MIDTERM EXAM

1. (5 pts) Give the definition of $\ln(x)$ for $x > 0$.

   SOLUTION: $\ln(x) = \int_1^x \frac{1}{t} dt.$

2. Solve for x:

   (a) (10 pts) $10^x = e^{x^2}$

   SOLUTION: $10^x = e^{x^2} \Rightarrow \ln(10^x) = \ln(e^{x^2})$

   $$\Rightarrow x\ln(10) = x^2 \Rightarrow x^2 - x\ln(10) = 0$$

   $$\Rightarrow x(x - \ln(10)) = 0 \Rightarrow x = 0 \text{ or } x = \ln(10).$$

   (b) (10 pts) $\log_x 8 = -3$

   SOLUTION: $\log_x 8 = -3 \Rightarrow x^{-3} = 8 \Rightarrow x = 8^{-1/3} = \frac{1}{2}.$

3. Calculate the derivatives of the following (you need not simplify):

   (a) (8 pts) $f(x) = \cosh(x) - \tan^{-1}(x)$

   SOLUTION: $f'(x) = \sinh(x) - \dfrac{1}{1+x^2}.$

   (b) (8 pts) $g(x) = (\sinh(x))(\tan(x))$

   SOLUTION: $g'(x) = (\cosh(x))(\tan(x)) + (\sinh(x))(\sec^2(x)).$

   (c) (8 pts) $h(x) = \dfrac{\ln(x)}{e^x + 1}$

   SOLUTION: $h'(x) = \dfrac{(e^x+1)x^{-1} + \ln(x)e^x}{(e^x+1)^2}.$

   (d) (8 pts) $i(x) = 2^{x\sin^{-1}(x)}$

   SOLUTION: $i'(x) = 2^{x\sin^{-1}(x)} \ln(2) \left(\sin^{-1}(x) + \dfrac{x}{\sqrt{1-x^2}} \right).$

4. (15 pts) Calculate the derivative of $M(x) = \dfrac{(\sin^2(x))(x^2+1)^{\pi}}{\sqrt{e^x \sec^2(x)}}$ (you may leave $M(x)$ in your answer).

SOLUTION: $\ln(M(x)) = \ln\left(\dfrac{(\sin^2(x))(x^2+1)^{\pi}}{\sqrt{e^x \sec^2(x)}}\right)$

$$\begin{aligned}\text{Therefore, } \ln(M(x)) &= 2\ln(\sin(x)) + \pi\ln(x^2+1) - \frac{1}{2}\ln(e^x\sec^2(x)) \\ &= 2\ln(\sin(x)) + \pi\ln(x^2+1) - \frac{1}{2}\ln(e^x) - \frac{1}{2}\ln(\sec^2(x)) \\ &= 2\ln(\sin(x)) + \pi\ln(x^2+1) - \frac{1}{2}x - \ln(\sec(x))\end{aligned}$$

Taking the derivatives of both sides, we get

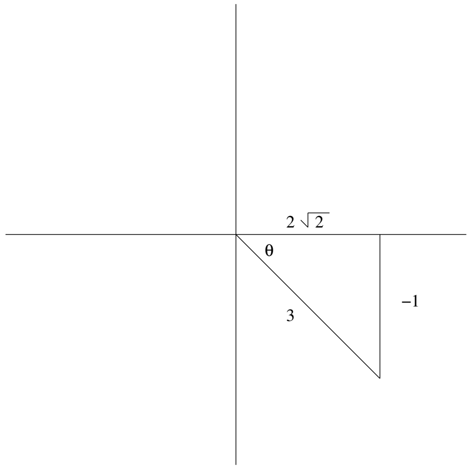
$$\frac{1}{M(x)}M'(x) = 2\frac{\cos(x)}{\sin(x)} + \pi\frac{2x}{x^2+1} - \frac{1}{2} - \frac{\sec(x)\tan(x)}{\sec(x)}$$

$$M'(x) = M(x)\left(2\cot(x) + \frac{2\pi x}{x^2+1} - \frac{1}{2} - \tan(x)\right).$$

5. Calculate the following:

(a) (4 pts) $\sin(\arcsin(-\frac{1}{3})) = -\frac{1}{3}$

(b) (4 pts) $\cos(\arcsin(-\frac{1}{3}))$

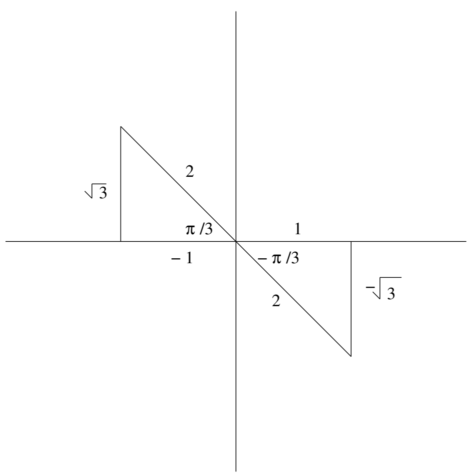


SOLUTION: Using the picture, let $\theta = \arcsin(-\frac{1}{3})$.

