Research Statement

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1. Introduction

My main research interest is combinatorics, with an emphasis on enumeration. My secondary interests are varied, ranging from operations research to the theory of determinants to computer algorithms all the way to mathematical models for social systems. The underlying focus for my work of the last year has been on counting domino tilings of various regions. A tiling consists of a union of non-overlapping dominos (unlabeled 2×1 or 1×2 rectangles) that entirely cover the region. Tiling problems were introduced in the 1930's in both organic chemistry as a model of carbon compounds and in statistical physics as a model for diatomic molecules. The enumeration of tilings explains various physical and chemical properties of these objects.

New combinatorial methods in the past thirty years produced a variety of results for different regions, but a theory that encompasses all regions is still elusive. Some milestones in this field include Kasteleyn's work in $\{0, +1, -1\}$ matrices in the 1960's [6], Gessel and Viennot's non-intersecting path method on acyclic graphs in the mid 1980's [3], and Jim Propp's urban renewal algorithm on graphs from the late 1990's [9]. My own work led me to explore related ideas in graph theory, matchings, and the theory of determinants. I explored a wide variety of directions, each leading to an increasingly more powerful result. In Section 2.3, I present the culmination of my research, an extension of the Gessel-Viennot method to some directed graphs with cycles.

First, a word about my choice of subject. I enjoy the subject of combinatorics for multiple reasons. For me, the goal of research is to make concepts understandable both to myself and to others. A truly satisfying proof is when I can take one set of objects, count it one way, and then after some manipulations count the same set of objects a different way. In this way, you really understand the "equals" in an identity. No other mathematical methods are as understandable as that to so many people. This allows me to share complex research projects with undergraduates so that they can jump in quickly, participate, and learn about the beauty of mathematics at an earlier point in life.

2. Motivation and Results

As I stated above, I work on counting tilings of various regions. In 1999, Jim Propp wrote the survey article "Enumerations of Matchings: Problems and Progress" [8] on domino tilings of various regions. While most of the presented problems have been either fully or partially solved, one problem that remained unsolved was the question of counting domino tilings of Aztec Pillows. I now explain my study in this area.

An Aztec diamond AD_n is a region that is the union of $2n$ rows of squares in the shape of a diamond. The top row has two squares centered on the next row of four squares, ... , through the n-th row with $2n$ squares, and the bottom n rows of squares are the reflection of the first n rows. An example of an Aztec Diamond is shown in Figure 1a. Elkies, Kuperberg, Larsen, and Propp [2] proved combinatorially that the number of domino tilings of an Aztec Diamond (represented by $\#AZ_n$) is exactly $\#AZ_n = 2^{n(n+1)/2}$.

Since then, many regions have been considered and analyzed, including a region called an Aztec Pillow. An Aztec Pillow is similar to an Aztec Diamond except that the boundary of the pillow has different size steps. If we think of an Aztec Diamond as having steps of type "up one and right one" as its upper left boundary, an Aztec Pillow has steps of type "up one and right three" as its upper left boundary. After a top row of either two or four squares, the upper right boundaries are the same and the region is centrally symmetric with a rotation of 180 degrees. An example of an Aztec Pillow is shown in Figure 1b.

In his 1999 survey article, Propp conjectured the number of domino tilings of an Aztec Pillow to be a product $b_n^2 s_n$ for some larger number b_n and some smaller number s_n that satisfies a simple generating function depending on whether the top step was of size 2 or of size 4. The respective generating functions for these step sizes are $(5+6x+3x^2-2x^3)/(1-2x-2x^2-2x^3+x^4)$ and $(5+3x+x^2-x^3)/(1-2x-2x^2-2x^3+x^4)$.

