Math 222 **Test 1** **September 25, 2006**

1. (20 points) Find the indicated derivatives. Show your work.

(a) $\dfrac{d}{dx}\tan^{-1}(x^2) =$
Solution: $\dfrac{d}{dx}\tan^{-1}(x^2) = \dfrac{1}{1+(x^2)^2}\dfrac{d}{dx}x^2 = \dfrac{2x}{1+x^4}$

(b) If $f(x) = x^2\ln(x)$ then $f'(e) =$
Solution: $f'(x) = \left(\dfrac{d}{dx}x^2\right)\cdot\ln(x) + x^2\cdot\dfrac{d}{dx}\ln(x) = 2x\cdot\ln(x) + x^2\cdot\dfrac{1}{x} = 2x\ \ln(x) + x,$
so $f'(e) = 2e\ \ln(e) + e = 2e\cdot 1 + e = 3e$

(c) $D_x \sin^{-1}(1-x) =$

(d) **Solution:** $D_x\sin^{-1}(1-x) = \dfrac{1}{\sqrt{1-(1-x)^2}}\cdot D_x(1-x) = \dfrac{-1}{\sqrt{1-(1-2x+x^2)}} = \dfrac{-1}{\sqrt{2x-x^2}}$

(e) $\dfrac{d}{dt}e^{3\sin(2t)} =$
Solution: $\dfrac{d}{dt}e^{3\sin(2t)} = e^{3\sin(2t)}\cdot\dfrac{d}{dt}(3\sin(2t)) = e^{3\sin(2t)}\cdot 6\cos(2t) = 6\cos(2t)e^{3\sin(2t)}$

2. (10 points) Find $f'(x)$. Show your work. If your answer involves $f(x)$ as a factor then you may leave the factor as $f(x)$. That is, you do not have to replace $f(x)$ with its formula.

(a) $f(x) = \dfrac{(1+x^3)^{12}(4+x^5)^9}{(1+7x)^{5/2}}$
Solution: Use logarithmic differentiation:

$$\begin{aligned} y &= \frac{(1+x^3)^{12}(4+x^5)^9}{(1+7x)^{5/2}} \\ \ln y &= 12\ln(1+x^3) + 9\ln(4+x^5) - \frac{5}{2}\ln(1+7x) \\ \frac{f'(x)}{f(x)} = \frac{y'}{y} &= 12\frac{3x^2}{1+x^3} + 9\frac{5x^4}{4+x^5} - \frac{5}{2}\frac{7}{1+7x} \\ f'(x) &= f(x)\cdot\left[\frac{36x^2}{1+x^3} + \frac{45x^4}{4+x^5} - \frac{35}{2+14x}\right] \end{aligned}$$

(b) $f(x) = x^{1/x}$

Solution: Use logarithmic differentiation:

$$\begin{aligned} y &= x^{1/x} \\ \ln y &= \frac{1}{x} \cdot \ln x \\ \frac{f'(x)}{f(x)} = \frac{y'}{y} &= -\frac{1}{x^2} \cdot \ln x + \frac{1}{x} \cdot \frac{1}{x} = \frac{1-\ln x}{x^2} \\ f'(x) &= \frac{1-\ln x}{x^2} f(x) = \frac{1-\ln x}{x^2} x^{1/x} \end{aligned}$$

3. (24 points) Integrate by first making a substitution. You will lose credit if you use a "quick formula" from the book instead of performing the substitution. Show all your work, and indicate clearly your substitutions.

(a) $\displaystyle\int \frac{(\ln x)^2}{x}\,dx =$

Solution: Substitute $u = \ln x,\ du = \dfrac{dx}{x}$:

$$\int \frac{(\ln x)^2}{x}\,dx = \int (\ln x)^2 \frac{dx}{x} = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C$$

(b) $\displaystyle\int_0^2 \frac{dx}{\sqrt{16-x^2}} =$

Solution: Substitute $x = 4u,\ dx = 4du,\ u = \frac{1}{4}x,\ x = 0 \implies u = 0,\ x = 2 \implies u = \frac{1}{2}$:

$$\int_0^2 \frac{dx}{\sqrt{16-x^2}} = \int_0^{\frac{1}{2}} \frac{4du}{\sqrt{16-16u^2}}\,du = \frac{4}{\sqrt{16}} \int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u)\Big|_0^{\frac{1}{2}} = \sin^{-1}\left(\tfrac{1}{2}\right) - \sin^{-1}(0) = \frac{\pi}{6}$$

(c) $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx =$

Solution: Substitute $u = e^x,\ du = e^x\,dx$:

$$\int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{e^x\,dx}{(e^x)^2+1} = \int \frac{du}{u^2+1} = \tan^{-1}(u) + C = \tan^{-1}(e^x) + C$$

4. (9 points)

(a) Define: $\ln 17 =$

Solution: $\displaystyle \ln 17 = \int_1^{17} \frac{dt}{t}$

(b) Solve for x: $\ln(x+1) + \ln(x-1) = \ln 2 \qquad x =$

Solution: First combine the logs on the left to get $\ln[(x+1)(x-1)] = \ln 2$, then exponentiate to get $(x+1)(x-1) = 2$. This is a quadratic equation for x with two solutions: $x = \pm\sqrt{3}$. However, $x = -\sqrt{3}$ must be rejected, since when you plug it into the original equation you get $\ln(x-1) = \ln(-\sqrt{3}-1)$, which doesn't exist since $-\sqrt{3}-1$ is negative. So the only possible solution is $x = \sqrt{3}$, and this satisfies the original equation.

