

1. (24 points) Put your answers in the blanks. You do not have to show any work.

(a)  $\sum_{n=1}^{\infty} \left( \frac{1}{n+2} - \frac{1}{n+3} \right) =$

**Solution:**  $\frac{1}{3}$

(b)  $\sum_{n=1}^{\infty} \left( \frac{-2}{5} \right)^{n-1} =$

**Solution:**  $\frac{5}{7}$

(c)  $\lim_{n \rightarrow \infty} \frac{1000 + 10n + n^2}{4n^2 - 330} =$

**Solution:**  $\frac{1}{4}$

(d)  $\lim_{n \rightarrow \infty} \sum_{k=0}^n \left( \frac{1}{5} \right)^k =$

**Solution:**  $\frac{5}{4}$

(e)  $\lim_{n \rightarrow \infty} \frac{(2n)!}{(2n-1)!} =$

**Solution:**  $+\infty$

(f) Give a power series for  $\sin x$ :

**Solution:**  $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

(g) Give a power series for  $e^{-x}$ :

**Solution:**  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$

(h) Give a power series for  $\frac{d}{dx} \left( \frac{1}{1-x} \right)$ :

**Solution:**  $\sum_{n=1}^{\infty} nx^{n-1}$

2. (5 points) This problem concerns the **sequence**  $\left\langle \frac{n^6}{3^n} \right\rangle$  for  $n \geq 1$ . Circle your answers, and fill in the box if you can. You do not have to show any work.

(a) Does this sequence have a limit?  yes  no If so, what is it?

**Solution:** Yes. The limit is 0.

(b) Does this sequence converge?  yes  no

**Solution:** Yes

(c) Is this sequence increasing?  yes  no

**Solution:** No

(d) Is this sequence decreasing?  yes  no

**Solution:** No

(e) Is this sequence bounded?  yes  no

**Solution:** Yes

3. (9 points) 
$$\sum_{n=1}^{\infty} \frac{n^2}{100n^2 - 1}$$

(a) (1 point) Does this series converge? Circle your answer:  yes  no

**Solution:** No

(b) (8 points) Justify your answer.

**Solution:** The limit of the  $n^{\text{th}}$  term is  $\lim_{n \rightarrow \infty} \frac{n^2}{100n^2 - 1} = \lim_{n \rightarrow \infty} \frac{1}{100 - 1/n^2} = \frac{1}{100}$ , and so the  $n^{\text{th}}$  term does not converge to 0. Therefore  $\sum_{n=1}^{\infty} \frac{n^2}{100n^2 - 1}$  diverges by the **Test for Divergence**

4. (9 points) 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

(a) (1 point) Does this series converge? Circle your answer:  yes  no

**Solution:** Yes

(b) (8 points) Justify your answer.

**Solution:** The following three conditions are satisfied for this series:

(1) The terms alternate in sign.

(2) The absolute value of the terms is  $\frac{1}{\sqrt{n}}$ , and this sequence is decreasing since the sequence  $\sqrt{n}$  is obviously increasing.

(3) The limit of the  $n^{\text{th}}$  term is 0.

Therefore  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$  converges by the **Alternating Series Test**.

5. (9 points) 
$$\sum_{n=1}^{\infty} \frac{100n - 1}{n^3}$$

(a) (1 point) Does this series converge? Circle your answer: 

yes	no
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**Solution:** Yes

(b) (8 points) Justify your answer.

**Solution:** Let  $a_n = \frac{100n - 1}{n^3}$  and  $b_n = \frac{1}{n^2}$ .

(1)  $a_n$  and  $b_n$  are sequences of positive terms.

(2) The limit of their ratios is

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{100n - 1/n^3}{1/n^2} = \lim_{n \rightarrow \infty} \frac{100n - 1}{n} = \lim_{n \rightarrow \infty} \left(100 - \frac{1}{n}\right) = 100.$$

Hence the limit of  $\frac{a_n}{b_n}$  exists and is neither 0 nor  $\infty$ .

(3)  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$  converges because it is a  $p$ -series, with  $p = 2$ , and a  $p$ -series converges if  $p > 1$ .

Therefore  $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{100n - 1}{n^3}$  converges by the **Limit Comparison Test**.

6. (9 points)  $\sum_{n=0}^{\infty} \frac{(-1)^n n^n}{(2n + 1)^n}$

(a) (1 point) Does this series converge absolutely? Circle your answer: 

yes	no
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**Solution:** Yes

(b) (8 points) Justify your answer.

**Solution:** If  $a_n$  is the  $n^{\text{th}}$  term of this series then

$\lim_{n \rightarrow \infty} |a_n|^{1/n} = \lim_{n \rightarrow \infty} \left( \frac{n^n}{(2n + 1)^n} \right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{2n + 1} = \lim_{n \rightarrow \infty} \frac{1}{2 + 1/n} = \frac{1}{2}$ . Since this limit exists and is

less than 1,  $\sum_{n=0}^{\infty} |a_n|$  converges by the **Root Test**. Then  $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{(-1)^n n^n}{(2n + 1)^n}$  converges absolutely.

7. (8 points) Find the radius of convergence of the power series  $\sum_{n=1}^{\infty} \frac{2^n x^n}{n^2}$ . Show your work.

