Math 222 **Test 3** **April 25, 2007**

1. (6 points) Give the power series expansion about 0, and give the radius of convergence:

(a) $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} \boxed{x^n}$ radius of convergence = $\boxed{1}$

(b) $\sin x = \sum_{n=0}^{\infty} \boxed{\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}}$ radius of convergence = $\boxed{\infty}$

(c) $e^{3x} = \sum_{n=0}^{\infty} \boxed{\dfrac{3^n}{n!} x^n}$ radius of convergence = $\boxed{\infty}$

2. (6 points) Suppose $f(x) = \sum_{n=0}^{\infty} n^2 x^n$, for $|x| < 1$.

(a) The power series centered at 0 for $\int_0^x f(t)\,dt$ is $\sum_{n=0}^{\infty} \boxed{\dfrac{n^2}{n+1} x^{n+1}}$

(b) The power series centered at 0 for $\dfrac{d}{dx} f(x)$ is $\sum_{n=0}^{\infty} \boxed{n^3 x^{n-1}}$

(c) $f^{(47)}(0) = \boxed{47! \cdot 47^2}$

3. (12 points) For the power series $\sum_{n=1}^{\infty} \dfrac{1}{2^n \sqrt{n}} (x-3)^n$

(a) (6 points) Find the radius of convergence. Show your work.
Solution: Apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{\frac{1}{2^{n+1}\sqrt{n+1}}(x-3)^{n+1}}{\frac{1}{2^n\sqrt{n}}(x-3)^n} \right| = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{1}{2} \cdot |x-3| = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} \cdot \frac{1}{2} \cdot |x-3| = \frac{1}{2}|x-3|$$

According to the ratio test the series converges if this limit is less than 1, and this means $\frac{1}{2}|x-3| < 1$, or $|x-3| < 2$. Similarly, the series diverges for $|x-3| > 2$. So the radius of convergence is $\boxed{2}$.

(b) (3 points) What is the left endpoint of the interval of convergence? Does the series converge at the left endpoint of the interval of convergence? | yes | no | Give a short justification for your answer.
Solution: The left endpoint is 2 units (the radius of convergence) less than 3 (the center), so it is 1. When $x = 1$ the series becomes $\sum_{n=1}^{\infty} \dfrac{1}{2^n\sqrt{n}}(1-3)^n = \sum_{n=1}^{\infty} \dfrac{1}{2^n\sqrt{n}}(-2)^n = \sum_{n=1}^{\infty} \dfrac{(-1)^n 2^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n}}$, and this **converges** by the Alternating Series Test.

(c) (3 points) What is the right endpoint of the interval of convergence? Does the series converge at the right endpoint of the interval of convergence? | yes | no | Give a short justification for your answer.

Solution: The right endpoint is 2 units (the radius of convergence) more than 3 (the center), so it is 5. When $x = 5$ the series becomes $\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n}}(5-3)^n = \sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n}} 2^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. This is a p-series, with $p = \frac{1}{2} < 1$, so it **diverges**.

4. (12 points) Let $f(x) = \sqrt{4+x}$.

(a) (6 points) Using Taylor's (or MacLaurin's) formula for the coefficients, write the sum of the first 3 terms in the power series expansion of $f(x)$ centered at the origin. [That is, up to the term involving x^2.]

Solution: Calculate the following table:

n	$f^{(n)}(x)$	$f^{(n)}(0)$	a_n
0	$(4+x)^{1/2}$	$4^{1/2} = 2$	2
1	$\frac{1}{2}(4+x)^{-1/2}$	$\frac{1}{2}(4)^{-1/2} = \frac{1}{4}$	$\frac{1}{4}$
2	$-\frac{1}{4}(4+x)^{-3/2}$	$-\frac{1}{4}(4)^{-3/2} = -\frac{1}{32}$	$-\frac{1}{32} \cdot \frac{1}{2!} = -\frac{1}{64}$
3	$\frac{3}{8}(4+x)^{-5/2}$		

Hence the second degree Taylor polynomial is $2 + \frac{1}{4}x - \frac{1}{64}x^2$

(b) (3 points) Using Taylor's formula for the remainder, find an expression for the remainder $R_2(x)$.

Solution: $R_2(x) = \frac{f^{(3)}(x^*)}{3!}x^3 = \frac{1}{6} \cdot \frac{3}{8}(4+x^*)^{-5/2}x^3 = \frac{x^3}{16(4+x^*)^{5/2}}$

where x^* is some number between 0 (the center) and x.