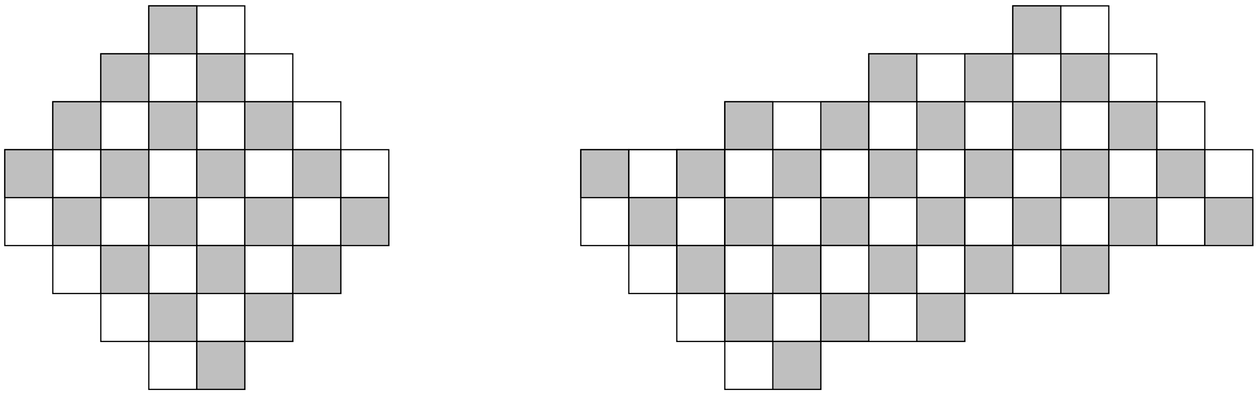

FIGURE 1. An Aztec Diamond and an Aztec Pillow

Approaching a problem of this type shows how experimental a science mathematics can be. To be able to grasp this formula, one has to get a sense of the data. Counting tilings by hand is infeasible since the regions quickly grow to become unmanageable. So we must rely on computer experiments to produce our data. Once again, the data is extremely large in size, so using computer software, we can break down the data (perhaps by factoring or other numerical procedures) and observe an otherwise unapparent structure. To make a concrete conjecture about the data and its structure, tools such as the Online Encyclopedia of Integer Sequences [10] or computer programs are essential once more. Computer analysis of the structure and manipulation of combinatorial constructs are both normally required to be able to acquire enough intuition to finally prove a new theorem.

This method proves a result from William Jockusch's thesis [4] in a new way, presented below.

2.1. Alternating Centrosymmetric Matrices.

Combinatorialists often study tilings of regions using a $\{0, +1, -1\}$ Kasteleyn matrix. Introduced in the 1960's [6], the determinant of the Kasteleyn matrix gives the number of tilings of the region. Gathering data as above allows us to conjecture that the family of Kasteleyn matrices for the Aztec Pillows all have the same form.

Define a matrix A to be *alternating centrosymmetric* if $KAK = A$, where K be the matrix that alternates 1's and -1's along the skew-diagonal.

Theorem 1. *The determinant of an alternating centrosymmetric matrix A equals* $\det(B+iC)\det(B-iC)$ *for certain submatrices B and C of A.*

Note that this implies that the determinant of an alternating centrosymmetric matrix is the sum of two squares. We define a region to be *2-even centrally symmetric* if its dual graph is symmetric after a rotation of 180 degrees, where this rotation sends each vertex to a vertex that is an even path length away. Aztec Diamonds and Aztec Pillows are examples of such regions; observing the form of their Kasteleyn matrices led to the discovery of the following theorem.

Theorem 2. *The Kasteleyn matrix of a 2-even centrally symmetric region is alternating centrosymmetric.*

These two theorems reprove the following result by Jockusch in a matrix-theoretic fashion.

