

INSTRUCTIONS: Show all necessary work for each problem. Partial credit is possible if work shows some understanding. Numerical answers should be expressed as exact mathematical expressions rather than decimal approximations.

- (1) (30 Points, 6 Points each) Evaluate the following integrals.

(a) $\int \cos^3(x) \sin^4(x) dx$

(b) $\int \tan^3(x) \sec^3(x) dx$

(c) $\int \cos^4(x) dx$

(d) $\int \frac{x^3 + x^2 + 6x + 3}{(x^2 + 3)x^2} dx$

(e) $\int \frac{1}{x^2 \sqrt{x^2 + 1}} dx$

- (2) (10 Points) Evaluate $\int_1^{\infty} \frac{dx}{1 + 4x^2}$

- (3) (10 Points) Evaluate $\int_1^4 \frac{dx}{x^2 + x - 6}$

- (4) (10 Points) Use the Comparison Theorem to determine whether the following improper integral converges or diverges. DO NOT COMPUTE THE EXACT VALUE OF THE INTEGRAL, but show all work needed for the Comparison Theorem.

$$\int_1^{\infty} \frac{1}{\sqrt[3]{x^5 + x + 1}} dx$$

- (5) (10 Points) Determine whether each sequence $\{a_n\}_{n=1}^{\infty}$ converges or diverges, and if it converges, then find its limit.

(a) $a_n = n^2 \left(\frac{1}{n+1} - \frac{1}{n+3} \right)$ (b) $a_n = \frac{2n^2 + 1}{3n^2 + 2}$

- (6) (15 Points) Determine whether each series converges or diverges. Explain the reasons for your answer. If the series converges, find its sum.

(a) $\sum_{n=1}^{\infty} \frac{2^{(3n)}}{3^{(2n)}}$ (b) $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{\ln(n^2+2)}$ (c) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

- (7) (15 Points) In each part determine whether the series converges or diverges. Explain all details of the test for convergence you used.

(a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ (b) $\sum_{n=1}^{\infty} \frac{1}{5n^{(3/2)}}$

(1) (30 Points)

(a) $\int \cos^3(x) \sin^4(x) dx = \int \cos^2(x) \sin^4(x) \cos(x) dx$ and after the substitution $u = \sin(x)$ with $du = \cos(x) dx$, this gives

$$\int (1 - u^2)u^4 du = \int (u^4 - u^6) du = \frac{u^5}{5} - \frac{u^7}{7} + C = \frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C.$$

(b) $\int \tan^3(x) \sec^3(x) dx = \int (\sec^2(x) - 1) \sec^2(x) \tan(x) \sec(x) dx$ and the substitution $u = \sec(x)$ with $du = \tan(x) \sec(x) dx$ gives

$$\int (u^2 - 1)u^2 du = \int (u^4 - u^2) du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + C.$$

(c) From the half-angle formula, we get

$$\begin{aligned} \int \cos^4(x) dx &= \frac{1}{4} \int (1 + \cos(2x))^2 dx = \frac{1}{4} \int (1 + 2\cos(2x) + \cos^2(2x)) dx \\ &= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{1}{8} \int (1 + \cos(4x)) dx = \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{1}{8} \left(x + \frac{\sin(4x)}{4} \right) + C \\ &= \frac{3x}{8} + \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C. \end{aligned}$$

(d) Using partial fractions we can write $\frac{x^3 + x^2 + 6x + 3}{(x^2 + 3)x^2} = \frac{Ax + B}{x^2 + 3} + \frac{C}{x} + \frac{D}{x^2}$. Then

$$x^3 + x^2 + 6x + 3 = (Ax + B)x^2 + C(x^2 + 3)x + D(x^2 + 3) = (A + C)x^3 + (B + D)x^2 + 3Cx + 3D$$

which gives the equations $A + C = 1$, $B + D = 1$, $3C = 6$, $3D = 3$ so $A = -1$, $B = 0$, $C = 2$, $D = 1$. Then

$$\begin{aligned} \int \frac{x^3 + x^2 + 6x + 3}{(x^2 + 3)x^2} dx &= \int \frac{-x dx}{x^2 + 3} + \int \frac{2 dx}{x} + \int \frac{dx}{x^2} \\ &= \frac{-1}{2} \ln|x^2 + 3| + 2 \ln|x| - \frac{1}{x} + C. \end{aligned}$$

