frac{kA(t)}{A(s)} = \frac{ka(t)}{a(s)}$.

Theorem 1

If an investment follows the amount function $A(\cdot)$, the present value at time t of a deposit of \$$k$ made at time s is $\$\frac{kA(t)}{A(s)} = \frac{ka(t)}{a(s)}$.

Proof. We need to invest $\frac{k}{A(s)}$ at time 0 to get a balance of k at time s. So, investing k at time s is equivalent to investing $\frac{k}{A(s)}$ at time 0. The future value at time t of an investment of $\frac{k}{A(s)}$ at time 0 is $\frac{kA(t)}{A(s)}$. Hence, investing k at time s is equivalent to investing $\frac{kA(t)}{A(s)}$ at time t.

Another way to see the previous theorem is as follows. The following three accounts have the same balance at any time bigger than t:

1. An account where a unique deposit of $A(0)$ is made at time zero.
2. An account where a unique deposit of $A(s)$ is made at time s.
3. An account where a unique deposit of $A(t)$ is made at time t.

Another way to see the previous theorem is as follows. The following three accounts have the same balance at any time bigger than t:

1. An account where a unique deposit of $A(0)$ is made at time zero.
2. An account where a unique deposit of $A(s)$ is made at time s.
3. An account where a unique deposit of $A(t)$ is made at time t.

Hence,

The present value at time t of an investment of $A(s)$ made at time s is $A(t)$.

Another way to see the previous theorem is as follows. The following three accounts have the same balance at any time bigger than t:

1. An account where a unique deposit of $A(0)$ is made at time zero.
2. An account where a unique deposit of $A(s)$ is made at time s.
3. An account where a unique deposit of $A(t)$ is made at time t.

Hence,

The present value at time t of an investment of $A(s)$ made at time s is $A(t)$.

This means that:

- If $t > s$, an investment of $A(s)$ made at time s has an accumulation value of $A(t)$ at time t.
- If $t < s$, to get an accumulation of $A(s)$ at time s, we need to invest $A(t)$ at time t.

We know that:
The present value at time t of an investment of $A(s)$ made at time s is $A(t)$, i.e.

$$A(s) \text{ at time } s \text{ is equivalent to } A(t) \text{ at time } t.$$

We know that:

The present value at time t of an investment of $A(s)$ made at time s is $A(t)$, i.e.

$$A(s) \text{ at time } s \text{ is equivalent to } A(t) \text{ at time } t.$$

By proportionality,

- The present value at time t of an investment of 1 made at time s is $\frac{A(t)}{A(s)}$, i.e.

$$1 \text{ at time } s \text{ is equivalent to } \frac{A(t)}{A(s)} \text{ at time } t.$$

We know that:
The present value at time t of an investment of $A(s)$ made at time s is $A(t)$, i.e.

$$A(s) \text{ at time } s \text{ is equivalent to } A(t) \text{ at time } t.$$

By proportionality,

- The present value at time t of an investment of 1 made at time s is $\frac{A(t)}{A(s)}$, i.e.

$$1 \text{ at time } s \text{ is equivalent to } \frac{A(t)}{A(s)} \text{ at time } t.$$

- The present value at time t of an investment of k made at time s is $\frac{kA(t)}{A(s)}$, i.e.

$$k \text{ at time } s \text{ is equivalent to } \frac{kA(t)}{A(s)} \text{ at time } t.$$

Example 2

The accumulation function of a fund follows the function $a(t) = 1 + \frac{t}{20}$, $t > 0$.

(i) Michael invests \$3500 into the fund at time 1*. Find the value of Michael's fund account at time* 4*.*

(ii) How much money needs Jason to invest at time 2 *to accumulate \$700 at time* 4*.*

Example 2

The accumulation function of a fund follows the function $a(t) = 1 + \frac{t}{20}$, $t > 0$.
(i) Michael invests \$3500 into the fund at time 1. *Find the value of Michael's fund account at time* 4.
(ii) How much money needs Jason to invest at time 2 *to accumulate \$700 at time* 4.

Solution: (i) The value of Michael's account at time 4 is
$3500\frac{a(4)}{a(1)} = (3500)\frac{1+\frac{4}{20}}{1+\frac{1}{20}} = (3500)\frac{1.20}{1.05} = 4000.$

Example 2

The accumulation function of a fund follows the function $a(t) = 1 + \frac{t}{20}$, $t > 0$.
(i) Michael invests \$3500 into the fund at time 1. *Find the value of Michael's fund account at time* 4.
(ii) How much money needs Jason to invest at time 2 *to accumulate \$700 at time* 4.

