Manual for SOA Exam MLC.

Chapter 6. Benefit premiums.

Section 6.4. Benefit premiums for fully continuous insurance.

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In section, we will consider the funding of insurance products paid at the time of death and funded continuously. This type of insurance is called **fully continuous**.

Whole life annuity

Suppose that an insurance company funds a continuous whole life insurance with unity payment with payments at a continuous rate of P while the individual is alive. The loss random variable is

$$\overline{L}(\overline{A}_x) = v^{T_x} - P\overline{a}_{\overline{T_x}|} = \overline{Z}_x - P\overline{Y}_x.$$

Using that $\overline{Y}_x = \frac{1-\overline{Z}_x}{\delta}$,

$$\overline{L}(\overline{A}_x) = \overline{Z}_x - \frac{1-\overline{Z}_x}{\delta} = \overline{Z}_x\left(1+\frac{P}{\delta}\right) - \frac{P}{\delta} = e^{-\delta T_x}\left(1+\frac{P}{\delta}\right) - \frac{P}{\delta}.$$

The biggest value of the loss is attained when $T_x = 0$ and it is one. The smallest value of the loss is attained when $T_x = \omega - x$ and it is $e^{-\delta(\omega-x)}\left(1+\frac{P}{\delta}\right) - \frac{P}{\delta}$. $\overline{L}(\overline{A}_x)$ is continuous random variable taking values on the interval $(e^{-\delta(\omega-x)}\left(1+\frac{P}{\delta}\right) - \frac{P}{\delta}, 1)$.

The actuarial present value of the loss is

$$E[\overline{L}(\overline{A}_x)] = \overline{A}_x - P\overline{a}_x = \overline{A}_x\left(1+\frac{P}{\delta}\right) - \frac{P}{\delta}.$$

The variance of the present value of the loss is

$$\mathrm{Var}(\overline{L}(\overline{A}_x)) = \left(1+\frac{P}{\delta}\right)^2 \mathrm{Var}(\overline{Z}_x) = \left(1+\frac{P}{\delta}\right)^2 \left({}^2\overline{A}_x - \overline{A}_x{}^2\right).$$

The probability that the loss is positive is

$$\mathbb{P}\{\overline{L}(\overline{A}_x) > 0\} = \mathbb{P}\{v^{T_x} - P\overline{a}_{\overline{T_x}|} > 0\} = \mathbb{P}\{v^{T_x} > P\overline{a}_{\overline{T_x}|}\}$$
$$= \mathbb{P}\left\{\frac{1}{\overline{s}_{\overline{T_x}|}} > P\right\}.$$

Example 1

An insurer offers a whole life insurance of 1000 paid at the time of death. To fund this insurance the policyholder must make continuous payments at the rate 125. The force of interest is 0.06. The force of mortality is 0.01.

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(i) Calculate the expected loss at issue.

Example 1

An insurer offers a whole life insurance of 1000 paid at the time of death. To fund this insurance the policyholder must make continuous payments at the rate 125. The force of interest is 0.06. The force of mortality is 0.01.

(i) Calculate the expected loss at issue.

Solution: (i) We have that

$$\overline{A}_x = \frac{0.01}{0.01 + 0.06} = \frac{1}{7},$$

$$E[(1000)\overline{L}(\overline{A}_x)] = (1000)\overline{A}_x \left(1 + \frac{P}{\delta}\right) - (1000)\frac{P}{\delta}$$

$$= \frac{1}{7}\left(1000 + \frac{125}{0.06}\right) - \frac{125}{0.06} = -35.71428571.$$

Example 1

An insurer offers a whole life insurance of 1000 paid at the time of death. To fund this insurance the policyholder must make continuous payments at the rate 125. The force of interest is 0.06. The force of mortality is 0.01.

(ii) Calculate the variance of the loss at issue random variable.

Example 1

An insurer offers a whole life insurance of 1000 paid at the time of death. To fund this insurance the policyholder must make continuous payments at the rate 125. The force of interest is 0.06. The force of mortality is 0.01.

(ii) Calculate the variance of the loss at issue random variable.

