Math 222 Calculus 2 Fall 2005 Exam 3 Solutions

(1) (8 Points) In each part test the series for convergence or divergence. Write all steps of the test you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5n^2+n+1}$

(4 Points) Since $0 \leq \frac{1}{5n^2+n+1} \leq \frac{1}{5n^2}$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \frac{1}{5n^2} = \frac{1}{5}\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a multiple of a convergent p-series with $p = 2$, the series converges by the Comparison test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4-n^2+1}}$

(4 Points) Apply the Limit Comparison test where the series $\sum b_n$ being compared to is the convergent p-series $\sum \frac{1}{n^{4/3}}$. We have

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n^4-n^2+1}}}{\frac{1}{\sqrt[3]{n^4}}} = \lim_{n\to\infty} \sqrt[3]{\frac{n^4}{n^4-n^2+1}} = \lim_{n\to\infty} \sqrt[3]{\frac{1}{(1-\frac{1}{n^2}+\frac{1}{n^4})}} = 1 > 0$$

so the Limit Comparison test says both series have the same behavior, they both converge.

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(2) (10 Points) Test the following series for absolute convergence, conditional convergence, or divergence. Explain what tests you are applying and how you apply them.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$

(5 Points) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ is an alternating series, and for $n \geq 1$ we have $\sqrt[3]{n+1} > \sqrt[3]{n}$ so $\frac{1}{\sqrt[3]{n+1}} < \frac{1}{\sqrt[3]{n}}$. Also, $lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$, so this series converges by the Alternating Series test. But the series whose terms are the absolute values of those terms is the p-series with $p = 1/3 < 1$, which diverges. So the given series converges conditionally.

(b) $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

(5 Points) Applying the root test to the series $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ gives $L = \lim_{n\to\infty} 1+\frac{1}{n} = 1$, which gives no information. But $\lim_{n\to\infty}(1+(1/n))^n = e$ so the n^{th} term of the series is converging to e which is nonzero. Therefore, the series diverges by the Test for Divergence.

(3) (12 Points) Find all values of x where the following power series converge (internal of convergence).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n10^n}$

(8 Points) The Ratio Test gives $\left|\frac{(x-1)^{n+1}}{(n+1)10^{n+1}}\frac{n10^n}{(x-1)^n}\right| = \frac{n}{n+1}\frac{|x-1|}{10}$. As n goes to infinity, this ratio goes to $L = \frac{|x-1|}{10}$ so that $L < 1$ iff $|x-1| < 10$ iff $-9 < x < 11$. We must check for convergence at the endpoints separately. At $x = 11$ the series is $\sum_{n=1}^{\infty} \frac{10^n}{n10^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is the harmonic series which diverges. At $x = -9$ the series is $\sum_{n=1}^{\infty} \frac{(-10)^n}{n10^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is the alternating harmonic series which converges. So the complete domain of convergence is $[-9, 11)$, that is, $-9 \le x < 11$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{5^n x^n}{n!}$

(4 Points) The Ratio Test gives $\left|\frac{5^{n+1}x^{n+1}}{(n+1)!}\frac{n!}{5^n x^n}\right| = \frac{5|x|}{n+1}$. As n goes to infinity, for any fixed x, this ratio goes to zero, so the series converges (absolutely) for all real x, that is, on $(-\infty, \infty)$.

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(4) (10 Points) Write a power series representation for each function and give its radius of convergence. Use summation notation to express all terms.

(a) $\displaystyle\frac{1}{(1-x)^3}$

(5 Points) The series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$. Taking the derivative of it, we get $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ and taking the derivative again, we get $\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$ so

$$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2} = \sum_{m=0}^{\infty} \frac{(m+2)(m+1)}{2} x^m$$

converges for $|x| < 1$. We can also get this from the binomial series formula

$$(1-x)^{-3} = \sum_{n=0}^{\infty} \binom{-3}{n} (-x)^n = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n.$$

(b) $e^{(x^3)}$

(5 Points) The series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for all real x. So

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} \text{ converges absolutely for all real } x.$$

(5) (10 Points) Find the second degree Taylor polynomial $T_2(x)$ which approximates the function $f(x) = \sqrt[3]{x} = x^{1/3}$ near $a = 8$.

(10 Points) We have $f^{(1)}(x) = \frac{1}{3}x^{-2/3}$, $f^{(2)}(x) = \frac{-2}{9}x^{-5/3}$, and $f^{(3)}(x) = \frac{10}{27}x^{-8/3}$. Then $f(8) = (8)^{1/3} = 2$, $f^{(1)}(8) = \frac{1}{3}8^{-2/3} = \frac{1}{12}$, $f^{(2)}(8) = \frac{-2}{9}8^{-5/3} = \frac{-1}{144}$ so

$$T_2(x) = f(8) + \frac{f^{(1)}(8)}{1!}(x-8) + \frac{f^{(2)}(8)}{2!}(x-8)^2 = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2.$$

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(6) (10 Points) The alternating series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ converges to S and has N^{th} partial sum $S_N = \displaystyle\sum_{k=1}^{N} \frac{(-1)^k}{\sqrt{k}}$. Find the smallest N such that you can be sure $|S - S_N| < \frac{1}{100}$.

