

(1) (30 Points, 6 Points Each) Evaluate the following integrals.

$$(a) \int x^3 \ln(x) dx$$

Using integration by parts with $u = \ln(x)$ and $dv = x^3 dx$, we have $du = \frac{1}{x} dx$ and $v = \frac{x^4}{4}$, so

$$\int x^3 \ln(x) dx = \ln(x) \frac{x^4}{4} - \int \frac{x^4}{4} \frac{1}{x} dx = \frac{x^4 \ln(x)}{4} - \int \frac{x^3}{4} dx = \frac{x^4 \ln(x)}{4} - \frac{x^4}{16} + C$$

$$(b) \int \sin^9(x) \cos^5(x) dx$$

Let $u = \sin(x)$ so $du = \cos(x) dx$, and the integral becomes

$$\begin{aligned} \int \sin^9(x) \cos^5(x) dx &= \int \sin^9(x) \cos^4(x) \cos(x) dx = \int u^9 (1 - u^2)^2 du \\ &= \int u^9 (1 - 2u^2 + u^4) du = \int (u^9 - 2u^{11} + u^{13}) du = \frac{u^{10}}{10} - \frac{u^{12}}{6} + \frac{u^{14}}{14} + C \\ &= \frac{\sin^{10}(x)}{10} - \frac{\sin^{12}(x)}{6} + \frac{\sin^{14}(x)}{14} + C \end{aligned}$$

$$(c) \int \sin^4(x) dx$$

From the half-angle formulas $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ and $\cos^2(x) = \frac{1 + \cos(2x)}{2}$, we get

$$\begin{aligned} \int \sin^4(x) dx &= \frac{1}{4} \int (1 - \cos(2x))^2 dx = \frac{1}{4} \int (1 - 2\cos(2x) + \cos^2(2x)) dx \\ &= \frac{x}{4} - \frac{\sin(2x)}{4} + \frac{1}{8} \int (1 + \cos(4x)) dx = \frac{x}{4} - \frac{\sin(2x)}{4} + \frac{1}{8} \left(x + \frac{\sin(4x)}{4} \right) + C \\ &= \frac{3x}{8} - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C. \end{aligned}$$

$$(d) \int \frac{x^2 + x + 2}{x(x^2 + 1)}$$

Using partial fractions we can write $\frac{x^2 + x + 2}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$. Then

$$x^2 + x + 2 = (A)(x^2 + 1) + (Bx + C)x = (A + B)x^2 + Cx + A$$

which gives the equations $A + B = 1$, $C = 1$, $A = 2$ so $A = 2$, $B = -1$, $C = 1$. Then the integral is

$$\begin{aligned} \int \frac{x^2 + x + 2}{x(x^2 + 1)} dx &= \int \frac{2dx}{x} + \int \frac{-xdx}{x^2 + 1} + \int \frac{dx}{x^2 + 1} \\ &= 2 \ln |x| - \frac{1}{2} \ln |x^2 + 1| + \tan^{-1}(x) + C \\ &= \ln \left| \frac{x^2}{\sqrt{x^2 + 1}} \right| + \tan^{-1}(x) + C. \end{aligned}$$

$$(e) \int \frac{\sqrt{x^2 - 1}}{x} dx$$

Since $\sec^2(u) - 1 = \tan^2(u)$, use the trig substitution $x = \sec(u)$ and $dx = \tan(u) \sec(u) du$, and get

$$\begin{aligned} \int \frac{\sqrt{x^2 - 1}}{x} dx &= \int \frac{\tan(u)}{\sec(u)} \tan(u) \sec(u) du = \int \tan^2(u) du = \int (\sec^2(u) - 1) du \\ &= \tan(u) - u + C = \sqrt{x^2 - 1} - \cos^{-1}(1/x) + C. \end{aligned}$$

$$(2) \text{ (10 Points) Evaluate } \int_0^{\infty} e^{-3x} dx$$

This integral is improper since the upper bound is infinity. We find the indefinite integral $\int e^{-3x} dx = \frac{-1}{3} e^{-3x} + C$ so the improper integral we want is then

$$\lim_{t \rightarrow \infty} \frac{-1}{3} (e^{-3t} - e^0) = \frac{-1}{3} (0 - 1) = \frac{1}{3}.$$

(3) (10 Points) Evaluate $\int_1^2 \frac{1}{x^2 - 4x + 3} dx$

This integral is improper since the denominator of the integrand equals $(x - 1)(x - 3)$, which is zero at $x = 1$. We don't have to worry about the zero at $x = 3$ since that is not in the bounds of the integration. Use partial fractions to write $\frac{1}{(x-1)(x-3)} = \frac{1}{2}(\frac{1}{x-3} - \frac{1}{x-1})$ so the indefinite integral is $\frac{1}{2} \ln \left| \frac{x-3}{x-1} \right| + C$. This improper integral diverges since

$$\frac{1}{2} \lim_{t \rightarrow 1^+} \ln \left| \frac{2-3}{2-1} \right| - \ln \left| \frac{t-3}{t-1} \right| = 0 - \infty.$$

(4) (15 Points) Use the Comparison Theorem to determine whether the following improper integral converges or diverges. DO NOT COMPUTE THE EXACT VALUE OF THE INTEGRAL, but show all work needed for the Comparison Theorem.