Solution: (ii) We have that

$$^2\overline{A}_x = \frac{0.01}{0.01 + (2)0.06} = \frac{1}{13},$$

$$\mathrm{Var}((1000)\overline{L}(\overline{A}_x)) = (1000)^2 \left(1 + \frac{P}{\delta}\right)^2 \left({}^2\overline{A}_x - \overline{A}_x{}^2\right)$$

$$= \left(1000 + \frac{125}{0.06}\right)^2 \left(\frac{1}{13} - \left(\frac{1}{7}\right)^2\right) = 82515.69859.$$

Example 1

An insurer offers a whole life insurance of 1000 paid at the time of death. To fund this insurance the policyholder must make continuous payments at the rate 125. The force of interest is 0.06. The force of mortality is 0.01.

(iii) Calculate the probability that the loss at issue is positive.

Example 1

An insurer offers a whole life insurance of 1000 paid at the time of death. To fund this insurance the policyholder must make continuous payments at the rate 125. The force of interest is 0.06. The force of mortality is 0.01.

(iii) Calculate the probability that the loss at issue is positive.

Solution: (iii) The probability that the loss is positive is

$$\begin{aligned}
&\mathbb{P}\{(1000)v^{T_x} - (125)\overline{a}_{\overline{T_x}|} > 0\} \\
=&\mathbb{P}\left\{(1000)e^{-(0.06)T_x} > (125)\frac{1-e^{-(0.06)T_x}}{0.06}\right\} \\
=&\mathbb{P}\left\{(60)e^{-(0.06)T_x} > 125 - 125e^{-(0.06)T_x}\right\} \\
=&\mathbb{P}\left\{(185)e^{-(0.06)T_x} > 125\right\} = \mathbb{P}\{1.48 > e^{(0.06)T_x}\} \\
=&\mathbb{P}\left\{T_x < \frac{\ln(1.48)}{0.06}\right\} = 1 - e^{-(0.01)\frac{\ln(1.48)}{0.06}} = 1 - (1.48)^{-1/6} = 0.0632514
\end{aligned}$$

The benefit premium of fund a whole life insurance is funded under the equivalence principle is denoted by $\overline{P}(\overline{A}_x)$. We have that

$$\overline{P}(\overline{A}_x) = \frac{\overline{A}_x}{\overline{a}_x} = \frac{\delta \overline{A}_x}{1 - \overline{A}_x} = \frac{1}{\overline{a}_x} - \delta.$$

In this case

$$\overline{L}(\overline{A}_x) = \overline{Z}_x - \overline{P}(\overline{A}_x)\overline{Y}_x = \overline{Z}_x \left(1 + \frac{\overline{P}(\overline{A}_x)}{\delta}\right) - \frac{\overline{P}(\overline{A}_x)}{\delta} = \frac{\overline{Z}_x - \overline{A}_x}{1 - \overline{A}_x},$$

$$\mathrm{Var}(\overline{L}(\overline{A}_x)) = \left(1 + \frac{\overline{P}(\overline{A}_x)}{\delta}\right)^2 \left({}^2\overline{A}_x - \overline{A}_x{}^2\right) = \frac{{}^2\overline{A}_x - \overline{A}_x{}^2}{(1 - \overline{A}_x)^2} = \frac{{}^2\overline{A}_x - \overline{A}_x{}^2}{(\delta \overline{a}_x)^2}$$

and the probability that the loss is positive is

$$\mathbb{P}\{\overline{L}(\overline{Z}_x) > 0\} = \mathbb{P}\left\{\frac{\overline{Z}_x - \overline{A}_x}{1 - \overline{A}_x} > 0\right\} = \mathbb{P}\{\overline{Z}_x > \overline{A}_x\}.$$

This is the probability that $\overline{A}_x$ is not adequate.

Theorem 1

Under constance force of mortality μ, $\overline{P}(\overline{A}_x) = \mu$.

Proof: We have that

$$\overline{P}(\overline{A}_x) = \frac{\delta \overline{A}_x}{1 - \overline{A}_x} = \frac{\delta \frac{\mu}{\mu+\delta}}{1 - \frac{\mu}{\mu+\delta}} = \mu.$$

Example 2

An insurer offers a whole life insurance of 1000 paid at the time of death. To fund this insurance the policyholder must make continuous payments. The rate of payments is found using the equivalence principle. $\delta = 0.06$. $\mu = 0.01$.

