Manual for SOA Exam FM/CAS Exam 2.

Chapter 5. Bonds.

Section 5.3. Book value and amortization schedules.

Extract from:
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Book value

The **book value** B_k of a bond at time k of a bond is the present value of the payments to be made, i.e. the present value at time k of the remaining $n-k$ coupons and the redemption value C. This is also the outstanding balance of the loan at that time. So, B_k can be found using any of the following expressions:

$$B_k = Fra_{\overline{n-k}|i} + C\nu^{n-k} = C + (Fr - Ci)a_{\overline{n-k}|i} = C + C(g-i)a_{\overline{n-k}|i}.$$

Notice that

$$Fra_{\overline{n-k}|i} + C\nu^{n-k} = Fra_{\overline{n-k}|i} + C(1 - ia_{\overline{n-k}|i})$$
$$= C + (Fr - Ci)a_{\overline{n-k}|i} = C + C(g-i)a_{\overline{n-k}|i}.$$

Of course, we have that $B_0 = P$ and $B_n = C$. The previous formula for B_k is that of the outstanding balance of a loan using the prospective method. Using the retrospective method, the book value of a bond is

$$B_k = P(1+i)^k - Frs_{\overline{k}|i}.$$

The previous formula is equivalent to

$$P = B_k(1+i)^{-k} + Fra_{\overline{k}|i}.$$

A way to interpret the previous formula is as follow. B_k is the balance in the loan after the first k payments are made. The present value of B_k and the payments made until that moment equals the initial balance.

Example 1

Zack buys a 20 year bond with a par value of 4000 and 10% semiannual coupons. He attains an annual yield of 5% convertible semiannually. The redemption value of the bond is 1200. Find the book value of the bond at the end of the 12–th year.

Example 1

Zack buys a 20 year bond with a par value of 4000 and 10% semiannual coupons. He attains an annual yield of 5% convertible semiannually. The redemption value of the bond is 1200. Find the book value of the bond at the end of the 12–th year.

Solution: We have that $F = 4000$, $r = 5\%$, $Fr = (4000)(0.05) = 200$, $n = 40$, $k = 24$ and $C = 1200$. Hence,

$$B_{24} = Fra_{\overline{n-k}|i} + C\nu^{n-k} = (200)a_{\overline{16}|5\%} + (1200)(1.05)^{-16}$$
$$=3419.350452.$$

Example 2

An n*–year* 5000 *par value bond pays* 6% *annual coupons. At annual yield of* 3%*, the book value of the bond at the end of year 7 is 5520. Calculate the price of the bond.*

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Solution: We have that $F = C = 5000$, $r = 6\%$, $Fr = (5000)(0.06) = 300$, $B_7 = 5520$. Hence,

$$P = B_k(1+i)^{-k} + Fra_{\overline{k}|i} = 5520(1.03)^{-7} + (300)a_{\overline{7}|3\%} = 6357.35.$$

Inductive relation for book value

Theorem 1

$$B_{k+1} = B_k(1+i) - Fr.$$

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Proof:

$$\begin{aligned} & B_k(1+i) - Fr = (Fra_{\overline{n-k}|i} + C\nu^{n-k})(1+i) - Fr \\ =& Fr((1+i)a_{\overline{n-k}|i} - 1) + C\nu^{n-k-1} \\ =& Fra_{\overline{n-k-1}|i} + C\nu^{n-k-1} = B_{k+1}. \end{aligned}$$

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B_k is the outstanding balance at time k. One period later, the principal has increased to $B_k(1+i)$, i.e. $B_k(1+i)$ is the outstanding balance immediately before the $(k+1)$–th payment is made. Immediately after the $(k+1)$–th payment to principal of Fr is made, the outstanding balance is $B_{k+1} = B_k(1+i) - Fr$.

Example 3

Consider a 30–year \$50,000 par–value bond with semiannual coupons, with $r = 0.03$*, and yield rate 10%, convertible semiannually.*
(i) Find the book value of the bond immediately after the 25–th coupon payment.
(ii) Find book price immediately before the 26–th coupon payment.
(iii) Find the book value of the bond immediately after the 26–th coupon payment.