Solution: (i) The value of Michael's account at time 4 is
$3500\frac{a(4)}{a(1)} = (3500)\frac{1+\frac{4}{20}}{1+\frac{1}{20}} = (3500)\frac{1.20}{1.05} = 4000.$
(ii) To accumulate \$700 at time 4, Jason needs to invest at time 2,
$700\frac{a(2)}{a(4)} = 700\frac{1.1}{1.2} = 641.67.$

Theorem 3

Present value of a cashflow. *If an investment account follows the amount function* $A(t)$, $t > 0$, *the* (**equation of value**) *present value at time t of the cashflow*

Deposits	C_1	C_2	$\cdots$	C_n
Time	t_1	t_2	$\cdots$	t_n

where $0 \le t_1 < t_2 < \cdots < t_n$, *is*

$$V(t) = \sum_{j=1}^{n} C_j \frac{A(t)}{A(t_j)}.$$

Proof. Let $s > t_n$.

Time	Balance before deposit	Balance after deposit
t_1	0	C_1
t_2	$C_1 \frac{a(t_2)}{a(t_1)} = \sum_{j=1}^{1} C_j \frac{a(t_2)}{a(t_j)}$	$C_1 \frac{a(t_2)}{a(t_1)} + C_2 = \sum_{j=1}^{2} C_j \frac{a(t_2)}{a(t_j)}$
t_3	$\sum_{j=1}^{2} C_j \frac{a(t_3)}{a(t_j)}$	$\sum_{j=1}^{3} C_j \frac{a(t_3)}{a(t_j)}$
t_4	$\sum_{j=1}^{3} C_j \frac{a(t_4)}{a(t_j)}$	$\sum_{j=1}^{4} C_j \frac{a(t_4)}{a(t_j)}$
$\cdots$	$\cdots$	$\cdots$
t_n	$\sum_{j=1}^{n-1} C_j \frac{a(t_n)}{a(t_j)}$	$\sum_{j=1}^{n} C_j \frac{a(t_n)}{a(t_j)}$

Since the balance after deposit at time t_1 is C_1, the balance before deposit at time t_2 is $\frac{a(t_2)}{a(t_1)} C_1$.

Since the balance after deposit at time t_2 is $\sum_{j=1}^{2} C_j \frac{a(t_2)}{a(t_j)}$, the balance before deposit at time t_3 is
$\frac{a(t_3)}{a(t_2)} \sum_{j=1}^{2} C_j \frac{a(t_2)}{a(t_j)} = \sum_{j=1}^{2} C_j \frac{a(t_3)}{a(t_j)}$.

Hence, the balance at time s is

$$\frac{a(s)}{a(t_n)}\sum_{j=1}^{n} C_j \frac{a(t_n)}{a(t_j)} = \sum_{j=1}^{n} C_j \frac{a(s)}{a(t_j)}.$$

The future value of the cashflow at time t is

$$\frac{a(t)}{a(s)}\sum_{j=1}^{n} C_j \frac{a(s)}{a(t_j)} = \sum_{j=1}^{n} C_j \frac{a(t)}{a(t_j)}.$$

end of proof.

Notice that the present value at time t of the cashflow

Deposits	C_1	C_2	$\cdots$	C_n
Time	t_1	t_2	$\cdots$	t_n

is the same as the sum of the present values at time t of n separated investment accounts each following the amount function A, with the j–the account having a unique deposit of C_j at time t_j.

Example 4

The accumulation function of a fund follows the function $a(t) = 1 + \frac{t}{20}$, $t > 0$. Jared invests \$1000 into the fund at time 1 and he withdraws \$500 at time 3. Find the value of Jared's fund account at time 5.

Example 4

The accumulation function of a fund follows the function $a(t) = 1 + \frac{t}{20}$, $t > 0$. Jared invests \$1000 into the fund at time 1 and he withdraws \$500 at time 3. Find the value of Jared's fund account at time 5.

Solution: The cashflow is

deposit/withdrawal	1000	−500
Time (in years)	1	3

The value of Jared's account at time 5 is

$$1000\frac{a(5)}{a(1)} - 500\frac{a(5)}{a(3)} = 1000\frac{1+\frac{5}{20}}{1+\frac{1}{20}} - 500\frac{1+\frac{5}{20}}{1+\frac{3}{20}}$$

$$=(1000)\frac{1.25}{1.05} - (500)\frac{1.25}{1.15} = 1190.48 - 543.48 = 647.00.$$