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An insurer offers a whole life insurance of 1000 paid at the time of death. To fund this insurance the policyholder must make continuous payments. The rate of payments is found using the equivalence principle. $\delta = 0.06$. $\mu = 0.01$.

(i) Calculate the rate of benefit premiums.

Example 2

An insurer offers a whole life insurance of 1000 paid at the time of death. To fund this insurance the policyholder must make continuous payments. The rate of payments is found using the equivalence principle. $\delta = 0.06$. $\mu = 0.01$.

(i) Calculate the rate of benefit premiums.

Solution: (i) The rate of benefit premiums is

$$(1000)\overline{P}_x = (1000)(0.01) = 10.$$

Example 2

An insurer offers a whole life insurance of 1000 paid at the time of death. To fund this insurance the policyholder must make continuous payments. The rate of payments is found using the equivalence principle. $\delta = 0.06$*.* $\mu = 0.01$*.*

(ii) Calculate the variance of the loss at issue random variable.

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An insurer offers a whole life insurance of 1000 paid at the time of death. To fund this insurance the policyholder must make continuous payments. The rate of payments is found using the equivalence principle. $\delta = 0.06$. $\mu = 0.01$.

(ii) Calculate the variance of the loss at issue random variable.

Solution: (ii) We have that

$$\overline{A}_x = \frac{0.01}{0.01 + 0.06} = \frac{1}{7},$$

$$^2\overline{A}_x = \frac{0.01}{0.01 + (2)0.06} = \frac{1}{13},$$

$$\mathrm{Var}((1000)\overline{L}(\overline{A}_x)) = (1000)^2 \frac{^2\overline{A}_x - \overline{A}_x{}^2}{(1 - \overline{A}_x)^2} = (1000)^2 \frac{\frac{1}{13} - \left(\frac{1}{7}\right)^2}{\left(1 - \frac{1}{7}\right)^2}$$

$$= 76923.07692.$$

Example 2

An insurer offers a whole life insurance of 1000 paid at the time of death. To fund this insurance the policyholder must make continuous payments. The rate of payments is found using the equivalence principle. $\delta = 0.06$. $\mu = 0.01$.

(iii) Calculate the probability that the loss at issue is positive.

Example 2

An insurer offers a whole life insurance of 1000 paid at the time of death. To fund this insurance the policyholder must make continuous payments. The rate of payments is found using the equivalence principle. $\delta = 0.06$. $\mu = 0.01$.

(iii) Calculate the probability that the loss at issue is positive.

Solution: (iii) The probability that the loss is positive is

$$\begin{aligned}
&\mathbb{P}\{(1000)v^{T_x} - (10)\bar{a}_{\overline{T_x}|} > 0\} \\
=&\mathbb{P}\left\{(1000)e^{-(0.06)T_x} > (10)\frac{1-e^{-(0.06)T_x}}{0.06}\right\} \\
=&\mathbb{P}\left\{(60)e^{-(0.06)T_x} > 10-(10)e^{-(0.06)T_x}\right\} = \mathbb{P}\left\{e^{-(0.06)T_x} > \frac{1}{7}\right\} \\
=&\mathbb{P}\{e^{(0.06)T_x} < 7\} = \mathbb{P}\left\{T_x < \frac{\ln(7)}{0.06}\right\} = 1 - e^{-(0.01)\frac{\ln(7)}{0.06}} \\
=&1-(7)^{-1/6} = 0.2769799736.
\end{aligned}$$

Theorem 2

The cumulative distribution function of $\overline{L}(\overline{A}_x)$ is

$$F_{\overline{L}(\overline{A}_x)}(u) = \begin{cases} 0 & \text{if } u < e^{-\delta(\omega-x)} - P\overline{a}_{\overline{\omega-x}|}, \\ 1 - F_{T_x}\left(-\frac{1}{\delta}\log\left(\frac{\delta u+P}{\delta+P}\right)\right) & \text{if } e^{-\delta(\omega-x)} - P\overline{a}_{\overline{\omega-x}|} < u \leq 1, \\ 1 & \text{if } 1 \leq u. \end{cases}$$

Proof: Consider the function $h : (0,\infty) \to \left(e^{-\delta(\omega-x)} - P\overline{a}_{\overline{\omega-x}|}, 1\right)$ defined by $h(x) = e^{-\delta x}\left(1 + \frac{P}{\delta}\right) - \frac{P}{\delta}$. We have that h is decreasing. If $u = h(x) = e^{-\delta x}\left(1 + \frac{P}{\delta}\right) - \frac{P}{\delta}$, then $e^{-\delta x} = \frac{\delta u+P}{\delta+P}$, and $x = -\frac{1}{\delta}\ln\left(\frac{\delta u+P}{\delta+P}\right)$. Hence, the inverse function of h is $h^{-1}(u) = -\frac{1}{\delta}\ln\left(\frac{\delta u+P}{\delta+P}\right)$, $e^{-\delta(\omega-x)} - P\overline{a}_{\overline{\omega-x}|} < u < 1$.