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(iii) Find the book value of the bond immediately after the 26–th coupon payment.

Solution: (i) We have that $F = C = 50000$, $i = 0.05$, $Fr = 50000(0.03) = 1500$ and $n = 60$. So,

$$B_{25} = Fra_{\overline{n-k}|i} + C\nu^{n-k} = 1500a_{\overline{35}|0.05} + 50000(1.05)^{-35} = 33625.80571.$$

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(ii) Just before the next coupon payment the book value is $33625.80571(1.05) = 35307.096$.

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(i) Find the book value of the bond immediately after the 25–th coupon payment.
(ii) Find book price immediately before the 26–th coupon payment.
(iii) Find the book value of the bond immediately after the 26–th coupon payment.

Solution: (i) We have that $F = C = 50000$, $i = 0.05$, $Fr = 50000(0.03) = 1500$ and $n = 60$. So,

$$B_{25} = Fra_{\overline{n-k}|i} + C\nu^{n-k} = 1500a_{\overline{35}|0.05} + 50000(1.05)^{-35} = 33625.80571.$$

(ii) Just before the next coupon payment the book value is $33625.80571(1.05) = 35307.096$.
(iii) The book value of the bond immediately after the 26–th coupon payment is $35307.096 - 1500 = 33807.096$.

General inductive relation for book value

By induction from the previous formula, we get that

$$B_{k+m} = B_k(1+i)^m - Frs_{\overline{m}|i}.$$

Notice that:

- B_{k+m} is the outstanding balance immediately after the $k+m$ payment.
- $B_k(1+i)^m$ is the accrued balance at time $k+m$ of the outstanding balance immediately after the k payment.
- $Frs_{\overline{m}|i}$ is the future value at time $k+m$ of the coupon payments $k+1, k+2, \ldots, k+m$, i.e. the coupons payments from when the outstanding was B_k until when the outstanding is B_{k+m}.

Example 4

An n–year 4000 *par value bond with 9% semiannual coupons has an annual nominal yield of i, $i > 0$, convertible semiannually. The book value of the bond at the end of year 4 is 3812.13 and the book value at the end of year 7 is 3884.27. Calculate i.*

Example 4

An n–year 4000 par value bond with 9% semiannual coupons has an annual nominal yield of i, $i > 0$, convertible semiannually. The book value of the bond at the end of year 4 is 3812.13 and the book value at the end of year 7 is 3884.27. Calculate i.

Solution: We have that $F = C = 4000$, $r = 4.5\%$ and $Fr = 4000(0.045) = 180$. The end of year 4 is the end of the 8–th period. The end of year 7 is the end of the 14–th period. Hence, $B_8 = 3812.13$ and $B_{14} = 3884.27$ and

$$3884.27 = 3812.13(1+i)^6 - (180)s_{\overline{6}|i}.$$

using the calculator with

6 | N | −3812.13 | PV | 180 | PMT | 3884.27 | FV | CPT | I/Y

we get that $i = 5\%$ (this is the effective interest rate per period). The annual nominal rate of interest convertible semiannually is $i = 10\%$.

Amount of interest contained in the k–th coupon.

The **amount of interest contained in the k–th coupon** is $I_k = iB_{k-1}$, which can obtained using any of the following formulas:

$$I_k = iB_{k-1} = Fr - (Fr - Ci)\nu^{n-k+1} = Cg - C(g - i)\nu^{n-k+1}.$$

Notice that

$$iB_{k-1} = iFra_{\overline{n+1-k}|i} + iC\nu^{n-k+1} = Fr(1 - \nu^{n-k+1}) + iC\nu^{n-k+1}$$
$$= Fr - (Fr - Ci)\nu^{n-k+1} = Cg - (Cg - Ci)\nu^{n-k+1}$$
$$= Cg - C(g - i)\nu^{n-k+1}.$$

Principal portion in the the k–th coupon.

