

Complex Matroids

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(joint work with Laura Anderson)

Combinatorics seminar
Binghamton University
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COMBINATORICS OF LINEAR DEPENDENCIES

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Maximal independent subsets of E :
Bases of $\text{Im } M$.

Fact: E contains a basis.

Theorem: Given bases $B_1, B_2 \subseteq E$,
 $e_1 \in B_1 \setminus B_2$, there is $e_2 \in B_2 \setminus B_1$
s.t. $(B_1 \setminus e_1) \cup e_2$ and $(B_2 \setminus e_2) \cup e_1$
both are bases of V .

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Consider circuits $C_1 \neq C_2$ and $e \in C_1 \cap C_2$.

With $E = \{e, v_2, \dots, v_n\}$, this means that there are coefficients $(\lambda_i)_{i \geq 2}$ and $(\mu_i)_{i \geq 2}$ with

$$\begin{array}{rcl} e + \lambda_2 v_2 + \dots + \lambda_n v_n & = & 0 \quad (\text{minimal lin. dep.,} \\ & & \lambda_i = 0 \text{ if } v_i \notin C_1) \\ -e + \mu_2 v_2 + \dots + \mu_n v_n & = & 0 \quad (\text{minimal lin. dep.}) \\ & & \mu_i = 0 \text{ if } v_i \notin C_2) \\ \hline (\lambda_2 + \mu_2)v_2 + \dots & = & 0 \quad (\text{lin. dep. among} \\ & & \text{elements of } C_1 \cup C_2) \end{array}$$

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there is a circuit C_3 with $C_3 \subseteq (C_1 \cup C_2) \setminus e$.

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Minimal supports of nonzero elements of $\ker M$: *Circuits*.

\emptyset is not a circuit.

Given minimal dependent subsets
 $C_1, C_2 \subset E$ and $e \in C_1 \cap C_2$,
 $(C_1 \cup C_2) \setminus e$ is dependent.
So, there is $C_3 \subseteq (C_1 \cup C_2) \setminus e$.

(Circuit elimination)

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B is a basis iff

$(B \cup e)$ contains a circuit
if and only if $e \in E \setminus B$



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- $C \not\subseteq B$ for all bases B ;
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Notice: If \mathcal{B} is the set of bases of a matroid \mathcal{M} on the ground set E , then

$$\mathcal{B}^* := \{E \setminus B \mid B \in \mathcal{B}\}$$

is the set of bases of another matroid \mathcal{M}^* called **dual** to the first.

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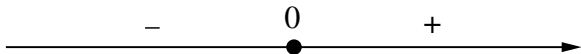
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Theorem.

$$\mathcal{C}^* = \min_{\subseteq} \{A \subseteq E \mid A \perp C \text{ for all } C \in \mathcal{C}\}.$$

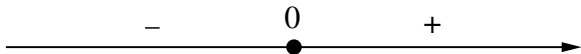
LINEAR DEPENDENCIES OVER \mathbb{R}

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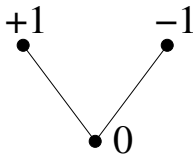
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So we consider the following set of signs.

Definition. The set $\{-1, 0, +1\}$ has a natural partial order coming from the stratification above



COMBINATORICS OF LINEAR DEPENDENCIES OVER \mathbb{R}

Let M be a $d \times n$ matrix with real coefficients.

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Every **ordered** element of \mathcal{B}
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Order $B \in \mathcal{B}$ as $\{v_1, \dots, v_d\}$,
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To every $C \in \mathcal{C}$ correspond
 $\lambda_i \in \mathbb{R}$ with $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$
where $\lambda_i \neq 0$ iff $v_i \in C$.

Given the λ_i s, define $X: E \rightarrow \{0, \pm\}$
as

$$X(v_i) := \text{sgn}(\lambda_i)$$

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Consider V as a d -dimensional linear subspace of \mathbb{R}^n so that, for all i , v_i is the orthogonal projection of the standard basis vector e_i on V .

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$$\sum_{j=0}^d (-1)^j \det(x_0, \dots, \hat{x}_j, \dots, x_d) \det(x_j, y_2, \dots, y_d) = 0$$

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Let $P := \{(-1)^l \chi(x_0, \dots, \widehat{x}_l, \dots, x_d) \chi(x_l, y_2, \dots, y_d) \mid 0 \leq l \leq d\}$.