Hence, if $e^{-\delta(\omega-x)} - P\overline{a}_{\overline{\omega-x}|} < u < 1$,

$$F_{\overline{L}(\overline{A}_x)}(u) = \mathbb{P}\{\overline{L}_x \le u\} = \mathbb{P}\{h(T_x) \le u\} = \mathbb{P}\left\{T_x \ge h^{-1}(u)\right\}$$
$$= \mathbb{P}\left\{T_x \ge -\frac{1}{\delta}\ln\left(\frac{\delta u + P}{\delta + P}\right)\right\} = 1 - F_{T_x}\left(-\frac{1}{\delta}\ln\left(\frac{\delta u + P}{\delta + P}\right)\right).$$

Theorem 3

The probability density function of $\overline{L}(\overline{A}_x)$ is

$$f_{\overline{L}(\overline{A}_x)}(u) = \begin{cases} \frac{f_{T_x}\left(-\frac{1}{\delta}\log\left(\frac{\delta u+P}{\delta+P}\right)\right)}{\delta u+P} & \text{if } e^{-\delta(\omega-x)} - P\overline{a}_{\overline{\omega-x}|} < u < 1, \\ 0 & \text{else.} \end{cases}$$

Proof.

For $e^{-\delta(\omega-x)} - P\overline{a}_{\overline{\omega-x}|} < u < 1$,

$$f_{\overline{L}(\overline{A}_x)}(u) = \frac{d}{du}F_{\overline{L}(\overline{A}_x)}(u) = 1 - \frac{d}{du}F_{T_x}\left(-\frac{1}{\delta}\log\left(\frac{\delta u+P}{\delta+P}\right)\right)$$

$$= \frac{f_{T_x}\left(-\frac{1}{\delta}\log\left(\frac{\delta u+P}{\delta+P}\right)\right)}{\delta u+P}.$$

□

Corollary 1

Under constant force of mortality μ, $\overline{L}(\overline{A}_x)$ is a continuous r.v. with cumulative distribution function

$$F_{\overline{L}(\overline{A}_x)}(u) = \left(\frac{\delta u + P}{\delta + P}\right)^{\frac{\mu}{\delta}}, \quad -\frac{P}{\delta} < u < 1$$

and density function

$$f_{\overline{L}(\overline{A}_x)}(u) = \mu \frac{(\delta u + P)^{\frac{\mu}{\delta} - 1}}{(\delta + P)^{\frac{\mu}{\delta}}}, \quad -\frac{P}{\delta} < u < 1.$$

Proof: Since $F_{T_x}(t) = 1 - e^{-\mu t}$, if $0 \le t < \infty$; we have that for $-\frac{P}{\delta} < u < 1$,

$$F_{\overline{L}(\overline{A}_x)}(u) = 1 - F_{T_x}\left(-\frac{1}{\delta}\ln\left(\frac{\delta u + P}{\delta + P}\right)\right) = \exp\left(\frac{\mu}{\delta}\ln\left(\frac{\delta u + P}{\delta + P}\right)\right)$$

$$= \left(\frac{\delta u + P}{\delta + P}\right)^{\frac{\mu}{\delta}}.$$

$\overline{L}(\overline{A}_x)$ has density

$$f_{\overline{L}(\overline{A}_x)}(u) = F'_{\overline{L}(\overline{A}_x)}(u) = \mu\frac{(\delta u + P)^{\frac{\mu}{\delta}-1}}{(\delta + P)^{\frac{\mu}{\delta}}}, \quad -\frac{P}{\delta} < u < 1.$$

