

1. (8 points) $\int_1^9 \frac{dx}{(x-1)^{1/3}} =$

Solution: This is improper at the lower endpoint. First find the indefinite integral by the simple substitution $u = x - 1$, $dx = du$: $\int \frac{dx}{(x-1)^{1/3}} = \int \frac{du}{u^{1/3}} = \int u^{-1/3} du = \frac{1}{2/3} u^{2/3} + C = \frac{3}{2}(x-1)^{2/3} + C$.

Hence:

$$\int_1^9 \frac{dx}{(x-1)^{1/3}} = \lim_{a \rightarrow 1^+} \int_a^9 \frac{dx}{(x-1)^{1/3}} = \lim_{a \rightarrow 1^+} \left[\frac{3}{2}(x-1)^{2/3} \right]_a^9 = \frac{3}{2} \left[8^{2/3} - \lim_{a \rightarrow 1^+} (a-1)^{2/3} \right] = \frac{3}{2} [4 - 0] = 6$$

2. (8 points) $\int_0^{\pi/2} x \sin x dx =$

Solution: Use integration by parts with $u = x$, $dv = \sin x dx$ so $du = dx$ and $v = \int \sin x dx = -\cos x$:

$$\begin{aligned} \int_0^{\pi/2} x \sin x dx &= \left[x(-\cos x) - \int (-\cos x) dx \right]_0^{\pi/2} = \left[-x \cos x + \int \cos x dx \right]_0^{\pi/2} = \\ & \left[-x \cos x + \sin x \right]_0^{\pi/2} = -\frac{\pi}{2} \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) - (-0 \cdot \cos 0 + \sin 0) = -\frac{\pi}{2} \cdot 0 + 1 - 0 = 1 \end{aligned}$$

3. (8 points) $\int_1^e x^3 \ln x dx =$

Solution: Use integration by parts with $u = \ln x$, $dv = x^3 dx$, so $du = \frac{dx}{x}$ and $v = \int x^3 dx = \frac{1}{4}x^4$:

$$\begin{aligned} \int_1^e x^3 \ln x dx &= \left[\ln x \cdot \frac{1}{4}x^4 \right]_1^e - \int_1^e \frac{1}{4}x^4 \frac{dx}{x} = \frac{1}{4} \ln e \cdot e^4 - \frac{1}{4} \ln 1 \cdot 1^4 - \frac{1}{4} \int_1^e x^3 dx = \\ & \frac{1}{4} \cdot 1 \cdot e^4 - \frac{1}{4} \cdot 0 \cdot 1 - \frac{1}{4} \left[\frac{1}{4}x^4 \right]_1^e = \frac{1}{4}e^4 - \frac{1}{16} [e^4 - 1^4] = \frac{1}{4}e^4 - \frac{1}{16}e^4 + \frac{1}{16} = \frac{3}{16}e^4 + \frac{1}{16} \end{aligned}$$

4. (8 points) $\int_9^\infty \frac{dx}{\sqrt{x}} =$

Solution: This is improper at the upper endpoint:

$$\int_9^\infty \frac{dx}{\sqrt{x}} = \lim_{b \rightarrow +\infty} \int_9^b \frac{dx}{\sqrt{x}} = \lim_{b \rightarrow +\infty} \int_9^b x^{-1/2} dx = \lim_{b \rightarrow +\infty} \left[\frac{x^{1/2}}{1/2} \right]_9^b = \lim_{b \rightarrow +\infty} 2b^{1/2} - 2 \cdot 9^{1/2} = +\infty$$

5. (20 points) Fill in the blanks. You do not have to show any work for this problem.