Corollary 1. *The number of tilings of a 2-even centrally symmetric region is the sum of two squares.*

2.2. Brualdi-Kirkland type argument.

In 2003, Richard Brualdi and Stephen Kirkland reproved the result $\#AZ_n = 2^{n(n+1)/2}$ in a new way [1]. They find a Toeplitz matrix (a matrix with constant diagonals) whose entries are large Schröder numbers $s_n = 1, 2, 6, 22, \ldots$ and whose determinant gives the number of tilings of an Aztec Diamond. This matrix is derived from the Aztec Diamond via an "Aztec digraph". An example of this graph for AD_4 is given in Figure 2.

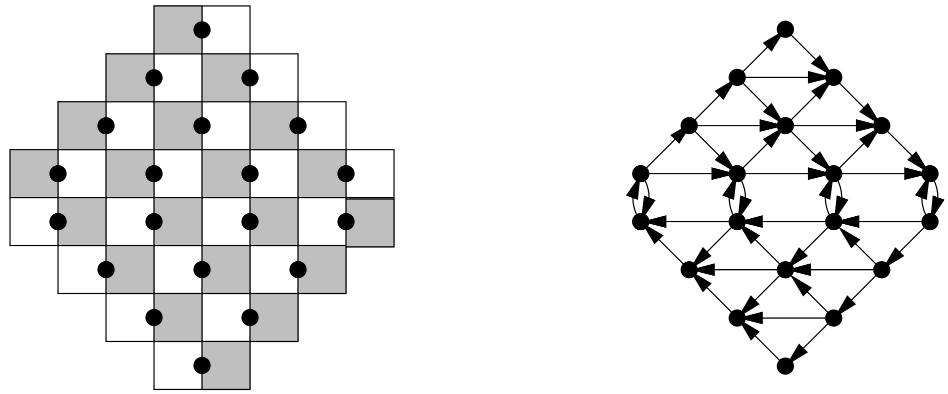

FIGURE 2. AZ_4 and its Digraph

If we adapt their method to Aztec Pillows, Schröder numbers do not appear, per se. Instead of a matrix with constant diagonals, the matrix has a two-dimensional array of integers that appeared for every Pillow. Restricted to its 6×6 principal submatrix, the array gives

$$A = \begin{bmatrix} 1 & 1 & 2 & 5 & 16 & 57 \\ 0 & 1 & 2 & 5 & 16 & 57 \\ 0 & 0 & 1 & 2 & 6 & 21 \\ 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1}$$

Let A_m be the $m \times m$ principal submatrix of this infinite array. Let J_m be the $m \times m$ transfer matrix, i.e. the matrix with 1's on the back-diagonal. Then the following theorem extension of Brualdi and Kirkland's method holds for the corresponding Aztec Pillow AP_m:

Theorem 3. *The number of domino tilings of AP_m is $\#AP_m = \det(A_m + J_m A_m^{-1} J_m)$.*

Unfortunately, the matrix $A_m + J_m A_m^{-1} J_m$ is not Toeplitz, so the method Brualdi and Kirkland used to find an explicit formula for Aztec Diamonds does not extend in the case of Aztec Pillows. However, this theorem does give a compact determinantal formula that counts the number of tilings of an Aztec Pillow.

2.3. **The Hamburger Theorem.** This compact matrix seems to be yelling out for a combinatorial interpretation. We now explain this interpretation as an expansion of the Gessel-Viennot method.

The Gessel-Viennot method [3] is a determinantal method for vertex-disjoint path systems in an acyclic directed graph with k sources s_i and k sinks t_j. A vertex-disjoint path system $\mathcal{P}$ is a collection of k vertex-disjoint paths from s_i to $t_{\sigma(i)}$ for some $\sigma \in S_k$. Let p^+ be the number of vertex-disjoint path systems $\mathcal{P}$ with $\operatorname{sgn}(\sigma) = +1$ and p^- be the number of vertex-disjoint path systems $\mathcal{P}$ with $\operatorname{sgn}(\sigma) = -1$. Define a $k \times k$ matrix $A = (a_{ij})$ where a_{ij} is the number of paths from s_i to t_j. The result of Gessel and Viennot states that $\det A = p^+ - p^-$. Note that this result can only hold on acyclic graphs since otherwise entries of A would be infinite. My main result is an analogous statement applying to a directed graph that we call a hamburger.