Then $\cos(\arcsin(-\frac{1}{3})) = \frac{2\sqrt{2}}{3}$.

(c) (4 pts) $\cos^{-1}(\cos(\frac{\pi}{4})) = \frac{\pi}{4}$

(d) (4 pts) $\tan^{-1}(\tan(\frac{2\pi}{3}))$



SOLUTION: Using the picture, we see that $\tan(\frac{2\pi}{3}) = -\sqrt{3}$.

Also, from the picture, we see that $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$.

6. Evaluate the following limits:

(a) (10 pts) $\lim_{x\to 0} \dfrac{\arctan(x)}{x}$

SOLUTION: $\lim_{x\to 0} \dfrac{\arctan(x)}{x} \to \dfrac{0}{0}$, so we use L'Hospital's rule to get

$$\lim_{x\to 0} \frac{\arctan(x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{1} = 1.$$

(b) (10 pts) $\lim_{x\to \ln(2)} \dfrac{\cosh(x)}{e^x}$

SOLUTION: $\lim_{x\to \ln(2)} \dfrac{\cosh(x)}{e^x} = \lim_{x\to \ln(2)} \dfrac{e^x + e^{-x}}{2e^x} = \dfrac{2+\frac{1}{2}}{2\cdot 2} = \dfrac{5}{8}.$

(c) (10 pts) $\lim_{x\to \frac{\pi}{4}} \dfrac{\sin(x)-\cos(x)}{x-\frac{\pi}{4}}$

SOLUTION: $\lim_{x\to \frac{\pi}{4}} \dfrac{\sin(x)-\cos(x)}{x-\frac{\pi}{4}} \to \dfrac{0}{0}$, so we use L'Hospital's rule to get

$$\lim_{x\to \frac{\pi}{4}} \frac{\sin(x)-\cos(x)}{x-\frac{\pi}{4}} = \lim_{x\to \frac{\pi}{4}} \frac{\cos(x)+\sin(x)}{1} = \sqrt{2}.$$

(d) (10 pts) $\lim_{x\to\infty} x^{1/\sqrt{x}}$

SOLUTION: $\lim_{x\to\infty} x^{1/\sqrt{x}} \to \infty^0$, so we let $y = \lim_{x\to\infty} x^{1/\sqrt{x}}$.

Therefore, $\ln(y) = \lim_{x\to\infty} \ln(x^{1/\sqrt{x}}) = \lim_{x\to\infty} \dfrac{\ln(x)}{\sqrt{x}} \to \dfrac{\infty}{\infty}$.

So we use L'Hospital's rule to get, $\lim_{x\to\infty} \dfrac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty} \dfrac{\frac{1}{x}}{\frac{1}{\sqrt{x}}}$

$$= \lim_{x\to\infty} \frac{\sqrt{x}}{x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

Thus, $\ln(y) = 0$, so $y = \lim_{x\to\infty} x^{1/\sqrt{x}} = e^0 = 1$.

(e) (10 pts) $\int_{\sqrt{3}}^{\infty} \frac{dx}{1+x^2}$

SOLUTION:
$$\begin{aligned}\int_{\sqrt{3}}^{\infty} \frac{dx}{1+x^2} &= \lim_{t\to\infty} \int_{\sqrt{3}}^{t} \frac{dx}{1+x^2} \\ &= \lim_{t\to\infty} \left(\tan^{-1}(x)\right)\Big|_{\sqrt{3}}^{t} \\ &= \lim_{t\to\infty} \left(\tan^{-1}(t) - \frac{\pi}{3}\right) \\ &= \frac{\pi}{2} - \frac{\pi}{3} \\ &= \frac{\pi}{6}\end{aligned}$$