(e) Use the trig substitution $x = \tan(\theta)$ and $dx = \sec^2(\theta) d\theta$, and get

$$\begin{aligned} \int \frac{1}{x^2 \sqrt{x^2 + 1}} dx &= \int \frac{\sec^2(\theta) d\theta}{\tan^2(\theta) \sqrt{\tan^2(\theta) + 1}} = \int \frac{\sec^2(\theta) d\theta}{\tan^2(\theta) \sec(\theta)} \\ &= \int \frac{\sec(\theta)}{\tan^2(\theta)} d\theta = \int \frac{\cos(\theta)}{\sin^2(\theta)} d\theta = \int \frac{dv}{v^2} \quad (\text{using } v = \sin(\theta)) \\ &= \frac{-1}{v} + C = \frac{-1}{\sin(\theta)} + C = \frac{-\sqrt{x^2 + 1}}{x} + C. \end{aligned}$$

(2) (10 Points) Evaluate $\int_1^{\infty} \frac{dx}{1+4x^2}$.

Solution: This integral is improper since the upper bound is infinity. We find the indefinite integral, by the substitution $u = 2x$, so $du = 2dx$.

$$\int \frac{dx}{1+4x^2} = \frac{1}{2} \int \frac{du}{1+u^2} = \frac{1}{2} \arctan(u) = \frac{1}{2} \arctan(2x) + C.$$

The improper integral we want is then

$$\lim_{t \rightarrow \infty} \frac{1}{2} (\arctan(2t) - \arctan(2)) = \frac{1}{2} \left(\frac{\pi}{2} - \arctan(2) \right).$$

(3) (10 Points) Evaluate $\int_1^4 \frac{dx}{x^2+x-6}$.

Solution: This integral is improper since the denominator of the integrand equals $(x+3)(x-2)$, which is zero at $x = 2$. Use partial fractions to write $\frac{1}{x^2+x-6} = \frac{1}{5} \left(\frac{1}{x-2} - \frac{1}{x+3} \right)$ so the indefinite integral is $\frac{1}{5} \ln \left| \frac{x-2}{x+3} \right| + C$. We must break up the improper integral into two pieces, from 1 to 2 and from 2 to 4, and if either one diverges, the whole integral diverges. The improper integral diverges since

$$\frac{1}{5} \lim_{t \rightarrow 2^-} \ln \left| \frac{t-2}{t+3} \right| - \ln \left| \frac{-1}{4} \right| \rightarrow -\infty.$$

(4) (10 Points) Use the Comparison Theorem to determine whether the following improper integral converges or diverges. DO NOT COMPUTE THE EXACT VALUE OF THE INTEGRAL, but show all work needed for the Comparison Theorem.

$$\int_1^{\infty} \frac{1}{\sqrt[3]{x^5+x+1}} dx$$

Solution: Since $x^5 + x + 1 > x^5 > 0$ for $x > 0$, we have $\sqrt[3]{x^5+x+1} > \sqrt[3]{x^5} = x^{5/3}$ so

$$0 < \frac{1}{\sqrt[3]{x^5+x+1}} < \frac{1}{x^{5/3}}.$$

We know that the integral $\int_1^{\infty} \frac{1}{x^p} dx$ converges for $p > 1$, and in this case $p = \frac{5}{3} > 1$. So by the Comparison Theorem, the given integral converges.

(5) (10 Points) Determine whether each sequence $\{a_n\}_{n=1}^{\infty}$ converges or diverges, and if it converges, then find its limit.