(c) (3 points) Using the fact that $4 + x^* \geq 4$ if $x^* \geq 0$, find an upper bound for the absolute value of $R_2(1)$.

Solution: Since x^* is between 1 and 0 it is greater than or equal to 0, so

$|R_2(1)| = \frac{1^3}{16(4+x^*)^{5/2}} \leq \frac{1}{16 \cdot 4^{5/2}} = \frac{1}{16 \cdot 32} = \frac{1}{512}$

5. (9 points) $\int_0^{\pi/8} \cos^2(2x)\,dx =$

Solution: $\int_0^{\pi/8} \cos^2(2x)\,dx = \frac{1}{2}\int_0^{\pi/8} (1 + \cos(4x))\,dx = \frac{1}{2}\left[x + \frac{1}{4}\sin(4x)\right]_0^{\pi/8} =$
$\frac{1}{2}\left[\frac{\pi}{8} + \frac{1}{4}\sin(\pi/2) - 0 - 0\right] = \frac{\pi}{16} + \frac{1}{8}$

6. (9 points) $\int \tan^3 x \sec^3 x\,dx =$

Solution: $\displaystyle\int \tan^3 x \sec^3 x\,dx = \int \tan^2 x \sec^2 x \tan x \sec x\,dx = \int \left(\sec^2 x - 1\right) \sec^2 x \tan x \sec x\,dx$
Now substitute $u = \sec x,\ du = \tan x \sec x\,dx$:
$\displaystyle\int \left(\sec^2 x - 1\right) \sec^2 x \tan x \sec x\,dx = \int (u^2-1)u^2\,du = \int (u^4 - u^2)\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C =$
$\displaystyle\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$

7. (9 points) $\displaystyle\int x^3\sqrt{9-x^2}\,dx =$

Solution: This can be done several different ways. This solution uses the substitution $x = 3\sin\theta,\ dx = 3\cos\theta\,d\theta$, so
$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$
(We are assuming that θ is in the first quadrant so we do not have to worry about absolute values when taking the square root of $\cos^2\theta$.) Hence $\cos\theta = \frac{1}{3}\sqrt{9-x^2}$. Now continue with the substitution:
$\displaystyle\int x^3\sqrt{9-x^2}\,dx = \int (3\sin x)^3 \cdot 3\cos x \cdot 3\cos\theta\,d\theta = 243\int \sin^3 x\cos^2 x\,d\theta =$
$\displaystyle 243\int \sin^2\theta\cos^2\theta\sin\theta\,d\theta = 243\int (1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta$
Next, make the substitution $u = \cos\theta,\ du = -\sin\theta\,d\theta$:
$\displaystyle 243\int (1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta = -243\int (1-u^2)u^2\,du = -243\int (u^2-u^4)\,du = -243\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right] + C =$
$\displaystyle -243\left[\frac{1}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta\right] + C = -243\left[\frac{1}{3}\left(\frac{1}{3}(9-x^2)^{1/2}\right)^3 - \frac{1}{5}\left(\frac{1}{3}(9-x^2)^{1/2}\right)^5\right] + C =$
$\displaystyle -243\left[\frac{1}{3}\cdot\frac{1}{27}(9-x^2)^{3/2} - \frac{1}{5}\cdot\frac{1}{243}(9-x^2)^{5/2}\right] + C = -3(9-x^2)^{3/2} + \frac{1}{5}(9-x^2)^{5/2} + C$