If $P \neq \{0\}$, then $\{+1, -1\} \subseteq P$.

SIGNED CIRCUITS

Consider real coefficients $(\lambda_i)_{i \geq 2}$ and $(\mu_i)_{i \geq 2}$ as above with

$$\begin{array}{rcl} v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n & = & 0 \quad (\text{minimal lin. dep.}) \\ -v_1 + \mu_2 v_2 + \dots + \mu_n v_n & = & 0 \quad (\text{minimal lin. dep.}) \\ \hline (\lambda_2 + \mu_2)v_2 + \dots & = & 0 \quad (\text{lin. dep.}) \end{array}$$

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Given signed circuits X, Y and i, j
with $X(v_i) = -Y(v_i) \neq 0$ and $X(v_j) \neq Y(v_j)$,
there is a signed circuit Z with $Z(v_i) = 0$, $Z(v_j) \neq 0$
and, for all i , $Z(v_i) \in \{0, X(v_i), Y(v_i)\}$.

ORIENTED MATROIDS

Definition. An *oriented matroid* on the finite ground set E is...

an alternating function

$$\chi : E^d \rightarrow \{-, 0, +\}$$

Such that:

For $x_0, \dots, x_d, y_2, \dots, y_d \in E$

and the set P given by

$$\{(-1)^i \chi(x_0, \dots, \hat{x}_i, \dots, x_d) \chi(x_i, y_2, \dots, y_d)\}$$

either $P = \{0\}$ or $P \supseteq \{+, -\}$

(Chirotope axioms)

a subset $\mathcal{C} \subseteq \{-, 0, +\}^E \setminus \underline{0}$
such that for $X, Y \in \mathcal{C}$:

- $\text{supp}(X) = \text{supp}(Y) \Rightarrow X = \pm Y$
- for $e, f \in \text{supp}(X) \cap \text{supp}(Y)$ with $X(e) = -Y(e)$, $X(f) \neq -Y(f)$, there is $Z \in \mathcal{C}$ with $f \in \text{supp}(Z) \not\ni e$ and $Z(g) \in \{0, X(g), Y(g)\}$ for all g .

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$$\underline{\mathcal{B}} = \{(x_1, \dots, x_d) \mid (x_1, \dots, x_d) \in \text{supp}(\chi)\} \quad \underline{\mathcal{C}} := \{\text{supp}(X) \mid X \in \mathcal{C}\}$$

are the set of bases, resp. circuits of the underlying matroid.

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Definition. Two *signed vectors* $X, Y \in E \rightarrow \{-, 0, +\}$ are *orthogonal* if for $P := \{X(e)Y(e) \mid e \in E\}$, either $P = \{0\}$ or $P \supseteq \{+, -\}$.

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Given $\mathcal{C} \subseteq (S^1 \cup \{0\})^E$, define

$$\mathcal{C}^\perp := \{X \in \{-, 0, +\}^E \mid X \perp Y \text{ for all } Y \in \mathcal{C}\}$$

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M^* is called dual to M , and if M represents $V \in G_{d,n}(\mathbb{R})$, then M^* represents the orthogonal complement $V^\perp \in G_{n-d,n}(\mathbb{R})$.

AN ORIENTED MATROID IS...

a pair of families $\mathcal{C}, \mathcal{D} \subset \{-, 0, +\}^E$ such that

- For $X, Y \in \mathcal{C}$ (or in \mathcal{D}), $\text{supp}(X) = \text{supp}(Y) \Rightarrow X = \pm Y$
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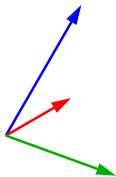
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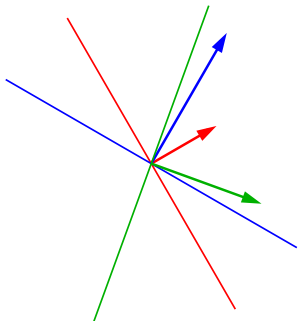
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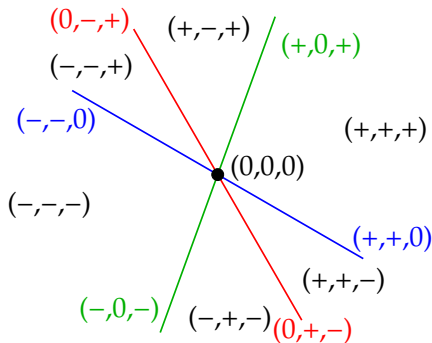
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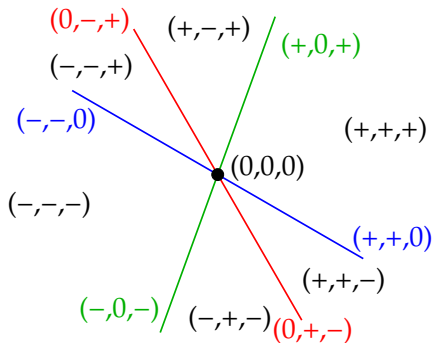