(a) Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right)$ as a single fraction:

Solution: The N^{th} partial sum, for $N \geq 6$, is $\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{N-2} - \frac{1}{N}\right) + \left(\frac{1}{N-1} - \frac{1}{N+1}\right) + \left(\frac{1}{N} - \frac{1}{N+2}\right)$

This is a telescoping sum; everything cancels out except $1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}$, so the limit of the N^{th} partial sum is $1 + \frac{1}{2} = \boxed{\frac{3}{2}}$

(b) Evaluate $\sum_{n=0}^{\infty} \left(-\frac{2}{5}\right)^n$ as a single fraction:

Solution: Geometric series: $\sum_{n=0}^{\infty} \left(-\frac{2}{5}\right)^n = \frac{1}{1 - (-\frac{2}{5})} = \boxed{\frac{5}{7}}$

(c) Evaluate $\sum_{n=2}^{\infty} \left(-\frac{2}{5}\right)^{n-1}$ as a single fraction:

Solution: Geometric series: Factor out the first term, which is $-\frac{2}{5}$, and then substitute $n = m + 2$:

$$\sum_{n=2}^{\infty} \left(-\frac{2}{5}\right)^{n-1} = -\frac{2}{5} \sum_{n=2}^{\infty} \left(-\frac{2}{5}\right)^{n-2} = -\frac{2}{5} \sum_{m=0}^{\infty} \left(-\frac{2}{5}\right)^m = -\frac{2}{5} \cdot \frac{1}{1 - (-\frac{2}{5})} = \boxed{-\frac{2}{7}}$$

(d) For what range of p values does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

Solution: This is the p -series, which converges for $\boxed{p > 1}$ and diverges for other values of p .

(e) For what range of p values does $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converge?

Solution: This diverges for $p \leq 0$, since the general term does not converge to 0 in this case. For other values the series converges by the Alternating Series Test, so the answer is $\boxed{p > 0}$

For the next three parts suppose $a_n > 0$ and $b_n > 0$ and $\sum_{n=1}^{\infty} b_n$ **diverges**, and suppose $L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ exists. In each part, be sure you indicate **all** values of L which answer the question. The answer may be “none”.

(f) For what values(s) of L can you conclude that $\sum_{n=1}^{\infty} a_n$ **diverges**?

Solution: According to the Limit Comparison Test (including the extended case $L = \infty$), $\sum_{n=1}^{\infty} a_n$ diverges when $\boxed{0 < L \leq \infty}$

(g) For what value(s) of L is it **impossible** to tell whether $\sum_{n=1}^{\infty} a_n$ converges or diverges?

Solution: It is impossible to conclude anything about convergence or divergence if $\boxed{L = 0}$

(h) For what value(s) of L can you conclude that $\sum_{n=1}^{\infty} a_n$ **converges**?

Solution: It is impossible to conclude that a series converges by comparing it to a divergent series, so the answer is none

6. (9 points) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{100n + 1}$

(a) (1 point) This series converges diverges (Circle your answer.)

(b) (8 points) Justify your answer.

Solution: This series diverges: If $a_n = \frac{(-1)^n n}{100n + 1}$ then $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{n}{n + 1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1$. Hence

it cannot be true that a_n converges to 0, so the series $\sum_{n=1}^{\infty} a_n$ diverges by the **Test for Divergence**.

7. (9 points) $\sum_{n=1}^{\infty} \frac{2^n}{3^n + 4^n}$

(a) (1 point) This series converges diverges (Circle your answer.)

(b) (8 points) Justify your answer.

Solution: This series converges: Let $a_n = \frac{2^n}{3^n + 4^n}$ and $b_n = \frac{2^n}{3^n}$. Then $a_n > 0$ and $b_n > 0$ and

$a_n < b_n$ (since the numerators are the same, but a_n has a larger denominator.) Also, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2^n}{3^n} =$

$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{2}{3} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1}$ and this converges because it is a geometric series with common ratio $\frac{2}{3}$,

which is less than 1 in absolute value. Hence the series $\sum_{n=1}^{\infty} a_n$ converges by the **Comparison Test**

8. (9 points) $\sum_{n=0}^{\infty} \frac{(-1)^n n^3}{n!}$

(a) (1 point) This series converges absolutely converges, but not absolutely diverges (Circle your answer.)