(10 Points) We have $|S - S_N| \leq \dfrac{1}{\sqrt{N+1}}$ from the Alternating Series Estimation theorem, so to guarantee the accuracy given, we need $\dfrac{1}{\sqrt{N+1}} < \dfrac{1}{100}$ which means $100 < \sqrt{N+1}$ which means $10000 < N+1$. The smallest N such that this is true is $N = 10000$.

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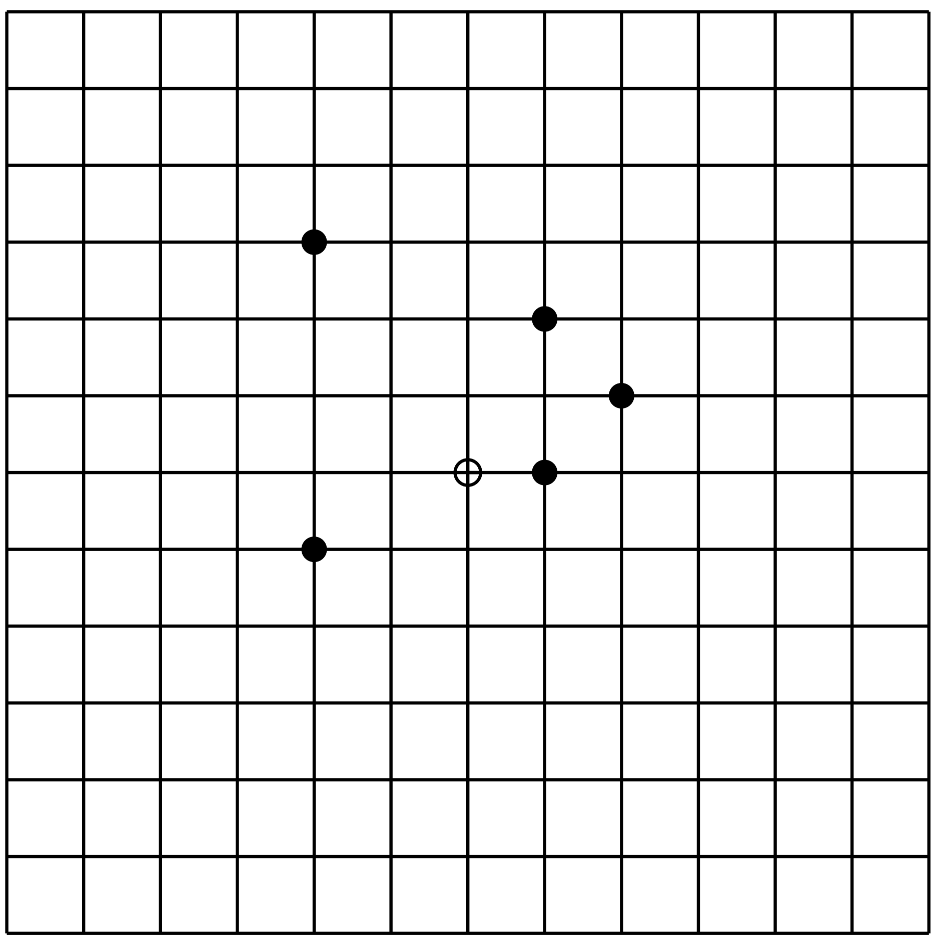
(7) (8 Points) The motion of a particle is given by the parametric equations $x = -t^2 + 2$ and $y = t + 1$ for $-2 \leq t \leq 2$.

(a) Find the equation relating x and y without parameter t.

(4 Points) Substituting $t = y - 1$ into the equation for x we get $x = -(y-1)^2 + 2$ which can be written as $x - 2 = -(y-1)^2$. This is a left-facing parabola with vertex at $(2, 1)$.

(b) Sketch the curve and indicate the direction of motion of the particle.

(4 Points) The motion of a particle given by the parametric equations $x = -t^2 + 2$ and $y = t + 1$ for $-2 \leq t \leq 2$ is along a parabola. As t hits the values -2, -1, 0, 1, 2 the points (x, y) hit are $(-2, -1)$, $(1, 0)$, $(2, 1)$, $(1, 2)$ and $(-2, 3)$. (See Sketch).



(8) (12 Points) Parametric equations for a curve are $x = t^2 + \sin(t)$ and $y = t + \cos(t)$ for $0 \le t \le 1$.

(a) Set up the integral for the length of that curve, but DO NOT EVALUATE OR SIMPLIFY THAT INTEGRAL. (4 Points)

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{(2t + \cos(t))^2 + (1 - \sin(t))^2}\, dt$$

(b) Set up the integral for the surface area obtained by rotating that curve around the y-axis, but DO NOT EVALUATE OR SIMPLIFY THAT INTEGRAL. (4 Points)

$$L = \int_0^1 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 2\pi(t^2 + \sin(t))\sqrt{(2t + \cos(t))^2 + (1 - \sin(t))^2}\, dt$$

(c) Find the equation of the tangent line to that curve at $t = 0$. (4 Points)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - \sin(t)}{2t + \cos(t)}$$

and at $t = 0$ this slope is 1. At $t = 0$ we also have $x = 0$ and $y = 1$, so the equation of the tangent line is $y - 1 = 1(x - 0)$, which can be written as $y - 1 = x$ or as $y = x + 1$.