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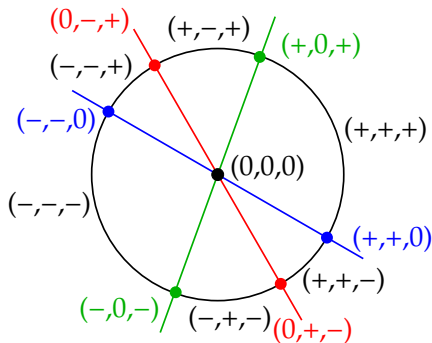


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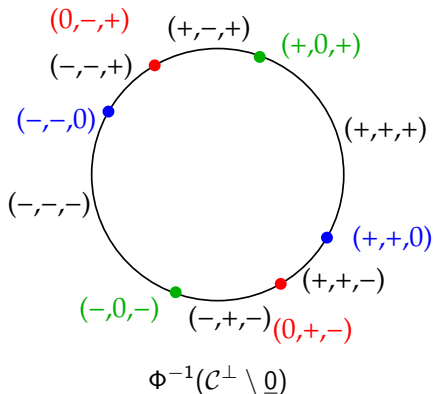


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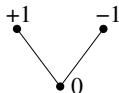


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$$(\sigma_1, \dots, \sigma_n) \leq (\sigma'_1, \dots, \sigma'_n) \Leftrightarrow \sigma_i \leq \sigma'_i \text{ for all } i,$$

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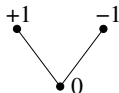


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Theorem [Folkman, Lawrence, '78]. *If \mathcal{C} is the set of signed circuits of an oriented matroid (of rank d), then*

$$\Delta(\mathcal{C}^\perp \setminus \underline{0}) \stackrel{hom}{\cong} S^{d-1}$$

(In fact, Oriented Matroids are cryptomorphic to arrangements of pseudospheres...)

CRYPTOMORPHISMS

Axioms for dual pairs



Chirotope axioms

Circuit axioms



“Covector Axioms”

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E. D.; *On generalizing complex matroids to a complex setting*.

Diploma thesis, ETH Zurich, 2003.

Focus: Orthogonality, Circuit duality, equivalence with Phirotopes.

THE TASK

? – Cryptomorphisms?

???



???

???



???

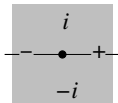
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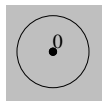
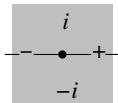


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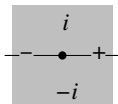
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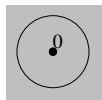
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Our choice: Consider $S^1 \cup \{0\} \subset \mathbb{C}$ and let

$$\text{ph} : \mathbb{C} \rightarrow S^1 \cup \{0\}$$

$$\text{ph}(z) := \begin{cases} 0 & \text{if } z = 0 \\ e^{i\theta} & \text{if } z = re^{i\theta} \text{ for } r \in \mathbb{R}_{>0} \end{cases}$$

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2 – How to express orthogonality?