The 100α–th percentile annual premium for a whole life fully continuous insurance to (x) is the value P_α such that

$$\alpha \geq \mathbb{P}\{\overline{Z}_x - P_\alpha \overline{Y}_x > 0\} = \mathbb{P}\{v^{T_x} - P_\alpha \overline{a}_{\overline{T_x}|} > 0\} = \mathbb{P}\left\{\frac{1}{\overline{s}_{\overline{T_x}|}} > P_\alpha\right\}.$$

P_α is a $100(1-\alpha)$–th percentile of $\frac{1}{\overline{s}_{\overline{T_x}|}}$. We have that $\frac{1}{\overline{s}_{\overline{t}|}} = \frac{1}{\int_0^t (1+i)^u \, du}$, $t \geq 0$, is a decreasing function of t. Hence, $P_\alpha = \frac{1}{\overline{s}_{\overline{t_\alpha}|}}$, where $t_\alpha \in \mathbb{R}$ is a 100α–th percentile of T_x. Indeed, suppose that $\mathbb{P}\{T_x < t_\alpha\} = \alpha$. Then,

$$\mathbb{P}\{\overline{Z}_x - P_\alpha \overline{Y}_x > 0\} = \mathbb{P}\left\{\frac{1}{\overline{s}_{\overline{T_x}|}} > \frac{1}{\overline{s}_{\overline{t_\alpha}|}}\right\} = \mathbb{P}\{T_x < t_\alpha\} = \alpha.$$

Example 3

An insurer offers a whole life insurance policy with face value of 40000 payable at the time of death to (35). This policy will be paid continuously while (35) is alive. Assume that $\delta = 0.045$ and death is modeled using the De Moivre model with terminal age 95. Calculate the 10–th percentile annual benefit premium for this policy.

Solution: Let $t_{0.10}$ be such that

$$0.10 = \mathbb{P}\{T_{35} \le t_{0.10}\} = \frac{t_{0.10}}{60},$$

i.e. $t_{0.10} = (0.10)(60) = 6$. Hence,

$$P_{0.10} = \frac{1}{\bar{s}_{\overline{6}|}} = \frac{\delta}{e^{\delta(6)} - 1} = \frac{0.045}{e^{(0.045)(6)} - 1} = 0.1451779386.$$

The benefit annual premium is
$(40000)(0.1451779386) = 5807.117544$.

Suppose that the funding scheme is limited to the first t years. The present value of the loss is

$$v^{T_x} - P\bar{a}_{\overline{\min(T_x,t)}|} = \overline{Z}_x - P\overline{Y}_{x:\overline{t}|}.$$

The present value of the loss is

$$\overline{A}_x - P\bar{a}_{x:\overline{t}|}.$$

The benefit premium which satisfies the equivalence principle is

$${}_t\overline{P}(\overline{A}_x) = \frac{\overline{A}_x}{\bar{a}_{x:\overline{t}|}}.$$

n–year term insurance.

The present value of the loss for a n–year term insurance is

$$\overline{L}(\overline{A}^{1}_{x:\overline{n}|}) = v^{T_x} I(T_x \le n) - P\overline{a}_{\overline{\min(T_x,n)}|} = \overline{Z}^{1}_{x:\overline{n}|} - P\overline{Y}_{x:\overline{n}|}.$$

The actuarial present value of the loss for a n–year term insurance is

$$\overline{A}^{1}_{x:\overline{n}|} - P\overline{a}_{x:\overline{n}|}.$$

The benefit premium which satisfies the equivalence principle is

$$\overline{P}(\overline{A}^{1}_{x:\overline{n}|}) = \frac{\overline{A}^{1}_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}}.$$

Theorem 4

We have that

$$\mathrm{Var}(\overline{L}(\overline{A}^{1}_{x:\overline{n}|})) = \left(1+\frac{P}{\delta}\right)^2 \cdot {}^2\overline{A}^{1}_{x:\overline{n}|} + \frac{P^2}{\delta^2} \cdot {}^2\overline{A}_{x:\overset{1}{\overline{n}|}} - \left(E\left[\overline{L}(\overline{A}^{1}_{x:\overline{n}|})\right] + \frac{P}{\delta}\right)^2$$