7. Evaluate the following integrals:

(a) (10 pts) $\int_1^e x\ln(x)dx$

SOLUTION: Using integration by parts with $u = \ln(x)$, $dv = xdx$,

$du = \frac{dx}{x}$, and $v = \frac{1}{2}x^2$, we get

$$\begin{aligned}\int_1^e x\ln(x)dx &= \left(\frac{1}{2}x^2\ln(x) - \frac{1}{2}\int xdx\right)\Big|_1^e \\ &= \left(\frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2\right)\Big|_1^e \\ &= \frac{e^2}{2} - \frac{e^2}{4} - \left(0 - \frac{1}{4}\right) \\ &= \frac{e^2}{4} + \frac{1}{4}\end{aligned}$$

(b) (10 pts) $\int_1^{\sqrt{\ln(3)}} xe^{x^2} dx$

SOLUTION: Using u-substitution with $u = x^2$ and $du = 2xdx$ and changing the bounds to $u = (1)^2 = 1$ and $u = (\sqrt{\ln(3)})^2 = \ln(3)$, we get

$$\begin{aligned}\int_1^{\sqrt{\ln(3)}} xe^{x^2} dx &= \frac{1}{2}\int_1^{\ln(3)} e^u du \\ &= \frac{1}{2}\left(e^u\right)\Big|_1^{\ln(3)} \\ &= \frac{1}{2}(3-e) \\ &= \frac{3-e}{2}\end{aligned}$$

(c) (10 pts) $\int \frac{dx}{x^2 - 2x - 15}$

SOLUTION: $\int \frac{dx}{x^2 - 2x - 15} = \int \frac{dx}{(x-5)(x+3)}$

$$\int \left(\frac{A}{x-5} + \frac{B}{x+3}\right) dx$$

$$\int \frac{A(x+3) + B(x-5)}{(x-5)(x+3)} dx$$

If $x = 5$, then we get $1 = 8A \Rightarrow A = \frac{1}{8}$

If $x = -3$, then we get $1 = -8B \Rightarrow B = -\frac{1}{8}$

$$= \frac{1}{8}\int \frac{dx}{x-5} - \frac{1}{8}\int \frac{dx}{x+3}$$

$$= \tfrac{1}{8}\ln|x-5| - \tfrac{1}{8}\ln|x+3| + C$$

(d) (10 pts) $\int \frac{x}{(4x^2+1)^2}dx$

SOLUTION: Using u-substitution with $u = 4x^2 + 1$ and $du = 8xdx$, we get

$$\begin{aligned}\int \frac{x}{(4x^2+1)^2}dx &= \frac{1}{8}\int u^{-2}du \\ &= \frac{1}{8}(-u^{-1}) + C \\ &= -\frac{1}{32x^2+8} + C\end{aligned}$$

(e) (10 pts) $\int \sin^3(x)\cos^5(x)dx$

SOLUTION: We can use u-substitution with either $u = \cos(x)$ or $u = \sin(x)$.

The better choice is $u = \cos(x)$, so $du = -\sin(x)dx$, and we get

$$\int \sin^3(x)\cos^5(x)dx = -\int \sin^2(x)u^5 du$$

$\sin^2(x) = 1 - \cos^2(x)$, so we get

$$\begin{aligned}&= -\int (1-u^2)u^5 du \\ &= -\int (u^5 - u^7)du \\ &= -\left(\tfrac{1}{6}u^6 - \tfrac{1}{8}u^8\right) + C \\ &= -\tfrac{1}{6}\cos^6(x) + \tfrac{1}{8}\cos^8(x) + C\end{aligned}$$

(f) (10 pts) $\int \sec^{10}(x)\tan(x)dx$

SOLUTION: Using u-substitution with $u = \sec(x)$ and $du = \sec(x)\tan(x)dx$, we get

$$\begin{aligned}\int \sec^{10}(x)\tan(x)dx &= \int u^9 du \\ &= \tfrac{1}{10}u^{10} + C \\ &= \tfrac{1}{10}\sec^{10}(x) + C\end{aligned}$$

(g) (10 pts) $\int \sin^4(x)dx$

SOLUTION: We use the identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ to get

$$\int \sin^4(x)dx = \frac{1}{4}\int(1 - 2\cos(2x) + \cos^2(2x))dx$$

Now we use the identity $\cos^2(2x) = \frac{1}{2}(1 + \cos(4x))$ to get

$$= \frac{1}{4}\int\left(1 - 2\cos(2x) + \frac{1}{2}(1 + \cos(4x))\right)dx$$

$$= \int\left(\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right)dx$$

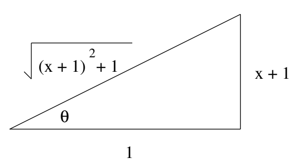
$$= \tfrac{3}{8}x - \tfrac{1}{4}\sin(2x) + \tfrac{1}{32}\sin(4x) + C$$

