Math 222 **Test 2** **October 23, 2006**

1. (9 points) $\displaystyle\int_0^{\frac{\pi}{6}} \sin^2(3x)\,dx =$

Solution: Use the half angle formula (also known as the double angle formula):

$$\begin{aligned}
\int_0^{\frac{\pi}{6}} \sin^2(3x)\,dx &= \int_0^{\frac{\pi}{6}} \frac{1}{2}\left(1-\cos(6x)\right)\,dx \\
&= \frac{1}{2}\left[x - \frac{1}{6}\sin(6x)\right]_0^{\frac{\pi}{6}} && (*) \\
&= \frac{1}{2}\left[\frac{\pi}{6} - \frac{1}{6}\sin(\pi) - \left(0 - \frac{1}{6}\sin(0)\right)\right] \\
&= \frac{\pi}{12}
\end{aligned}$$

The step flagged with $(*)$ used the substitution $u = 6x$, $x = \frac{1}{6}u$, $dx = \frac{1}{6}\,du$ to get

$$\int \cos(6x)\,dx = \frac{1}{6}\int \cos(u)\,du = \frac{1}{6}\sin(u) + C = \frac{1}{6}\sin(6x) + C$$

2. (9 points) $\displaystyle\int \tan^3 x \sec^4 x\,dx =$

Solution:

$$\begin{aligned}
\int \tan^3 x \sec^4 x\,dx &= \int \tan^3 x \sec^2 x \sec^2 x\,dx \\
&= \int \tan^3 x(\tan^2 x + 1)\sec^2 x\,dx && \text{using } \sec^2 x = \tan^2 x + 1 \\
&= \int u^3(u^2+1)\,du && \text{substitute } u = \tan x,\ du = \sec^2 x\,dx \\
&= \int (u^5 + u^3)\,du \\
&= \frac{1}{6}u^6 + \frac{1}{4}u^4 + C \\
&= \frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + C && \text{replace } u \text{ with } \tan x
\end{aligned}$$

3. (9 points) $\displaystyle\int_2^9 \frac{dx}{\sqrt{x-2}} =$

Solution: First, calculate the indefinite integral using the substitution $u = x-2$, $du = dx$:

$$\int \frac{dx}{\sqrt{x-2}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{u} + C = 2\sqrt{x-2} + C$$

Now

$$\begin{aligned}\int_2^9 \frac{dx}{\sqrt{x-2}} &= \lim_{a\to 2^+}\int_a^9 \frac{dx}{\sqrt{x-2}} && \text{it's improper at the lower endpoint}\\ &= \lim_{a\to 2^+}\Big[2\sqrt{x-2}\Big]_a^9\\ &= \lim_{a\to 2^+}\Big[2\sqrt{7}-2\sqrt{a-2}\Big]\\ &= 2\sqrt{7}\end{aligned}$$

4. (9 points) $\displaystyle\int \frac{dx}{\sqrt{x^2+6x+10}} =$

Solution: First complete the square: $x^2+6x+10 = x^2+6x+9-9+10 = (x+3)^2+1$. Now this is of the form "variable squared + constant squared", so a tangent substitution is called for:

$$x+3=\tan u,\quad dx=\sec^2 u\,du,\quad \sec u=\sqrt{\tan^2 u+1}=\sqrt{(x+3)^2+1}=\sqrt{x^2+6x+10}$$

Making this substitution,

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+6x+10}} &= \int \frac{\sec^2 u\,du}{\sec u}\\ &= \int \sec u\,du\\ &= \ln|\tan u+\sec u|+C\\ &= \ln\left|x+3+\sqrt{x^2+6x+10}\right|+C\end{aligned}$$

5. (9 points) $\displaystyle\int_0^1 \tan^{-1}(x)\,dx =$

Solution: First calculate the indefinite integral:

$$\begin{aligned}\int \tan^{-1}(x)\,dx &= x\tan^{-1}x-\int\frac{x\,dx}{1+x^2} && \text{integrate by parts: } u=\tan^{-1}x,\ dv=dx;\ du=\frac{dx}{1+x^2},\ v=x\\ &= x\tan^{-1}x-\frac{1}{2}\int\frac{dz}{z} && \text{substitute } z=1+x^2,\ dz=2x\,dx,\ x\,dx=\frac{1}{2}dz\\ &= x\tan^{-1}x-\frac{1}{2}\ln|z|+C\\ &= x\tan^{-1}x-\frac{1}{2}\ln\left(1+x^2\right)+C\end{aligned}$$

Now put in the limits:

$$\begin{aligned}\int_0^1 \tan^{-1}(x)\,dx &= \left[x\tan^{-1}x-\frac{1}{2}\ln(1+x^2)\right]_0^1 = 1\cdot\tan^{-1}(1)-\frac{1}{2}\ln 2-\left[0\cdot\tan^{-1}(0)-\frac{1}{2}\ln 1\right]\\ &= \frac{\pi}{4}-\frac{1}{2}\ln 2-[0-0] = \frac{\pi}{4}-\frac{1}{2}\ln 2\end{aligned}$$

6. (9 points) $\displaystyle\int \frac{dx}{(9-x^2)^{3/2}} =$

Solution: Use a trig substitution:

$$x = 3\sin u, \quad dx = 3\cos u\,du, \quad \sqrt{9-x^2} = \sqrt{9-9\sin^2 u} = 3\sqrt{1-\sin^2 u} = 3\cos u$$