Two vectors $v, w \in \mathbb{C}^n$ are orthogonal if

$$0 = \langle v | w \rangle = \sum_{i=1}^n v_i \bar{w}_i = \sum_{i=1}^n \lambda_i \text{ph}(v_i \bar{w}_i)$$

For positive real numbers λ_i with $\sum_i \lambda_i = 1$
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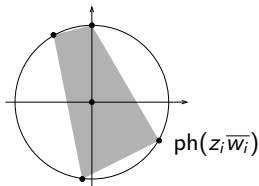
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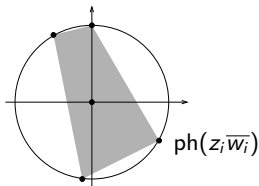
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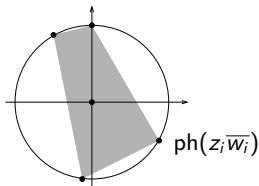
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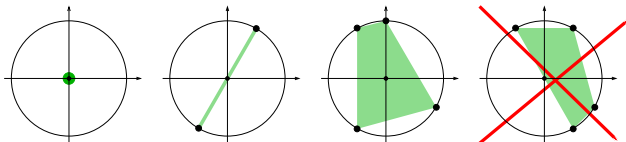
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We start by mimicking the Grassmann-Plücker relations in $G_{d,n}(\mathbb{C})$.

Definition [B.,K.,R.-G.'03]. A *complex matroid* of rank d on the ground set E is an alternating function

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such that for all $x_0, x_1, \dots, x_d, y_2, \dots, y_d$,

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- There are nonrealizable complex matroids
- A *dual* phirotope φ^* can be defined in terms of φ (as in O.M. theory)

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Consider an O.M. with chirotope χ and set of signed circuits \mathcal{C} .

Let $X \in \mathcal{C}$ with $\text{supp}(X) = \{x_0, \dots, x_k\}$, let x_{k+1}, \dots, x_d s.t. $\chi(x_1, \dots, x_d) \neq 0$.

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This uniquely defines the set \mathcal{C}_φ of **phased circuits** X_C associated to φ

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Remark. It follows that, as is the case for oriented matroids,

$$\mathcal{C}_{\varphi^*} = \min_{\text{supp}} \mathcal{C}^\perp$$

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$$v_4 + \left(\frac{7}{13} + \frac{4}{13}i\right)v_3 + \left(\frac{25}{26} + \frac{5}{26}i\right)v_5 + \left(\frac{8}{13} - \frac{1}{13}i\right)v_6 + \left(\frac{5}{13} + \frac{i}{13}\right)v_7 = 0,$$

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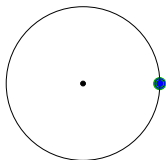
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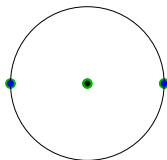
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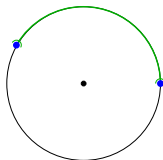
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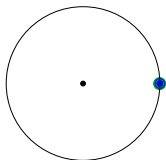
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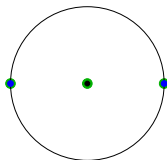
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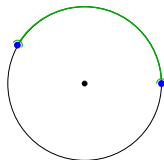
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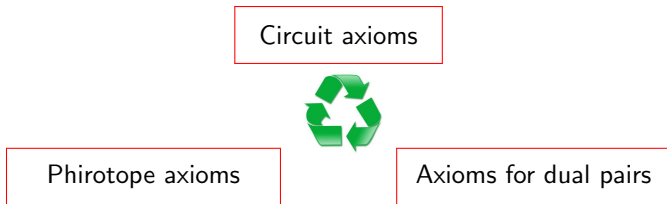
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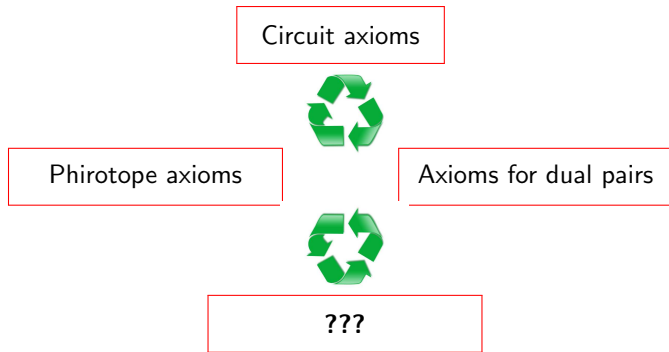
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In fact, there are two configurations V_1, V_2 (each of 4 vectors in \mathbb{C}^2) such that

- The two configurations have the same sets of signed circuits ($\mathcal{C}_{V_1} = \mathcal{C}_{V_2}$)
- For Φ as above, $\text{Im}(\Phi_{V_1}) \neq \text{Im}(\Phi_{V_2})$.

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