A hamburger graph H is made up of two acyclic graphs G_1 and G_2 and a connecting edge set E_3 with the following properties. The graph G_1 has k distinguished vertices $\{v_1, \cdots, v_k\}$ with paths from v_i to v_j only if $i < j$. The graph G_2 has k distinguished vertices $\{w_1, \cdots, w_k\}$ with paths from w_i to w_j only if $i > j$. The edge set E_3 connects each vertex v_i to vertex w_i and vice versa, giving H the structure as in Figure 3, hence its name. As an analogue to the Gessel-Viennot method, the goal is to count vertex-disjoint cycle systems in the hamburger graph. Define a vertex-disjoint cycle system $\mathcal{C}$ to be a collection of vertex-disjoint edge cycles. Let ℓ be the number of edges in $\mathcal{C}$ that travel from G_2 to G_1 and let m be the number of vertex-disjoint edge cycles in $\mathcal{C}$. Let c^+ be the number of vertex-disjoint cycle systems with $(-1)^{\ell+m} = +1$ and c^- be the number of vertex-disjoint cycle systems with $(-1)^{\ell+m} = -1$.

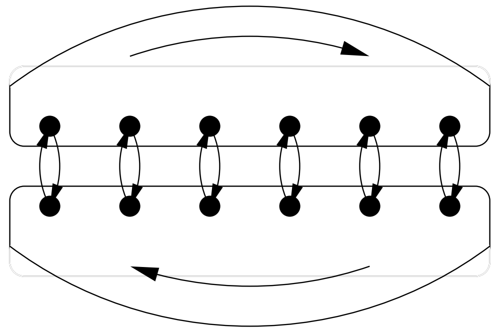

FIGURE 3. A Hamburger Graph

Given a hamburger graph H, define the hamburger matrix M_H to be the $2k \times 2k$ block matrix $M_H = \begin{bmatrix} A & I_k \\ -I_k & B \end{bmatrix}$, where the upper triangular matrix $A = (a_{ij})$ represents the number of paths from v_i to v_j in G_1 and the lower triangular matrix $B = (b_{ij})$ represents the number of paths from w_i to w_j in G_2. For each hamburger graph H and corresponding hamburger matrix M_H, the following "Hamburger Theorem" holds.

Theorem 4. *The determinant of the hamburger matrix* M_H *equals* $c^+ - c^-$.

The proof of the Hamburger Theorem, like the proof of the Gessel-Viennot method, hinges on terms canceling in the permutation decomposition of the determinant of M_H. Whereas the canceling in the Gessel-Viennot method is by pairs determined by where the first intersection point is, the canceling in the proof of the Hamburger Theorem relies on the appearance of families of intersecting cycle systems that cancel each other out. The same method also proves a weighted version of the Hamburger Theorem.

The beauty of the Hamburger Theorem comes from its purely combinatorial nature. It introduces a new counting method for cycle systems in non-planar graphs. Previous methods of counting cycle systems rely on the Kasteleyn matrix of a graph, which exists only for planar graphs; by contrast, the Hamburger Theorem applies even when G_1 and G_2 are non-planar. In addition, it simplifies Brualdi and Kirkland's novel proof on the enumeration of domino tilings of an Aztec Diamond and applies more generally to Aztec Pillows and other regions.

