Math 222 Calculus 2 Fall 2005 Exam 1 Solutions

(1) (10 Points)

(a) (2 points) Write down the formula relating the derivatives $f'(x)$ and $(f^{-1})'(y)$. (No work needs to be shown.)

The relationship between them is $(f^{-1})'(y) = \dfrac{1}{f'(x)}$.

(b) (4 points) For the specific function $f(x) = \sqrt[3]{x}$ USE YOUR ANSWER FROM PART (a) to get $(f^{-1})'(2)$.

Since $f(8) = 2$, $f^{-1}(2) = 8$, and $f'(x) = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3(\sqrt[3]{x})^2}$ so $f'(8) = \dfrac{1}{3(\sqrt[3]{8})^2} = \dfrac{1}{12}$ and then the answer in part (a) says $(f^{-1})'(2) = \frac{1}{f'(8)} = 12$.

(c) (4 points) For $y = f(x) = \sqrt[3]{x}$, find $x = (f^{-1})(y)$ and use it to get a formula for the derivative $(f^{-1})'(y)$. IS THIS CONSISTENT WITH YOUR ANSWER TO PART (b)?

The function $y = f(x) = \sqrt[3]{x}$ has inverse function $x = f^{-1}(y) = y^3$ whose derivative is $(f^{-1})'(y) = 3y^2$. At $y = 2$ this gives $(f^{-1})'(2) = 3(2^2) = 12$, consistent with the answer to part (b).

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(2) (10 Points) Let $f(x) = \dfrac{(x^2+1)^{17}(x^8+4)^{20}}{(x^4+4)^8(x^6+2)^9}$. Use logarithmic differentiation to express the derivative $f'(x)$ as $f(x)$ times an expression. You do not need to write out $f(x)$ in the answer.

To use logarithmic differentiation, first write

$$\ln(f(x)) = 17\ln(x^2+1) + 20\ln(x^8+4) - 8\ln(x^4+4) - 9\ln(x^6+2)$$

then take the derivative: $$\frac{f'(x)}{f(x)} = \frac{17(2x)}{x^2+1} + \frac{20(8x^7)}{x^8+4} - \frac{8(4x^3)}{x^4+4} - \frac{9(6x^5)}{x^6+2}$$

so $$f'(x) = f(x)\left[\frac{34x}{x^2+1} + \frac{160x^7}{x^8+4} - \frac{32x^3}{x^4+4} - \frac{54x^5}{x^6+2}\right].$$

(3) (10 Points) Use the fact that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ to find $\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n$.

Let $x = n/5$. Since $x \to \infty$ as $n \to \infty$, the limit equals

$$\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{5x} = \left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right)^5 = e^5.$$

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(4) (15 Points) Find each of these derivatives.

(a) $\frac{d}{dx}\ \tan^{-1}(x^2)$

$$\frac{d}{dx}\ \tan^{-1}(x^2) = \frac{2x}{1+(x^2)^2} = \frac{2x}{1+x^4}.$$

(b) $\frac{d}{dx}\ 2^{\sin(x)}$

$$\frac{d}{dx}\ 2^{\sin(x)} = 2^{\sin(x)}\cos(x)\ln(2)$$

(c) $\frac{d}{dx}\ \log_3(x^5+x)$

$$\frac{d}{dx}\ \log_3(x^5+x) = \frac{5x^4+1}{(x^5+x)\ln(3)}$$

(5) (15 Points) Find each of these integrals.

(a) $\int \frac{dx}{\sqrt{4-9x^2}}$

Using the substitution $x = 2u/3$ we get

$$\frac{2}{3}\int \frac{du}{\sqrt{4-4u^2}} = \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{3}\ \sin^{-1}(u) + C = \frac{1}{3}\ \sin^{-1}(3x/2) + C.$$

(b) $\int_e^{e^2} \frac{dx}{x(\ln(x))^3}$

Using substitution $u = \ln(x)$ we get $du = \frac{dx}{x}$ and the bounds of the integral are from $u = \ln(e) = 1$ to $u = \ln(e^2) = 2$, so

$$\int_e^{e^2} \frac{dx}{x(\ln(x))^3} = \int_1^2 u^{-3}\ du = u^{-2}/(-2)\Big]_1^2 = \frac{-1}{8} - \frac{-1}{2} = \frac{3}{8}.$$

(c) $\int_0^{\pi/2} \cos(x)\ e^{\sin(x)}\ dx$

$$\int_0^{\pi/2} \cos(x)\ e^{\sin(x)}dx = e^{\sin(x)}\Big]_{x=0}^{x=\pi/2} = e^{\sin(\pi/2)} - e^{\sin(0)} = e^1 - e^0 = e - 1.$$

(6) (20 Points) Evaluate the following limits. If you use L'Hospital's Rule, show where you use it and explain what type of limit you are using it on.

(a) $\lim_{x\to 0} \frac{\sin(3x^2)}{\sin(4x^2)}$

(7 points) By L'Hospital's Rule for a $\frac{0}{0}$ type indeterminate form,

$$\lim_{x\to 0} \frac{\sin(3x^2)}{\sin(4x^2)} \overset{L'H}{=} \lim_{x\to 0} \frac{6x\cos(3x^2)}{8x\cos(4x^2)} = \lim_{x\to 0} \frac{(6)(1)}{(8)(1)} = \frac{3}{4}.$$

(b) $\lim_{x\to 1} \frac{x^2+3x+1}{2x^2-3x+4}$

(6 points) $\lim_{x\to 1} \frac{x^2+3x+1}{2x^2-3x+4} = \frac{5}{3}$ is not a L'Hospital's Rule problem, not an indeterminate form.

(c) $\lim_{x\to 0} \left(1+x+x^2\right)^{1/x}$.

(7 points) Let $y = \left(1+x+x^2\right)^{1/x}$ so

$$\ln(y) = \frac{\ln(1+x+x^2)}{x}.$$

Then

$$\lim_{x\to 0} \ln(y) = \lim_{x\to 0} \frac{\ln(1+x+x^2)}{x}$$

is a type $\frac{0}{0}$ indeterminate form. L'Hospital's Rule gives

$$\lim_{x\to 0} \frac{\ln(1+x+x^2)}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{1+2x}{1+x+x^2}}{1} = 1$$

so $y \to e^1 = e$ is the limit we seek.