Math 222 Calculus 2 Fall 2005 Practice Exam 1

INSTRUCTIONS: Show all necessary work for each problem. Partial credit is possible if work shows some understanding. Numerical answers should be expressed as exact mathematical expressions rather than decimal approximations.

(1) (10 Points) Find $\int_1^2 x\, 3^{x^2}\, dx$

(2) (10 Points) The function $y = f(x) = x^3 + x + 1$ has an inverse function $x = f^{-1}(y)$ because $f'(x) = 3x^2 + 1 > 0$ for all real x, so $f(x)$ is strictly increasing. Find the derivatives $(f^{-1})'(11)$ and $(f^{-1})'(3)$.

(3) (10 Points) Find $\lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^n$

(4) (30 Points)

(a) Find $\frac{d}{dx} \sin^{-1}(e^{\tan(x)})$ (b) Find $\int \frac{1}{9x^2+25}\, dx$ (c) Find $\int \frac{(\ln(x))^2}{x}\, dx$

(5) (20 Points) Evaluate the following limits.

(a) $\lim_{x\to\infty} \frac{\ln(x^2+1)}{\ln(x^3+1)}$ (b) $\lim_{x\to\pi/4^-} \tan(x)^{\tan(2x)}$

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(1) (10 Points)

(a) $\displaystyle\int_1^2 x3^{x^2}\,dx = \frac{3^{x^2}}{2\ln(3)}\Bigg]_{x=1}^{x=2} = \frac{3^4-3^1}{2\ln(3)} = \frac{78}{2\ln(3)} = \frac{39}{\ln(3)}.$

(2) (10 Points) If $b = f(a)$ then $(f^{-1})'(b) = \frac{1}{f'(a)}$. Since $f(2) = 11$ and $f(1) = 3$, we have $(f^{-1})'(11) = \dfrac{1}{f'(2)} = \dfrac{1}{13}$ and $(f^{-1})'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{4}$.

(3) (10 Points) Let $m = 2n$, the limit equals

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m/2} = \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right)^{1/2} = e^{1/2} = \sqrt{e}.$$

(4) (30 Points)

(a) $\dfrac{d}{dx}\ \sin^{-1}(e^{\tan(x)}) = \dfrac{e^{\tan(x)}\sec^2(x)}{\sqrt{1-e^{2\tan(x)}}}.$

(b) Using the substitution $x = 5u/3$ we get

$$\int \frac{dx}{9x^2+25} = \frac{5}{3}\int\frac{du}{25(u^2+1)} = \frac{1}{15}\ \tan^{-1}(u) + C = \frac{1}{15}\ \tan^{-1}(3x/5) + C.$$

(c) Using substitution $u = \ln(x)$ we get

$$\int \frac{(\ln(x))^2}{x}\ dx = \int u^2\ du = u^3/3 + C = (\ln(x))^3/3 + C.$$

(5) (20 Points) (a) By L'Hospital's Rule for a $\frac{\infty}{\infty}$ indeterminate form,

$$\lim_{x\to\infty}\frac{\ln(x^2+1)}{\ln(x^3+1)} = \lim_{x\to\infty}\frac{\frac{2x}{x^2+1}}{\frac{3x^2}{x^3+1}} = \lim_{x\to\infty}\frac{(2x)(x^3+1)}{(x^2+1)(3x^2)}$$

$$= \lim_{x\to\infty}\frac{(2)(x^3+1)}{3(x^3+x)} = \lim_{x\to\infty}\frac{(2)(3x^2)}{3(3x^2+1)} = \lim_{x\to\infty}\frac{(2)(6x)}{3(6x)} = \frac{2}{3}$$

(b) Let $y = \tan(x)^{\tan(2x)}$ so $\ln(y) = \tan(2x)\ln(\tan(x)) = \frac{\ln(\tan(x))}{\cot(2x)}$. Then $\displaystyle\lim_{x\to\pi/4^-}\ln(y) = \lim_{x\to\pi/4^-}\frac{\ln(\tan(x))}{\cot(2x)}$ is a type $\frac{0}{0}$ indeterminate form. L'Hospital's Rule gives

$$\lim_{x\to\pi/4^-}\frac{\frac{\sec^2(x)}{\tan(x)}}{-2\csc^2(2x)} = \lim_{x\to\pi/4^-}\frac{\cos(x)\sin^2(2x)}{-2\cos^2(x)\sin(x)} = \lim_{x\to\pi/4^-}\frac{4\sin^2(x)\cos^2(x)}{-2\cos(x)\sin(x)}$$

$$= \lim_{x\to\pi/4^-} -2\cos(x)\sin(x) = -2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = -1$$

so $y \to e^{-1} = \dfrac{1}{e}$.