The principal portion in the k–th coupon is $P_k = B_{k-1} - B_k = Fr - B_{k-1}i = Fr - I_k$ and it can be obtained using any of the following formulas:

$$P_k = (Fr - Ci)\nu^{n-k+1} = C(g - i)\nu^{n-k+1}.$$

P_k is the change in the book value of the bond (principal adjustment) between times $k - 1$ and k. P_k could be either negative, or zero or positive. P_k is the **amortization** in the k–th payment.

Example 5

Kendal buys a 5000 par–value 10 year bond with 8% semiannual coupons to yield 4% converted semiannually. Find the amount of interest and principal in the 5–th coupon.

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Solution: We have that $F = C = 5000$, $r = g = 0.04$, $Fr = 200$, $n = 20$ and $i = 2\%$. Using that $I_k = Fr - (Fr - Ci)\nu^{n+1-k}$ and $P_k = (Fr - Ci)\nu^{n+1-k}$, we get that

$$I_5 = 200 - (200 - (5000)(0.02))(1.02)^{-(20+1-5)} = 127.1554$$

and

$$P_5 = (200 - (5000)(0.02))(1.02)^{-(20+1-5)} = 72.8446.$$

Example 6

For a bond pays annual coupons. The principal amortized in the first coupon is half that amortized in the 10–th coupon. Calculate the annual yield rate.

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Solution: We know that $(Fr - Ci)\nu^n = \frac{1}{2}(Fr - Ci)\nu^{n-9}$. Hence, $(1 + i)^9 = 2$ and $i = 8.005973889\%$.

The amortization schedule of a bond is

Time	Payment	Interest paid (iB_{k-1})	Principal repaid	Book value (B_k)
0	–	–	–	$C + (Fr - Ci)a_{\overline{n}\vert i}$
1	Fr	$Fr - (Fr - Ci)\nu^n$	$(Fr - Ci)\nu^n$	$C + (Fr - Ci)a_{\overline{n-1}\vert i}$
2	Fr	$Fr - (Fr - Ci)\nu^{n-1}$	$(Fr - Ci)\nu^{n-1}$	$C + (Fr - Ci)a_{\overline{n-2}\vert i}$
...	...	...	...	...
k	Fr	$Fr - (Fr - Ci)\nu^{n-k+1}$	$(Fr - Ci)\nu^{n+1-k}$	$C + (Fr - Ci)a_{\overline{n-k}\vert i}$
...	...	...	...	...
$n-1$	Fr	$Fr - (Fr - Ci)\nu^2$	$(Fr - Ci)\nu^2$	$C + (Fr - Ci)a_{\overline{1}\vert i}$
n	$Fr + C$	$Fr - (Fr - Ci)\nu$	$(Fr - Ci)\nu + C$	0

The total payments in a bond are $nFr + C$.
The total coupon interest (sum of the column of interest payments) is $nFr + C - P$.
The total coupon principal (sum of the column of payments to principal) is P.

Notice that for each time k the interest paid plus the principal paid equal to the total payment. We also have that

$$I_k = iB_{k-1} = i\left(C + (Fr - Ci)a_{\overline{n-k+1}|i}\right) = iC + i(Fr - Ci)a_{\overline{n-k+1}|i}$$
$$= Fr + (Fr - Ci)(1 - \nu^{n-k+1}) = Fr - (Fr - Ci)\nu^{n-k+1}.$$

and

$$B_k = B_{k-1} - (Fr - Ci)\nu^{n+1-k}$$
$$= C + (Fr - Ci)a_{\overline{n-k+1}|i} - (Fr - Ci)\nu^{n+1-k}$$
$$= C + (Fr - Ci)(a_{\overline{n-k+1}|i} - \nu^{n+1-k}) = C + (Fr - Ci)a_{\overline{n-k}|i}.$$

Example 7

A 1000 par–value 3–year bond pays 6%, convertible semiannually, and has a yield rate of 8%, convertible semiannually.
(i) What is the interest paid in the 3^{rd} *coupon?*
(ii) What is the change in book value contained in the 3^{rd} *coupon?*
(iii) Construct a bond amortization schedule.