Proof: We have that

$$\overline{L}(\overline{A}^{1}_{x:\overline{n}|}) = \overline{Z}^{1}_{x:\overline{n}|} - P\frac{1-\overline{Z}_{x:\overline{n}|}}{\delta} = -\frac{P}{\delta} + \left(1+\frac{P}{\delta}\right)\overline{Z}^{1}_{x:\overline{n}|} + \frac{P}{\delta}\overline{Z}_{x:\overline{n}|}^{\;\;1}$$

and

$$\overline{Z}^{1}_{x:\overline{n}|}\overline{Z}_{x:\overline{n}|}^{\;\;1} = v^{T_x}I(T_x \le n)v^n I(n < T_x) = 0.$$

Hence,

$$E\left[\left(\overline{L}\left(\overline{A}^{1}_{x:\overline{n}|}\right) + \frac{P}{\delta}\right)^2\right] = E\left[\left(1+\frac{P}{\delta}\right)^2\left(\overline{Z}^{1}_{x:\overline{n}|}\right)^2 + \frac{P^2}{\delta^2}\left(\overline{Z}_{x:\overline{n}|}^{\;\;1}\right)^2\right]$$

$$= \left(1+\frac{P}{\delta}\right)^2 \cdot {}^2\overline{A}^{1}_{x:\overline{n}|} + \frac{P^2}{\delta^2}\cdot {}^2\overline{A}_{x:\overline{n}|}^{\;\;1}$$

and

$$\mathrm{Var}(\overline{L}(\overline{A}^{1}_{x:\overline{n}|})) = \mathrm{Var}\left(\overline{L}(\overline{A}^{1}_{x:\overline{n}|}) + \frac{P}{\delta}\right)$$

$$= \left(1+\frac{P}{\delta}\right)^2 \cdot {}^2\overline{A}^{1}_{x:\overline{n}|} + \frac{P^2}{\delta^2}\cdot {}^2\overline{A}_{x:\overline{n}|}^{\;\;1} - \left(E\left[\overline{L}(\overline{A}^{1}_{x:\overline{n}|})\right] + \frac{P}{\delta}\right)^2.$$

The present value of the loss for a t–year funded n–year term insurance is

$$v^{T_x} I(K_x \le n) - P\overline{a}_{\overline{\min(K_x,t)}|} = \overline{Z}^{1}_{x:\overline{n}|} - P\overline{Y}_{x:\overline{t}|}.$$

The actuarial present value of the loss for a t–year funded n–year term insurance is

$$\overline{A}^{1}_{x:\overline{n}|} - P\overline{a}_{x:\overline{t}|}.$$

The benefit premium which satisfies the equivalence principle is

$${}_t\overline{P}(\overline{A}^{1}_{x:\overline{n}|}) = \frac{\overline{A}^{1}_{x:\overline{n}|}}{\overline{a}_{x:\overline{t}|}}.$$

n–year pure endowment.

The present value of the loss for a n–year pure endowment is

$$\overline{L}(\overline{A}_{x:\overline{n}|}^{\;\;1}) = v^n I(T_x > n) - P\overline{a}_{\overline{\min(T_x,n)}|} = \overline{Z}_{x:\overline{n}|}^{\;\;1} - P\overline{Y}_{x:\overline{n}|}.$$

The actuarial present value of the loss for a n–year term insurance is

$$\overline{A}_{x:\overline{n}|}^{\;\;1} - P\overline{a}_{x:\overline{n}|}.$$

The benefit premium which satisfies the equivalence principle is

$$\overline{P}(\overline{A}_{x:\overline{n}|}^{\;\;1}) = \frac{\overline{A}_{x:\overline{n}|}^{\;\;1}}{\overline{a}_{x:\overline{n}|}}.$$

Theorem 5

We have that

$$\mathrm{Var}(\overline{L}(\overline{A}_{x:\overline{n}|}^{\,1})) = \frac{P^2}{\delta^2} \cdot {}^2\overline{A}_{x:\overline{n}|}^{1} + \left(1 + \frac{P}{\delta}\right)^2 \cdot {}^2\overline{A}_{x:\overline{n}|}^{\,1} - \left(E\left[\overline{L}(\overline{A}_{x:\overline{n}|}^{\,1})\right] + \frac{P}{\delta}\right)^2$$

Proof: We have that

$$\overline{L}(\overline{A}_{x:\overline{n}|}^{1}) = \overline{Z}_{x:\overline{n}|}^{\,1} - P\frac{1 - \overline{Z}_{x:\overline{n}|}}{\delta} = -\frac{P}{\delta} + \frac{P}{\delta}\overline{Z}_{x:\overline{n}|}^{1} + \left(1 + \frac{P}{\delta}\right)\overline{Z}_{x:\overline{n}|}^{\,1}.$$

The rest of the proof follows similarly to that of the previous theorem.