(b) (8 points) Justify your answer.

Solution: This series converges absolutely: Let $a_n = \frac{(-1)^n n^3}{n!}$. This is never 0, and

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^3}{(n+1)!}}{\frac{n^3}{n!}} = \lim_{n \rightarrow \infty} \frac{(n+1)^3}{n^3} \cdot \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^3 \cdot \frac{n!}{n! \cdot (n+1)} =$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^3 \cdot \frac{1}{n+1} = 1 \cdot 0 = 0$$

Since this limit exists and is less than 1 the **Ratio Test** implies that $\sum_{n=0}^{\infty} |a_n|$ converges. Hence, by

definition, $\sum_{n=0}^{\infty} a_n$ converges absolutely.

9. (9 points) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n^2+1}\right)^n$

(a) (1 point) This series

converges	diverges
-----------	----------

 (Circle your answer.)

(b) (8 points) Justify your answer.

Solution: This series converges. Let $a_n = \left(\frac{n+1}{n^2+1}\right)^n$. Then $a_n > 0$ and

$$\lim_{n \rightarrow \infty} a_n^{1/n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n^2+1}\right) = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} + \frac{1}{n^2}}{1 + \frac{1}{n^2}} = \frac{0+0}{1+0} = 0$$

Since this limit exists and is less than 1 the **Root Test** implies that $\sum_{n=1}^{\infty} a_n$ converges.

10. (12 points) This problem concerns the convergent series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$, and the partial

sums $S_N = \sum_{n=1}^N \frac{(-1)^{n+1}}{\sqrt{n}}$.

(a) (5 points) State the Alternating Series Test precisely, and indicate why it applies to this series.

Solution: The Alternating Series Test says that a series $\sum_{n=1}^{\infty} a_n$ converges if it satisfies these three conditions:

- (i) The signs of the terms alternate.
- (ii) $\langle |a_n| \rangle$ is a decreasing sequence.
- (iii) $\lim_{n \rightarrow \infty} |a_n| = 0$.

In this case, $a_n = \frac{(-1)^{n+1}}{\sqrt{n}}$. Clearly the signs alternate; $|a_n| = \frac{1}{\sqrt{n}}$ is decreasing (since the denominators are increasing); and $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$

(b) (6 points) Find the smallest value of N for which you can guarantee that $|S - S_N| \leq \frac{1}{7}$, and explain why your answer is correct.

Solution: Since the series satisfies the conditions of the alternating series test, $|S - S_N| \leq |a_{N+1}|$. That is, the remainder, in absolute value, is bounded by the absolute value of the first omitted term. We now solve the inequality $|a_{N+1}| = \frac{1}{\sqrt{N+1}} \leq \frac{1}{7}$. Inverting both sides, this is equivalent

to $\sqrt{N+1} \geq 7$. Since $N+1$ is positive, we can square both sides to get the equivalent inequality $N+1 \geq 49$, so $N \geq 48$. That is, the smallest value of N satisfying $|a_{N+1}| \leq \frac{1}{7}$ is $\boxed{N = 48}$.

This value of N satisfies the requirement that $|S - S_N| \leq \frac{1}{7}$ since $|S - S_N| \leq |a_{N+1}| \leq \frac{1}{7}$. We can't guarantee that any smaller value of N will work since we don't have any other way of bounding $|S - S_N|$.

(c) (1 point) For the value of N that you determined in (b), is $S - S_N$ positive or negative?

positive negative (Circle your answer.)

Solution: $S - S_N$ has the same sign as the first omitted term. Since $N = 48$, the first omitted term is $a_{49} = \frac{(-1)^{49+1}}{\sqrt{49}} = \frac{1}{\sqrt{49}}$, which is $\boxed{\text{positive}}$.