So you get:

$$\begin{aligned}\int \frac{dx}{(9-x^2)^{3/2}} &= \int \frac{3\cos u\,du}{(3\cos u)^3} = \frac{1}{9}\int \frac{du}{\cos^2 u} = \frac{1}{9}\int \sec^2 u\,du \\ &= \frac{1}{9}\tan u + C = \frac{1}{9}\cdot\frac{\sin x}{\cos x} + C = \frac{1}{9}\cdot\frac{\frac{1}{3}x}{\frac{1}{3}\sqrt{9-x^2}} + C \\ &= \frac{x}{9\sqrt{9-x^2}} + C\end{aligned}$$

7. (9 points) $\displaystyle\int_2^\infty \frac{dx}{x^2+4} =$

Solution: First, calculate the indefinite integral using the substitution $x = 2u$, $dx = 2\,du$:

$$\int \frac{dx}{x^2+4} = \int \frac{2\,du}{4u^2+4} = \frac{2}{4}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1}(u) + C = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Now

$$\begin{aligned}\int_2^\infty \frac{dx}{x^2+4} &= \lim_{b\to\infty}\int_a^b \frac{dx}{x^2+4} && \text{it's improper at the upper endpoint} \\ &= \lim_{b\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_2^b \\ &= \lim_{b\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{b}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{2}{2}\right)\right] \\ &= \frac{1}{2}\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{4} && \text{since } \frac{b}{2}\to\infty \text{ and } \lim_{x\to\infty}\tan^{-1}(x) = \frac{\pi}{2} \\ &= \frac{\pi}{8}\end{aligned}$$

8. (9 points) $\displaystyle\int \frac{1}{\sqrt{x}(x-1)}\,dx =$

Solution: Make the substitution $u = \sqrt{x}$, so $x = u^2$ and $dx = 2u\,du$:

$$\int \frac{1}{\sqrt{x}(x-1)}\,dx = \int \frac{2u\,du}{u(x^2-1)} = \int \frac{2}{u^2-1}\,du$$

This integral can be handled by a trig substitution (either $\sec\theta$ or $\sin\theta$), but it is easier to use partial fractions:

$$\begin{aligned}\frac{2}{u^2-1} &= \frac{2}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1} \\ 2 &= A(u+1) + B(u-1) && \text{cross multiply} \\ 2 &= 2A \implies A = 1 && \text{plug in } u = 1 \\ 2 &= -2B \implies B = -1 && \text{plug in } u = -1\end{aligned}$$

So now finish the integral:

$$\begin{aligned}
\int \frac{1}{\sqrt{x}(x-1)}\,dx &= \int \frac{2}{u^2-1}\,du \\
&= \int \left(\frac{A}{u-1} + \frac{B}{u+1}\right) du \\
&= \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du \\
&= \ln|u-1| - \ln|u+1| + C \\
&= \ln\left|\sqrt{x}-1\right| - \ln\left|\sqrt{x}+1\right| + C && \text{plug in } u = \sqrt{x}
\end{aligned}$$

9. (9 points) $\int x^2 e^{3x}\,dx =$

Solution: Double integration by parts, using several times $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$:

$$\begin{aligned}
\int x^2 e^{3x}\,dx &= && u = x^2,\ dv = e^{3x}\,dx;\ du = 2x\,dx,\ v = \tfrac{1}{3}e^{3x} \\
&= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}\,dx && u = x,\ dv = e^{3x}\,dx;\ du = dx,\ v = \tfrac{1}{3}e^{3x} \\
&= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left[\frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx\right] \\
&= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left[\frac{1}{3}xe^{3x} - \frac{1}{3}\cdot\frac{1}{3}e^{3x}\right] + C \\
&= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C
\end{aligned}$$

10. (9 points) Find the partial fraction expansion for $\frac{x^3+2}{x^2(x^2+1)}$. You must determine the unknown constants, and show your work. **Do not** integrate anything.

Solution: The bottom is already fully factored, since x^2+1 is irreducible. So

$$\begin{aligned}
\frac{x^3+2}{x^2(x^2+1)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} \\
x^3+2 &= Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2 && \text{cross multiply} \\
x^3+2 &= Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2 && \text{multiply out} \\
x^3+2 &= (A+C)x^3 + (B+D)x^2 + Ax + B && \text{collect powers of } x \\
1 &= A+C && \text{coefficients of } x^3 \\
0 &= B+D && \text{coefficients of } x^2 \\
0 &= A && \text{coefficients of } x \\
2 &= B && \text{constant terms}
\end{aligned}$$

The last four lines, reading from the bottom, give $B = 2$, $A = 0$, $2 + D = 0 \implies D = -2$, and $0 + C = 1 \implies C = 1$, so the partial fraction decomposition is

$$\frac{x^3+2}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} = \frac{2}{x^2} + \frac{x-2}{x^2+1}$$

11. (10 points) The curve C is given by $x = 1 + y^3$ for $0 \le y \le 2$. In each part, give your answer as an integral involving only one variable.

Do not try to do any of the integrations.

(a) What is the length of the curve C?

Solution: $\dfrac{dx}{dy} = 3y^2$ so the arc length is

$$L = \int_0^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^2 \sqrt{1 + 9y^4}\, dy$$

(b) What is the area of the surface which is obtained when C is rotated about the X-axis?

Solution: The radius of the circle traced by a point on the curve is given by y, so the surface area is

$$S = 2\pi \int_0^2 y\sqrt{1 + 9y^4}\, dy$$

(c) What is the area of the surface which is obtained when C is rotated about the Y-axis?

Solution: The radius of the circle traced by a point on the curve is given by $x = 1 + y^3$, so the surface area is

$$S = 2\pi \int_0^2 (1 + y^3)\sqrt{1 + 9y^4}\, dy$$