$$\int_2^{\infty} \frac{1}{\sqrt[3]{x^2 + 4}} dx$$

For $2 \leq x$ we have $4 \leq x^2$ so $x^2 + 4 \leq x^2 + x^2 = 2x^2$ so $\sqrt[3]{x^2 + 4} \leq \sqrt[3]{2x^2} = \sqrt[3]{2}x^{2/3}$ and then

$$\frac{1}{\sqrt[3]{x^2 + 4}} \geq \frac{1}{\sqrt[3]{2}x^{2/3}} \geq 0.$$

We know that $\int_2^{\infty} \frac{1}{x^p} dx$ diverges for $p \leq 1$, so

$$\int_2^{\infty} \frac{1}{\sqrt[3]{2}x^{2/3}} = \frac{1}{\sqrt[3]{2}} \int_2^{\infty} \frac{1}{x^{2/3}} dx$$

diverges. By the Comparison Theorem, the given integral diverges.

- (5) (10 Points) Determine whether each sequence $\{a_n\}_{n=1}^{\infty}$ converges or diverges, and if it converges, then find its limit.

$$(a) a_n = \frac{\arctan(n^2 - n)}{\sqrt{2 - \frac{1}{n}}}$$

Since $n^2 - n = n(n-1) \rightarrow \infty$ as $n \rightarrow \infty$, and we know that $\arctan(x) \rightarrow \pi/2$ as $x \rightarrow \infty$, the numerator of a_n is going to $\pi/2$ as $n \rightarrow \infty$. Since $\frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$, the denominator of a_n is going to $\sqrt{2}$ as $n \rightarrow \infty$, so the sequence converges to $\frac{\pi}{2\sqrt{2}}$.

$$(b) a_n = \frac{5000 + 100e^n + 3e^{2n}}{7000 + 200e^n + 5e^{2n}}$$

After dividing numerator and denominator of a_n by e^{2n} , we get

$$a_n = \frac{5000/e^{2n} + 100/e^n + 3}{7000/e^{2n} + 200/e^n + 5} \rightarrow \frac{3}{5}$$

as $n \rightarrow \infty$, so the sequence converges to $\frac{3}{5}$.

- (6) (15 Points) Determine whether each series converges or diverges. Explain the reasons for your answer. If the series converges, find its sum.

$$(a) \sum_{n=1}^{\infty} \frac{e^{(n+1)}}{\pi^{(n-1)}} = e\pi \sum_{n=1}^{\infty} \frac{e^{(n)}}{\pi^{(n)}} = e\pi \sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$$

is a geometrical series with ratio $0 < \frac{e}{\pi} < 1$ and first term e^2 , which converges to $\frac{e^2}{1 - \frac{e}{\pi}} = \frac{e^2\pi}{\pi - e}$.

$$(b) \sum_{n=1}^{\infty} \frac{n^2 + 1}{5n^2 + 2}$$

diverges by the Test for Divergence because the limit of the n^{th} term

$$\text{is } \lim_{n \rightarrow \infty} \frac{n^2 + 1}{5n^2 + 2} = \lim_{n \rightarrow \infty} \frac{1 + 1/n^2}{5 + 2/n^2} = \frac{1}{5} \neq 0.$$

$$(c) \sum_{n=1}^{\infty} \frac{1}{n^2 + 5n + 6} = \sum_{n=1}^{\infty} \left(\frac{1}{n+2} - \frac{1}{n+3} \right)$$

is a telescoping series with partial sums $S_k = \left(\frac{1}{3} - \frac{1}{k+3} \right)$ which converge to $\frac{1}{3}$ as $k \rightarrow \infty$.

- (7) (15 Points) Apply the Integral Test to determine whether the series below converges or diverges. Explain all details of the test.

$$\sum_{n=1}^{\infty} \frac{1}{n(n^2 + 1)}$$

Let $f(x) = \frac{1}{x(x^2 + 1)} = \frac{1}{x^3 + x}$, so $f(n)$ is the n^{th} term of the series. To use the Integral test we must check that $f(x)$ is positive, continuous, and decreasing on some interval $[N, \infty)$. Since $x(x^2 + 1) > 0$ for $x > 0$ it is clear that $f(x)$ is positive and continuous for any $x > 0$. We have the derivative

$$f'(x) = \frac{-(3x^2 + 1)}{(x^3 + x)^2} < 0$$

So we may take $N = 1$. Then the convergence of the series is equivalent to the convergence of the improper integral

$$\int_1^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x(x^2 + 1)} dx.$$

This integral can be done by partial fractions in a similar way to problem 1 (d). We find the indefinite integral

$$\int \frac{1}{x(x^2 + 1)} dx = \int \left(\frac{1}{x} - \frac{x}{x^2 + 1} \right) dx = \ln|x| - \frac{1}{2} \ln|x^2 + 1| + C = \ln \left(\frac{x}{\sqrt{x^2 + 1}} \right) + C$$

so the improper integral is $\lim_{t \rightarrow \infty} \ln \left(\frac{t}{\sqrt{t^2 + 1}} \right) - \ln \left(\frac{1}{\sqrt{1^2 + 1}} \right)$. Since $\frac{t}{\sqrt{t^2 + 1}} = \sqrt{\frac{t^2}{t^2 + 1}} = \sqrt{\frac{1}{1 + 1/t^2}} \rightarrow 1$ as $t \rightarrow \infty$, the limit giving the improper integral goes to $\ln(1) - \ln \left(\frac{1}{\sqrt{2}} \right)$ which is finite, so the series converges by the integral test.