Example 7

A 1000 par–value 3–year bond pays 6%, convertible semiannually, and has a yield rate of 8%, convertible semiannually.
(i) What is the interest paid in the 3^{rd} coupon?
(ii) What is the change in book value contained in the 3^{rd} coupon?
(iii) Construct a bond amortization schedule.

Solution:

(i) We know that $F = C = 1000$, $r = 0.03$, $Fr = 30$, $n = 6$, $i = 0.04$ and $Ci = 40$. The interest paid in the 3^{rd} coupon is

$$I_3 = Fr - (Fr - Ci)\nu^{n-k+1} = 30 - (30 - 40)(1.04)^{-4} = 38.54804191.$$

We also can do

$$B_2 = 30a_{\overline{4}|0.04} + 1000(1.04)^{-4} = 963.7010478$$

and $I_3 = B_2(0.04) = 38.54804191$.

Example 7

A 1000 par–value 3–year bond pays 6%, convertible semiannually, and has a yield rate of 8%, convertible semiannually.
(i) What is the interest paid in the 3^{rd} coupon?
(ii) What is the change in book value contained in the 3^{rd} coupon?
(iii) Construct a bond amortization schedule.

Solution:

(ii) Since $I_3 > Fr$, the book value increases in the 3^{rd} coupon. The increase in the book value in the 3^{rd} coupon is $I_3 - Fr = 38.54804191 - 30 = 8.54804191$. We also can do

$$B_3 = 30a_{\overline{3}|0.04} + 1000(1.04)^{-3} = 972.2490897$$

and $B_3 - B_2 = 972.2490897 - 963.7010478 = 8.5480419$.

(iii) Here is the bond amortization schedule:

Time	Coupon	Interest	Principal Adjustment	Book Value
0				947.58
1	30	37.90	7.90	955.48
2	30	38.22	8.22	963.70
3	30	38.59	8.59	972.25
4	30	38.89	8.89	981.14
5	30	39.25	9.25	990.37
6	30	39.62	9.62	1000.00

The price of a typical bond changes in the opposite direction from a change in interest rates.
As interest rates rise, the price of a bond falls. A bond assures a determined number of payments in the future, if interest rates rise, the present value of these payments decreases. We make an *unrealized capital loss*, if the market value is less than the book value.
Reciprocally, if interest rates decline, the price of a bond rises. The market value of a bond is the price at which a bond is bought/sold. When rates of interest change, the market value of a bond changes. We make an **unrealized capital gain**, if the market value is bigger than the book value.

Example 8

Oliver buys a ten–year 5000 face value bond with semiannual coupons at annual rate of 6%. He buys his bond to yield 8% compounded semiannually and immediately sell them to an investor to yield 4% compounded semiannually. What is Oliver's profit in this investment?

Example 8

Oliver buys a ten–year 5000 face value bond with semiannual coupons at annual rate of 6%. He buys his bond to yield 8% compounded semiannually and immediately sell them to an investor to yield 4% compounded semiannually. What is Oliver's profit in this investment?

Solution: We have that $F = C = 5000$, $r = 0.03$, $Fr = 150$ and $n = 20$. Oliver buys his bond for

$$(150)a_{\overline{20}|0.04} + 5000(1.04)^{-20} = 4320.484$$

Oliver sells his bond for

$$(150)a_{\overline{20}|0.02} + 5000(1.02)^{-20} = 5817.572$$

Oliver's profit is $5817.572 - 4320.484 = 1497.088$.

Example 9

On January 1, 2000, Maxwell bought a 10–year \$5000 non–callable bond with coupons at 7% convertible semiannually. Maxwell bought the bond to yield 7%, compounded semiannually. On July 1, 2005, the market value of bonds is based on a 5% interest rate, compounded semiannually. Calculate the unrealized capital gain on July 1, 2005.

Example 9

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Solution: We have that $F = 5000$, $r = 0.035$ and $Fr = 175$. On July 1, 2005, there are nine remaining coupons. On July 1, 2005, the book value of the bond is

$$175a_{\overline{9}|3.5\%} + 5000(1.035)^{-9} = 5000.$$

On July 1, 2005, the market value of the bond is

$$175a_{\overline{9}|2.5\%} + 5000(1.025)^{-9} = 5398.543276.$$

The unrealized capital gain is $5398.543276 - 5000 = 398.543276$.