**Solution:** This is a power series with center 0. Write the general term as  $a_n x^n$ , where  $a_n = \frac{2^n}{n^2}$ . If

$x = 0$  the series converges, so suppose  $x \neq 0$ . Then  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1} x^{n+1} / (n+1)^2}{2^n x^n / n^2} \right| =$

$\lim_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} \cdot 2|x| \right| = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^2 \cdot 2|x| = \lim_{n \rightarrow \infty} \left( \frac{1}{1 + 1/n} \right)^2 \cdot 2|x| = 2|x|$ . By the **Ratio Test**,

$\sum_{n=1}^{\infty} a_n x^n$  converges absolutely if this limit is less than 1 and diverges if this limit is greater than 1. That

is, it converges absolutely if  $2|x| < 1$ , or  $|x| < \frac{1}{2}$  and, similarly, it diverges if  $|x| > 1$ . Hence the radius of convergence is  $\frac{1}{2}$ .

8. (4 points) The power series  $\sum_{n=1}^{\infty} (-1)^n \frac{(x-2)^n}{n}$  has radius of convergence 1. **You do not need to check this.** You do not need to show any work for this problem.

(a) What is the center of this power series?

**Solution:** 2

(b) What are the endpoints of the interval of convergence?

**Solution:** 1, 3

(c) Does the series converge at the left endpoint? Circle your answer: 

yes	no
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**Solution:** No

(d) Does the series converge at the right endpoint? Circle your answer: 

yes	no
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**Solution:** Yes

9. (15 points) In this problem you may use the formula

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots$$

(a) Find a power series for  $\cos(x^2)$ , using sigma notation.

**Solution:** Substitute  $x^2$  for  $x$  in the series for  $\cos x$ :  $\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$

(b) Find a series for the value of  $\int_0^1 \cos(x^2) dx$ , using sigma notation.

**Solution:** The power series for  $\cos x$  has infinite radius of convergence, so the power series for  $\cos(x^2)$  also has infinite radius of convergence. Any power series may be integrated term-by-term over any closed interval which lies in the interior of the interval of convergence, so

$$\begin{aligned} \int_0^1 \cos(x^2) dx &= \int_0^1 \left( \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} \right) dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{4n}}{(2n)!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int_0^1 x^{4n} dx = \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{1}{4n+1} x^{4n+1} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)(2n)!} \cdot (1-0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)(2n)!} \end{aligned}$$

(c) Write out the first 3 *non-zero* terms of your series in part (b). Do not simplify your answer.

**Solution:**  $1 - \frac{1}{5 \cdot 2!} + \frac{1}{9 \cdot 4!} = 1 - \frac{1}{10} + \frac{1}{216} = \frac{977}{1080}$

(d) How close is your answer in part (c) to the value of the integral? (In other words, what bound can you put on the error?) Do not simplify your answer.

**Solution:** The series for the integral satisfies the conditions of the **Alternating Series Test**. Therefore the absolute value of the difference between the integral and the sum of the first three terms is no more than the absolute value of the first omitted term; that is, the fourth non-zero

term. This is  $\frac{1}{13 \cdot 6!} = \frac{1}{13 \cdot 720} = \frac{1}{9360}$ .

(e) Find  $f^{(20)}(0)$  if  $f(x) = \cos(x^2)$ . Do not simplify your answer.

**Solution:** The term in  $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$  containing  $x^{20}$  occurs when  $n = 5$ . The coefficient of  $x^{20}$  is  $\frac{(-1)^n}{(2n)!}$  when  $n = 5$ , or  $-\frac{1}{10!}$ . By **Taylor's Formula**, the coefficient of  $x^{20}$  is  $\frac{f^{(20)}(0)}{20!}$ . Equating these gives  $\frac{f^{(20)}(0)}{20!} = -\frac{1}{10!}$ , so  $f^{(20)}(0) = -\frac{20!}{10!} = 670442572800$ .

10. (8 points)

(a) (6 points) Using Taylor's (or MacLaurin's) formula for the coefficients, find the first 3 terms in the power series expansion of  $f(x) = \ln(3+x)$  centered at the origin. [That is, up to the term involving  $x^2$ .]

**Solution:** **Taylor's Formula** says that the  $n^{\text{th}}$  coefficient is  $\frac{f^{(n)}(0)}{n!}$ . Here are the calculations:

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$	$a_n$
0	$\ln(3+x)$	$\ln 3$	$\frac{\ln 3}{0!} = \ln 3$
1	$\frac{1}{3+x}$	$\frac{1}{3}$	$\frac{1}{1!} \cdot \frac{1}{3} = \frac{1}{3}$
2	$-\frac{1}{(3+x)^2}$	$-\frac{1}{9}$	$\frac{1}{2!} \left(-\frac{1}{9}\right) = -\frac{1}{18}$

Hence the sum of the first three terms is  $\ln 3 + \frac{1}{3}x - \frac{1}{18}x^2$ .

(b) (2 points) Using Taylor's formula for the remainder, find an expression for the remainder  $R_N$  when  $N = 2$ .

**Solution:** **Taylor's Formula for the Remainder** is  $R_N(x) = \frac{f^{(N+1)}(x^*)}{(N+1)!}(x-c)^{N+1}$  where  $c$  is the center of the series and  $x^*$  is between  $c$  and  $x$ . In this case  $c = 0$  and  $N = 2$  and  $f^{(3)}(x) = \frac{2}{(3+x)^3}$ , so the remainder is

$$\frac{2}{(3+x^*)^3} \cdot \frac{1}{3!} \cdot x^3 = \frac{x^3}{3(3+x^*)^3} \text{ where } x^* \text{ is between } 0 \text{ and } x.$$