If a n–year pure endowment is funded only t years where $1 \leq t \leq n$, then the present value of the loss for a n–year pure endowment is

$$v^n I(T_x > n) - P\overline{a}_{\overline{\min(T_x,t)}|} = \overline{Z}_{x:\overline{n}|}^{\ \ 1} - P\overline{Y}_{x:\overline{t}|}.$$

The actuarial present value of the loss for a t–year funded n–year term insurance is

$$\overline{A}_{x:\overline{n}|}^{\ \ 1} - P\overline{a}_{x:\overline{t}|}.$$

The benefit premium which satisfies the equivalence principle is

$${}_t\overline{P}(\overline{A}_{x:\overline{n}|}^{\ \ 1}) = \frac{\overline{A}_{x:\overline{n}|}^{\ \ 1}}{\overline{a}_{x:\overline{t}|}}.$$

n–year endowment.

The present value of the loss for a n–year endowment is

$$\overline{L}(\overline{A}_{x:\overline{n}|}) = v^{\min(n,T_x)} - P\overline{a}_{\overline{\min(T_x,n)}|} = \overline{Z}_{x:\overline{n}|} - P\overline{Y}_{x:\overline{n}|}$$

$$= \overline{Z}_{x:\overline{n}|} - P\frac{1-\overline{Z}_{x:\overline{n}|}}{\delta} = -\frac{P}{\delta} + \left(1+\frac{P}{\delta}\right)\overline{Z}_{x:\overline{n}|}.$$

The actuarial present value of the loss for a n–year term insurance is

$$\overline{A}_{x:\overline{n}|} - P\overline{a}_{x:\overline{n}|}.$$

The variance of the present value of the loss for a n–year term insurance is

$$\mathrm{Var}(\overline{L}(\overline{A}_{x:\overline{n}|})) = \left(1+\frac{P}{\delta}\right)\left({}^2\overline{A}_{x:\overline{n}|} - \left(\overline{A}_{x:\overline{n}|}\right)^2\right).$$

The annual benefit premium which satisfies the equivalence principle is

$$\overline{P}(\overline{A}_{x:\overline{n}|}) = \frac{\overline{A}_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}} = \frac{1-\delta \overline{a}_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}} = \frac{\delta \overline{A}_{x:\overline{n}|}}{1-\overline{A}_{x:\overline{n}|}}.$$

When the annual benefit premium is used

$$\mathrm{Var}(L) = \frac{{}^2\overline{A}_{x:\overline{n}|} - \overline{A}_{x:\overline{n}|}^{\,2}}{\left(1-\overline{A}_{x:\overline{n}|}\right)^2}.$$

The present value of the loss for a t–year funded n–year endowment is

$$v^{\min(n,T_x)} - P\overline{a}_{\overline{\min(T_x,n)}|} = \overline{Z}_{x:\overline{n}|} - P\overline{Y}_{x:\overline{t}|}.$$

The actuarial present value of the loss for a t–year funded n–year term insurance is

$$\overline{A}_{x:\overline{n}|} - P\overline{a}_{x:\overline{t}|}.$$

The benefit premium which satisfies the equivalence principle is

$${}_t\overline{P}(\overline{A}_{x:\overline{n}|}) = \frac{A_{x:\overline{n}|}}{\overline{a}_{x:\overline{t}|}}.$$

Theorem 6

$$\overline{P}(\overline{A}_{x:\overline{n}|}) = \overline{P}(\overline{A}^{1}_{x:\overline{n}|}) + \overline{P}(\overline{A}_{x:\overset{1}{\overline{n}|}}).$$

Proof.