(h) (10 pts) $\int \frac{dx}{(x^2 + 2x + 2)^{3/2}}$

SOLUTION: We need to complete the square in the denominator.

$$(x^2 + 2x) + 2 = (x^2 + 2x + 1) - 1 + 2 = (x + 1)^2 + 1$$

So we have $\int \frac{dx}{(x^2 + 2x + 2)^{3/2}} = \int \frac{dx}{((x + 1)^2 + 1)^{3/2}}$



Using trig substitution, we have $((x + 1)^2 + 1)^{3/2} = sec^3(\theta)$, $x = \tan(\theta) - 1$, and $dx = \sec^2(\theta)d\theta$.

Thus, $\int \frac{dx}{((x + 1)^2 + 1)^{3/2}} = \int \frac{\sec^2(\theta)d\theta}{sec^3(\theta)}$

$$= \int \cos(\theta)d\theta$$

$$= \sin(\theta) + C$$

$$= \frac{x + 1}{\sqrt{(x + 1)^2 + 1}} + C$$

8. Let C be the curve given parametrically by $x(t) = e^t$ and $y(t) = t^2$ for $0 \leq t \leq 2$.

(a) (6 pts) Find $\frac{dy}{dx}$ as a function of t.

SOLUTION: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{e^t}$

(b) (6 pts) Find the equation of the line tangent to the curve at the point where $t = 1$.

SOLUTION: When $t = 1$, then $x = e$, $y = 1$, and $\frac{dy}{dx} = \frac{2}{e}$.

So the equation of the line is $y - 1 = \frac{2}{e}(x - e)$.

(c) (6 pts) Calculate $\frac{d^2y}{dx^2}$ as a function of t.

SOLUTION:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{2e^t - 2te^t}{e^{2t}}}{e^t} = \frac{2 - 2t}{e^{2t}}$$

(d) (6 pts) [**Set Up Only**] Set up the integral for the arc length of C.

SOLUTION: $L = \int_0^2 \sqrt{e^{2t} + 4t^2}dt$

(e) (6 pts) [**Set Up Only**] Set up the integral for the surface area of the surface generated when C is revolved about the y-axis.

SOLUTION: $A = 2\pi \int_0^2 e^t\sqrt{e^{2t} + 4t^2}dt$

9. (15 pts) Evaluate $\displaystyle\int \frac{x^{-1/3}+x^{-1/2}}{x^{1/3}-2x^{1/6}+1}dx$

SOLUTION: Let $u=\sqrt[6]{x}$, so $u^6=x \Rightarrow 6u^5du=dx$.

Also, $x^{-1/3}=(u^6)^{-1/3}=u^{-2}$, $x^{-1/2}=(u^6)^{-1/2}=u^{-3}$, and $x^{1/3}=(u^6)^{1/3}=u^2$.

Therefore, we get $\displaystyle\int \frac{x^{-1/3}+x^{-1/2}}{x^{1/3}-2x^{1/6}+1}dx = \int \frac{u^{-2}+u^{-3}}{u^2-2u+1}(6u^5du)$

$$= 6\int \frac{u^3+u^2}{u^2-2u+1}du$$

Using long division of polynomials, we get

$$\begin{array}{r|l} & u+3 \\ \hline u^2-2u+1 & u^3+u^2 \\ & -u^3+2u^2-u \\ \hline & 3u^2-u \\ & -3u^2+6u-3 \\ \hline & 5u-3 \end{array}$$

So, $\displaystyle 6\int \frac{u^3+u^2}{u^2-2u+1}du = 6\int\left(u+3+\frac{5u-3}{(u-1)^2}\right)du$

$$= 6\int\left(u+3+\frac{5u-5}{(u-1)^2}+\frac{2}{(u-1)^2}\right)du$$

$$= 6\int\left(u+3+\frac{5}{u-1}+\frac{2}{(u-1)^2}\right)du$$

$$= 6\left(\frac{u^2}{2}+3u+5\ln|u-1|-\frac{2}{u-1}\right)+C$$

$$= 3\sqrt[3]{x}+18\sqrt[6]{x}+30\ln|\sqrt[6]{x}-1|-\frac{12}{\sqrt[6]{x}-1}+C$$