(c) Solve for x: $3^x = 5 \cdot 2^x \qquad x =$

Solution: Take logs of both sides to get $\ln(3^x) = \ln(5 \cdot 2^x)$, then simplify to get $x \ln 3 = \ln 5 + x \ln 2$. Gather the x terms on one side, factor, and solve for x:

$$x \ln 3 - x \ln 2 = \ln 5 \qquad \Longrightarrow \qquad (\ln 3 - \ln 2)\, x = \ln 5 \qquad \Longrightarrow \qquad x = \frac{\ln 5}{\ln 3 - \ln 2}$$

5. (21 points) Find the limits. Show all your work. Identify any use of L'Hôpital's Rule, and indicate briefly why it is applicable.

(a) $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x}}{\tan x} =$

Solution: This is a $\frac{0}{0}$ indeterminate form, so L'Hôpital's Rule applies:

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{\tan x} = \lim_{x\to 0} \frac{e^x - e^{-x}(-1)}{\sec^2 x} = \frac{1-(-1)}{1} = 2$$

(b) $\displaystyle\lim_{x\to 0} \frac{e^x - \ln(e+x)}{x - x^2} =$

Solution: This is a $\frac{0}{0}$ indeterminate form, so L'Hôpital's Rule applies:

$$\lim_{x\to 0} \frac{e^x - \ln(e+x)}{x - x^2} = \lim_{x\to 0} \frac{e^x - \dfrac{1}{e+x}}{1-2x} = \frac{1 - \dfrac{1}{e}}{1} = 1 - \frac{1}{e}$$

(c) $\displaystyle\lim_{x\to\infty} (1 + e^{3x})^{1/x} =$

Solution: This is a ∞^0 indeterminate form. Let $y = (1+e^{3x})^{1/x}$ so $\ln y = \dfrac{1}{x}\ln(1+e^{3x}) = \dfrac{\ln(1+e^{3x})}{x}$. This is a $\frac{\infty}{\infty}$ indeterminate form as $x \to \infty$ so we can apply L'Hôpital's Rule:

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+e^{3x})}{x} = \lim_{x\to\infty} \frac{\dfrac{3e^{3x}}{1+e^{3x}}}{1} = \lim_{x\to\infty} \frac{3e^{3x}}{1+e^{3x}}.$$

This is still a $\frac{\infty}{\infty}$ indeterminate form so we can apply L'Hôpital's Rule again:

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{3e^{3x}}{1+e^{3x}} = \lim_{x\to\infty} \frac{9e^{3x}}{3e^{3x}} = 3.$$

Now we can solve the original problem: $\displaystyle\lim_{x\to\infty} (1+e^{3x})^{1/x} = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty} \ln y} = e^3$

6. (10 points) A sample of the rare radioactive element Linoleum decayed from 16 grams to 4 grams in 12 years. It is known that the amount, y, of Linoleum at time t satisfies the equation $\dfrac{dy}{dt} = ky$. (y is measured in grams, and t is measured in years.)

(a) Find k.

Solution: The equation $\dfrac{dy}{dt} = ky$ has the solution $y = y_0 e^{kt}$ where y_0 is the value of y when $t = 0$. So in this problem $y_0 = 16$ and so $y(t) = 16e^{kt}$. Plugging in $y = 4, t = 12$ gives $4 = 16e^{12k}$. Now take logs and solve for k:

$$\ln 4 = \ln\left(16e^{12k}\right) = \ln(16) + 12k \qquad \Longrightarrow \qquad k = \frac{\ln 4 - \ln(16)}{12} = \frac{2\ln 2 - 4\ln 2}{12} = -\frac{\ln 2}{6}$$

(b) How long will it take for the sample to decay from 4 grams to 2 grams?

Solution: This uses the same k as in part (a), but we restart the clock so that $y = 4$ corresponds to $t = 0$, so $y(t) = 4e^{kt}$. We need to plug in $y = 2$ and solve for t:

$$4e^{kt} = 2 \qquad \Longrightarrow \qquad \ln\left(4e^{kt}\right) = \ln 4 + kt = \ln 2 \qquad \Longrightarrow \qquad t = \frac{\ln 2 - \ln 4}{k} = \frac{\ln 2 - 2\ln 2}{k} = \frac{-\ln 2}{-\frac{\ln 2}{6}} = 6$$

7. (6 points) A function $f(x)$ is defined for $0 \leq x \leq 1$ and satisfies $f'(x) = 1 + [f(x)]^4$. That is, if $y = f(x)$ then $y' = 1 + y^4$.

(a) Why does this function have an inverse? [Hint: Is $f(x)$ increasing or decreasing? Why?]
Solution: Since $[f(x)]^4$ is a square it is never negative, so $f'(x) = 1 + [f(x)]^4$ is positive, so f is strictly increasing, so it is one to one. Therefore it has an inverse.

(b) Find the derivative of $f^{-1}(x)$. Your answer should involve x, but not f or y.
Solution: Method 1: Use the formula $(f^{-1})'(y) = \dfrac{1}{f'(x)}$ where $y = f(x)$. Plugging in $f'(x) = 1 + [f(x)]^4 = 1 + y^4$ gives $(f^{-1})'(y) = \dfrac{1}{1+y^4}$. Now replace the variable y with x to get $(f^{-1})'(x) = \dfrac{1}{1+x^4}$.

Method 2: Start with $y = f(x)$ and apply implicit differentiation with respect to y:

$$y = f(x) \qquad \Longrightarrow \qquad 1 = \frac{d}{dy}y = \frac{d}{dy}f(x) = f'(x)\frac{dx}{dy} = \left(1 + [f(x)]^4\right)\frac{dx}{dy} = \left(1 + y^4\right)\frac{dx}{dy}$$

Now solve this for the derivative: $(f^{-1})'(y) = \dfrac{dx}{dy} = \dfrac{1}{1+y^4}$. Finally, replace y with x to get $(f^{-1})'(x) = \dfrac{1}{1+x^4}$.