3. Proposed Future Research

In terms of continuing research in tilings, the following questions are natural extensions of my current research and would help to further the understanding of domino tilings in general:

- **Generalize and prove Propp's Conjecture**. Much more is now known about Aztec Pillows, but Jim Propp's conjecture proposed in 1999 remains unsolved. Calculating the squares that arise in Corollary 1 for generalized Aztec Pillows suggests that a more general conjecture than Propp's is true since the number of tilings for each of these regions is a large number squared times a small number. In addition, there is structure in the small numbers but for odd step sizes larger than 3, no simple generating function has been found.
- **Understand random tilings of Aztec Pillows**. An interesting phenomenon occurs when you try to choose a random tiling of an Aztec Diamond. A picture of one is shown in Figure 4a, and you can see that there are subregions of the Aztec Diamond that contain either only horizontal dominoes or only vertical dominoes. Jockusch, Propp, and Shor [5] proved that the border between these "frozen" parts of the tiling and the mixing subregion is in the limit the inscribed circle. A phenomenon that is similar but not as simple occurs in Aztec Pillows. An example of a random tiling of an Aztec Pillow is shown in Figure 4b. There are still regions containing one type of domino, but the structure of where the boundary is is not clear. Probabilistic methods are required to find out exactly what structure there is.

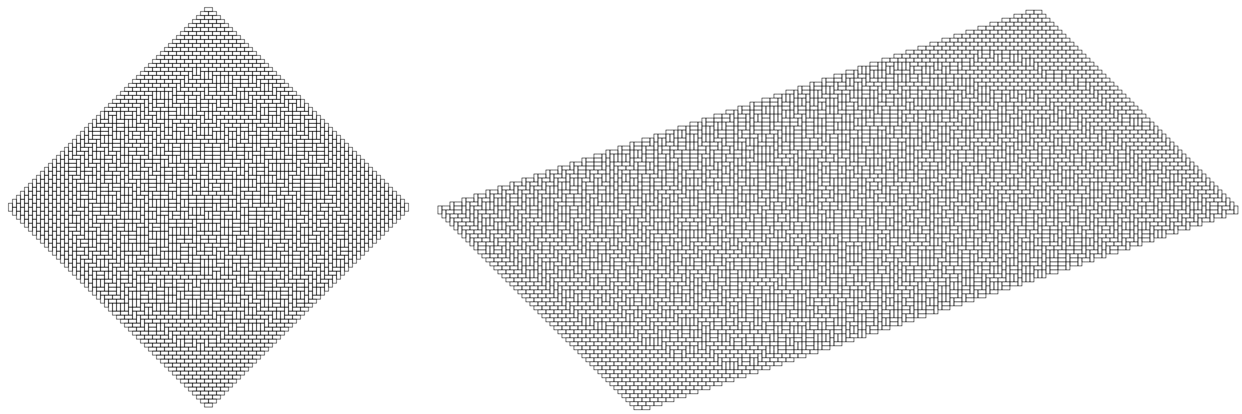

FIGURE 4. Random Tilings of an Aztec Diamond and an Aztec Pillow

As I stated in the introduction, I have wide reaching interest areas, for which I now provide some future research directions.

- **Theory of Determinants**: The structure inherent in combinatorial problems consistently yields determinantal formulas to count sets of objects. Christian Krattenthaler has written a survey article encompassing methods in determinantal calculus [7] that have been adapted to evaluate these determinants explicitly and are particularly useful in combinatorics research. One specific question in the theory of determinants comes from the matrix in Theorem 3. The analogous Toeplitz determinant that Brualdi and Kirkland found was calculated using a "J-fraction expansion". The added structure of the non-Toeplitz matrices arising in Aztec Pillow theory may lead to a generalization of this method.
- **Discrete Mathematical Modeling in the Social Sciences**: I am working with a colleague at the Center for Research in Health Care in Pittsburgh, PA. We presented a comparison between current utility assessments used in the medical profession. All the current models have been developed anecdotally and never analyzed mathematically. In particular, the assumptions in the statistical theory behind the utility assessments do not hold. Approaching the problem combinatorially produced a utility assessment that performed better than all the current models and satisfies these statistical assumptions. This collaboration has been fascinating and further collaborations will certainly be part of my research agenda over the coming years.

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