8. (9 points) $\displaystyle\int_1^5 \frac{dx}{x^2-6x+13} =$

Solution: First complete the square in the denominator: $x^2-6x+13 = x^2-6x+9-9+13 = (x-3)^2+4$.
We will substitute $x-3 = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta$, so $\tan\theta = \dfrac{x-3}{2}$, or $\theta = \tan^{-1}\left(\dfrac{x-3}{2}\right)$.
Hence the limits $x=1$ and $x=5$ become $\theta = \tan^{-1}\left(\dfrac{1-3}{2}\right) = \tan^{-1}(-1) = -\dfrac{\pi}{4}$ and
$\theta = \tan^{-1}\left(\dfrac{5-3}{2}\right) = \tan^{-1}(1) = \dfrac{\pi}{4}$.
Now make this substitution:
$\displaystyle\int_1^5 \frac{dx}{x^2-6x+13} = \int_1^5 \frac{dx}{(x-3)^2+4} = \int_{-\pi/4}^{\pi/4} \frac{2\sec^2\theta\,d\theta}{(2\tan\theta)^2+4} = \int_{-\pi/4}^{\pi/4} \frac{2\sec^2\theta\,d\theta}{4(\tan^2\theta+1)} = \int_{-\pi/4}^{\pi/4} \frac{2\sec^2\theta\,d\theta}{4\sec^2\theta} =$
$\displaystyle\frac{1}{2}\int_{-\pi/4}^{\pi/4} d\theta = \frac{1}{2}\Big[\theta\Big]_{-\pi/4}^{\pi/4} = \frac{1}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$

9. (9 points) On this problem you have a choice:

$$\int \frac{dx}{1+\cos x+3\sin x} = \qquad \textbf{OR} \qquad \int \frac{dx}{(\sqrt{x}-1)^2} =$$

Do only one of these. Make sure that you cross out any work that is not related to your chosen integral.

Solution: The first integral uses the $z = \tan(x/2)$ substitution:

$$\int \frac{dx}{1+\cos x + 3\sin x} = \int \frac{\dfrac{2\,dz}{1+z^2}}{1+\dfrac{1-z^2}{1+z^2}+3\cdot\dfrac{2z}{1+z^2}} = \int \frac{\dfrac{2\,dz}{1+z^2}\cdot(1+z^2)}{\left(1+\dfrac{1-z^2}{1+z^2}+3\cdot\dfrac{2z}{1+z^2}\right)\cdot(1+z^2)} =$$

$$\int \frac{2\,dz}{1+z^2+1-z^2+6z} = \int \frac{2\,dz}{2+6z} = \int \frac{dz}{1+3z} = \frac{1}{3}\ln|1+3z| + C = \frac{1}{3}\ln\left|1+3\tan\left(\frac{x}{2}\right)\right| + C$$

The second integral uses the substitution $u = \sqrt{x}$, $x = u^2$, $dx = 2u\,du$, and then the substitution $v = u-1$, $u = v+1$, $du = dv$:

$$\int \frac{dx}{(\sqrt{x}-1)^2} = \int \frac{2u\,du}{(u-1)^2} = \int \frac{2(v+1)\,dv}{v^2} = 2\int\left(\frac{1}{v}+\frac{1}{v^2}\right)dv = 2\left(\ln|v| - \frac{1}{v}\right) + C =$$

$$2\ln|u-1| - \frac{2}{u-1} + C = 2\ln\left|\sqrt{x}-1\right| - \frac{2}{\sqrt{x}-1} + C$$

10. (9 points) $\displaystyle\int \frac{2x+8}{x^2-4}\,dx =$

Solution: First find the partial fraction decomposition of the integrand:

$$\frac{2x+8}{x^2-4} = \frac{2x+8}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

Clear fractions to get $2x+8 = A(x+2)+B(x-2)$. If you plug in $x=2$ the equation becomes $2\cdot 2+8 = 4A$, so $A = 3$. Similarly, plugging in $x=-2$ leads to $B=-1$. Now integrate:

$$\int \frac{2x+8}{x^2-4}\,dx = \int\left(\frac{3}{x-2} - \frac{1}{x+2}\right)dx = 3\ln|x-2| - \ln|x+2| + C$$

11. (10 points) Find the partial fraction expansion for $\dfrac{3x^3+x-1}{x^2(x^2+1)}$. You must determine the unknown constants, and show your work. **Do not** integrate anything.

Solution: First write the form for the partial fraction decomposition:

$$\frac{3x^3+x-1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

Next, clear fractions by multiplying by $x^2(x^2+1)$ and collect the terms of the same degree:

$$3x^3+x-1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2 = (A+C)x^3 + (B+D)x^2 + Ax + B$$

Equate coefficients of like powers:

$$3 = A+C \qquad 0 = B+D \qquad 1 = A \qquad -1 = B$$

The last two equations give $A = 1$ and $B = -1$. Then the first two equations become $3 = 1+C$ and $0 = -1+D$, so $C = 2$ and $D = 1$. So the partial fraction decomposition is

$$\frac{3x^3+x-1}{x^2(x^2+1)} = \frac{1}{x} - \frac{1}{x^2} + \frac{2x+1}{x^2+1}$$