(a) (5 Points) $a_n = n^2 \left(\frac{1}{n+1} - \frac{1}{n+3} \right)$

Solution: The sequence converges to 2 because as $n \rightarrow \infty$ we have

$$n^2 \left(\frac{1}{n+1} - \frac{1}{n+3} \right) = n^2 \frac{(n+3) - (n+1)}{n^2 + 4n + 3} = \frac{2n^2}{n^2 + 4n + 3} = \frac{2}{1 + 4/n + 3/n^2} \rightarrow 2.$$

(b) (5 Points) $a_n = \frac{2n^2 + 1}{3n^2 + 2}$

Solution: Let $f(x) = \frac{2x^2+1}{3x^2+2}$ so that $f(n) = a_n$, and use L'Hospital's rule for type $\frac{\infty}{\infty}$.

$$\lim_{x \rightarrow \infty} \frac{2x^2 + 1}{3x^2 + 2} = \lim_{x \rightarrow \infty} \frac{4x}{6x} = \frac{4}{6} = \frac{2}{3}.$$

(6) (15 Points) Determine whether each series converges or diverges. Explain the reasons for your answer. If the series converges, find its sum.

(a) (5 Points) $\sum_{n=1}^{\infty} \frac{2^{(3n)}}{3^{(2n)}}$

Solution: Writing $\frac{2^{(3n)}}{3^{(2n)}} = \frac{8^n}{9^n} = \left(\frac{8}{9}\right)^n$ we see this is a geometric series with ratio $0 < \frac{8}{9} < 1$ and first term $\frac{8}{9}$, which converges to $\frac{8/9}{1-8/9} = \frac{8}{9-8} = 8$.

(b) (5 Points) $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{\ln(n^2+2)}$

Solution: This series diverges by the Test for Divergence because by L'Hospital

$$\lim_{x \rightarrow \infty} \frac{\ln(x+1)}{\ln(x^2+2)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x+1}}{\frac{2x}{x^2+2}} = \lim_{x \rightarrow \infty} \frac{x^2+2}{2x(x+1)} = \lim_{x \rightarrow \infty} \frac{1+2/x^2}{2+2/x} = \frac{1}{2} \neq 0.$$

(c) (5 Points) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

Solution: Since $\frac{1}{n(n+2)} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right)$, this is a telescoping series whose partial sums are $S_k = \frac{1}{2} \sum_{n=1}^k \left(\frac{1}{n} - \frac{1}{n+2} \right) = \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2} \right)$ so the series converges to $\lim_{k \rightarrow \infty} S_k = \frac{3}{4}$.

(7) (15 Points) In each part determine whether the series converges or diverges. Explain all details of the test for convergence you used.

$$(a) \sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2} \qquad (b) \sum_{n=1}^{\infty} \frac{1}{5n^{(3/2)}}$$

(a) (10 Points) Let $f(x) = \frac{1}{x(\ln(x))^2}$, so $f(n)$ is the n^{th} term of the series. To use the Integral test we must check that $f(x)$ is positive, continuous, and decreasing on some interval $[N, \infty)$. It is clearly positive and continuous for any $x > 1$. We have the derivative

$$f'(x) = -x^{-2}(\ln(x))^{-2} - 2x^{-1}(\ln(x))^{-3}x^{-1} = \frac{-1}{x^2(\ln(x))^2} \left[1 + \frac{2}{\ln(x)} \right]$$

which is negative when $1 + \frac{2}{\ln(x)} > 0$. This is certainly true if $\ln(x) > 0$, which happens for all $x > 1$, and we may take $N = 2$. Then the convergence of the series is equivalent to the convergence of the improper integral

$$\int_2^{\infty} f(x)dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x(\ln(x))^2} dx.$$

After the substitution $u = \ln(x)$, $du = dx/x$, we find the indefinite integral

$$\int \frac{1}{x(\ln(x))^2} dx = \int u^{-2} du = -u^{-1} = \frac{-1}{\ln(x)} + C$$

so the improper integral is $\lim_{t \rightarrow \infty} \frac{-1}{\ln(t)} - \frac{-1}{\ln(2)} = \frac{1}{\ln(2)}$ which is finite, so the series converges.

(b) (5 Points) Since $\sum_{n=1}^{\infty} \frac{1}{5n^{(3/2)}} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{n^{(3/2)}}$ is a multiple of a convergent p -series with $p = 3/2 > 1$, the series converges by the p -test.