Premium.

If the investor is paying more than the redemption value, i.e. if $P > C$, we say that the bond has being bought at **premium**. The premium is $P - C$. We have that

$$P - C = (Fr - Ci)a_{\overline{n}|i} = C(g - i)a_{\overline{n}|i}.$$

A bond has been bought at premium if and only if $g > i$. For a bond bought at premium $Fr > Ci$ and $P = B_0 > B_1 > B_2 > \cdots > B_n = C$. Notice that $B_k - C = (Fr - Ci)a_{\overline{n-k}|i}$. $P_k = B_{k-1} - B_k = (Fr - Ci)\nu^{n+1-k}$ is the write–up in premium in the k–th coupon. The premium is $P - C = \sum_{j=1}^{n}(B_{j-1} - B_j)$.

Discount.

If the investor is paying less than the redemption value, i.e. if $P < C$, we say that the bond has being bought at **discount**. The discount is $C - P$. We have that

$$C - P = (Ci - Fr)a_{\overline{n}|i} = C(i - g)a_{\overline{n}|i}.$$

A bond has been bought at discount if and only if $g < i$. For a bond bought at discount $Fr < Ci$ and
$P = B_0 < B_1 < B_2 < \cdots < B_n = C$.
$|P_k| = B_k - B_{k-1} = (Ci - Fr)\nu^{n+1-k}$ is the write–up in discount in the k–th coupon. The discount is $C - P = \sum_{j=1}^{n}(B_j - B_{j-1})$.

Example 10

A 10 year 50000 face value bond pays semi–annual coupons of 3000. The bond is bought to yield a nominal annual interest rate of 6% convertible semi–annually. Calculate the premium paid for the bond.

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Solution: We know that $F = C = 50000$, $Fr = 3000$, $n = 20$, and $i = 3\%$. The price of the bond is

$$P = Fra_{\overline{n}|i} + C(1+i)^{-n} = (3000)a_{\overline{20}|3\%} + (50000)(1.03)^{-20}$$
$$= 72316.21229.$$

The premium of the bond is
$P - C = 72316.21229 - 50000 = 22316.21229.$

Example 11

Oprah buys a 10000 par–value 15 year bond with 9% semiannual coupons to yield 5% converted semiannually.
(i) Find the premium in the bond.
(ii) Find the write up in premium in the 8–th coupon.

Example 11

Oprah buys a 10000 par–value 15 year bond with 9% semiannual coupons to yield 5% converted semiannually.
(i) Find the premium in the bond.
(ii) Find the write up in premium in the 8–th coupon.

Solution: (i) We have that $F = C = 10000$, $r = 0.045$, $Fr = 450$ and $n = 30$. Oprah buys her bond for

$$(450)a_{\overline{30}|2.5\%} + 10000(1.025)^{-30} = 14186.06$$

The premium of the bond is $P - C = 14186.06 - 10000 = 4186.06$.

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(i) Find the premium in the bond.
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Solution: (i) We have that $F = C = 10000$, $r = 0.045$, $Fr = 450$ and $n = 30$. Oprah buys her bond for

$$(450)a_{\overline{30}|2.5\%} + 10000(1.025)^{-30} = 14186.06$$

The premium of the bond is $P - C = 14186.06 - 10000 = 4186.06$.
(ii) The write–up in premium in the 8–th coupon is

$$(Fr - Ci)\nu^{n+1-k} = (10000)(0.045 - 0.025)(1.025)^{-23} = 113.34.$$

We also can do the problem in the following way:

$$B_7 = (450)a_{\overline{23}|2.5\%} + 10000(1.025)^{-23} = 13466.42$$

$$B_8 = (450)a_{\overline{22}|2.5\%} + 10000(1.025)^{-22} = 13353.08$$

and $B_7 - B_8 = 13466.42 - 13353.08 = 113.34$.