We have that

$$\overline{P}(\overline{A}^{1}_{x:\overline{n}|}) + \overline{P}(\overline{A}_{x:\overset{1}{\overline{n}|}}) = \frac{\overline{A}^{1}_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}} + \frac{\overline{A}_{x:\overset{1}{\overline{n}|}}}{\overline{a}_{x:\overline{n}|}} = \frac{\overline{A}_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}} = \overline{P}(\overline{A}_{x:\overline{n}|}).$$

□

Theorem 7

$$ {}_n\overline{P}(\overline{A}_x) = \overline{P}(\overline{A}^{1}_{x:\overline{n}|}) + \overline{P}(\overline{A}_{x:\overline{n}|}^{\ \ \ 1})\overline{A}_{x+n}. $$

Proof.

We have that

$$ \overline{P}(\overline{A}^{1}_{x:\overline{n}|}) + \overline{P}(\overline{A}_{x:\overline{n}|}^{\ \ \ 1})\overline{A}_{x+n} = \frac{\overline{A}^{1}_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}} + \frac{\overline{A}_{x:\overline{n}|}^{\ \ \ 1}}{\overline{a}_{x:\overline{n}|}}\overline{A}_{x+n} = \frac{\overline{A}^{1}_{x:\overline{n}|} + {}_{n}|\overline{A}_x}{\overline{a}_{x:\overline{n}|}} $$

$$ = \frac{\overline{A}_x}{\overline{a}_{x:\overline{n}|}} = {}_n\overline{P}(\overline{A}_x). $$

□

n–year deferred insurance.

The present value of the loss for a n–year deferred insurance funded continuously at rate P is

$$\overline{L}({}_{n}|\overline{A}_x) = v^{T_x} I(T_x > n) - P\overline{a}_{\overline{\min(T_x,n)}|} = {}_{n}|\overline{Z}_x - P\overline{Y}_x.$$

The actuarial present value of the loss for a n–year term insurance is

$$_{n}|\overline{A}_x - P\overline{a}_x.$$

The benefit premium which satisfies the equivalence principle is

$$\overline{P}({}_{n}|\overline{A}_x) = \frac{{}_{n}|\overline{A}_x}{\overline{a}_x}.$$

Theorem 8

We have that

$$\mathrm{Var}(\overline{L}({}_{n}|\overline{A}_{x})) = \frac{P^2}{\delta^2} \cdot {}^2\overline{A}^{1}_{x:\overline{n}|} + \left(1 + \frac{P}{\delta}\right)^2 \cdot {}^2{}_{n}|\overline{A}_{x} - \left(E\left[\overline{L}({}_{n}|\overline{A}_{x})\right] + \frac{P}{\delta}\right)^2$$

Proof: Using that $\overline{Z}_x = \overline{Z}^{1}_{x:\overline{n}|} + {}_{n}|\overline{Z}_x$, we get that

$$\overline{L}({}_{n}|\overline{A}_{x}) = {}_{n}|\overline{Z}_x - P\overline{Y}_x = {}_{n}|\overline{Z}_x - P\frac{1-\overline{Z}_x}{\delta}$$
$$= -\frac{P}{\delta} + \frac{P}{\delta}\overline{Z}^{1}_{x:\overline{n}|} + \left(1+\frac{P}{\delta}\right){}_{n}|\overline{Z}_x.$$

Now, the proof follows similarly to that of previous theorems.

Plan	Premium
Whole life insurance	$\bar{P}(\bar{A}_x) = \bar{A}_x / \bar{a}_x$
n–year term insurance	$\bar{P}(\bar{A}_{x^1:\overline{n}\rvert}) = \bar{A}_{x^1:\overline{n}\rvert} / \bar{a}_{x:\overline{n}\rvert}$
n –year endowment insurance	$\bar{P}(\bar{A}_{x:\overline{n}\rvert}) = \bar{A}_{x:\overline{n}\rvert} / \bar{a}_{x:\overline{n}\rvert}$
h–payment whole life insurance	${}_h\bar{P}(\bar{A}_x) = \bar{A}_x / \bar{a}_{x:\overline{h}\rvert}$
h–payment n–year endowment insurance	${}_h\bar{P}(\bar{A}_{x:\overline{n}\rvert}) = \bar{A}_{x:\overline{n}\rvert} / \bar{a}_{x:\overline{h}\rvert}$
n–year pure endowment	$\bar{P}(\bar{A}_{x:\overline{n}\rvert^1}) = \bar{A}_{x:\overline{n}\rvert 1} / \bar{a}_{x:\overline{n}\rvert}$

Table: Benefit premiums in the fully continuous case

