

# COMPLEX MATROIDS

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*Phirotopes, circuits, duality, and a sad but inevitable absence of vectors*

*(Preliminary version)*

ABSTRACT. We define complex matroids in terms of cryptomorphic circuit axioms, duality axioms, and phirotope axioms. Our phirotopes are the same as those studied previously by Below, Krummeck, and Richter-Gebert [2] and Delucchi [5]. We further show that complex matroids cannot have vector axioms analogous to those for oriented matroids.

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## INTRODUCTION

The aim of the present paper is to explore the axiomatics of an approach to “matroids with complex structure”, and to see the extent to which this approach

can lead to multiple, equivalent axiom systems analogous to those for matroids and for oriented matroids (“matroids with real structure”).

*Matroid theory* was initiated by Whitney [14] as the study of combinatorial data derived from linear dependencies among the columns  $v_1, \dots, v_n$  of a matrix  $M$  over an arbitrary field. Call  $E$  the set of columns and consider the following data sets:

- (1) the set  $\mathbf{B}(M)$  of all  $A \subseteq E$  such that  $\{v_a : a \in A\}$  are bases for the column space of  $M$
- (2)  $\mathbf{V}(M) := \{\text{supp}(x) : x \in \ker(M)\}$
- (3) the set  $\mathbf{C}(M)$  of minimal nonempty elements of  $\mathbf{V}(M)$
- (4)  $\mathbf{V}^*(M) := \{\text{supp}(x) : x \in \text{row}(M)\}$
- (5) the set  $\mathbf{C}^*(M)$  of minimal nonempty elements of  $\mathbf{V}^*(M)$ .

For each of these data sets, there is a set of combinatorial axioms satisfied by that data set:  $\mathbf{B}(M)$  satisfies the *basis axioms*,  $\mathbf{V}(M)$  and  $\mathbf{V}^*(M)$  satisfy what we call *vector axioms*,  $\mathbf{C}(M)$  and  $\mathbf{C}^*(M)$  satisfy the *circuit axioms*. The axioms will be stated in Section 1.1. The basis and circuit axioms are well-known to any student of matroids. The vector axioms are the obvious modification of the well-known *flat axioms*, which describe  $\{E - \text{supp}(x) : x \in \ker(M)\}$  and  $\{\text{supp}(x) : x \in \text{row}(M)\}$ . A detailed introduction to the axiomatics of matroids can be found in [10, Chapter 1 and 2].

Two important points about these data sets:

- (1) Each of these data sets associated to  $M$  determine each of the other data sets. The *matroid* of  $M$  is defined to be the information about  $M$  encoded by any one of these data sets.
- (2) The sets  $\mathbf{V}(M)$  and  $\mathbf{V}^*(M)$  arise from a vector space  $\ker(M)$  and its orthogonal complement  $\text{row}(M)$ . Given the matrix  $M$ , there is a matrix  $N$  whose row space is  $\ker(M)$  and whose kernel is  $\text{row}(M)$ , and thus  $\mathbf{V}(N) = \mathbf{V}^*(M)$  and  $\mathbf{C}(N) = \mathbf{C}^*(M)$ .

There are sets satisfying the various axiom systems discussed above which do not arise from matrices. However, just as for those arising from matrices, each set satisfying one of the axiom systems determines sets satisfying the other axiom systems. Thus we can refer to a *matroid with basis set  $\mathbf{B}$ , vector set  $\mathbf{V}$  circuit set  $\mathbf{C}$ , covector set  $\mathbf{V}^*$ , and cocircuit set  $\mathbf{C}^*$* . Matroid theory has a peculiar term to express this equivalence: we say that these axiom systems are *cryptomorphic* - a *cryptomorphism* being the translation rule proving the equivalence between two such systems. . Also, there is a notion of duality for matroids: every matroid  $M$ , say, with vector set  $\mathbf{V}$  and covector set  $\mathbf{V}^*$ , has a dual matroid with vector set  $\mathbf{V}^*$  and covector set  $\mathbf{V}$ . Thus a more streamlined presentation might be to say that a matroid can be given, equivalently, by its bases, its vectors, its circuits, or its dual matroid (given by its own set of bases, vectors, or circuits).

If a matroid arises from a matrix with coefficients in a field  $\mathbb{K}$ , it is called *realizable over  $\mathbb{K}$* .

Inspired by the well-developed theory of matroids (cf. [10, 13, 12]), one might consider some specific field  $\mathbb{K}$  and look for stronger axiom systems that reflect properties special to matrices over  $\mathbb{K}$ .

In the case  $\mathbb{K} = \mathbb{R}$  this search has been wildly successful: the result is *oriented matroids*, introduced by Folkman and Lawrence [8]. Oriented matroids are matroids with extra structure. Broadly put, each data set described above for matroids realized by a matrix  $M$  over a field  $\mathbb{K}$  says whether various elements of  $\mathbb{K}$

are zero or nonzero, while the corresponding data set for oriented matroids realized over  $\mathbb{R}$  describes whether these elements of  $\mathbb{R}$  are zero, positive, or negative. (As a shorthand for this we shall say that the *structure set* for matroids is  $\{0, \neq 0\}$ , while the structure set for oriented matroids is  $\{0, +, -\}$ .) Thus oriented matroids have cryptomorphic axiom systems:

- *signed basis axioms*, better known as *chirotope axioms*, which in the case of a matroid arising from a matrix  $M$  over  $\mathbb{R}$  describe the signs of all nonzero maximal minors of  $M$ ;
- *signed vector axioms* which in the case of a matroid arising from a matrix  $M$  over  $\mathbb{R}$  describe  $\{\text{sign}(x) : x \in \ker(M)\}$ .
- *signed circuit axioms* which in the case of a matroid arising from a matrix  $M$  over  $\mathbb{R}$  describe the elements of  $\{\text{sign}(x) : x \in \ker(M)\}$  of minimal nonempty support (where  $\text{sign}(x_1, \dots, x_n) = (\text{sign}(x_1), \dots, \text{sign}(x_n))$ );

Further, oriented matroids have a notion of duality that is compatible with duality of ordinary matroids and reflects orthogonality of subspaces of  $\mathbb{R}^n$ . If  $\mathcal{M}$  is an oriented matroid with set of vectors  $\mathcal{V}$ , then the set  $\mathcal{V}^*$  of vectors of the dual  $\mathcal{M}^*$  is called the set of *covectors* of  $\mathcal{M}$ , and the set  $\mathcal{C}^*$  of circuits of  $\mathcal{M}^*$  is called the set of *cocircuits* of  $\mathcal{M}^*$ .

Perhaps the most wonderful thing about oriented matroids is the Topological Representation Theorem. The set of vectors of a rank  $d$  oriented matroids can be partially ordered by componentwise extension of the natural partial order on signs (see Definition 1.16). This defines a poset whose order complex is PL-homeomorphic to the  $(d-1)$ -sphere and subdivides a stratification of  $S^{d-1}$  given by a set of centrally symmetric pseudospheres of codimension 1 satisfying some additional conditions [3, Section 5.1 and 5.2]. In fact such *arrangements of pseudospheres* give (yet another) cryptomorphic description of oriented matroids.

Now consider the case  $\mathbb{K} = \mathbb{C}$ : what is the right notion of “complex matroid”? In contrast to oriented matroids, the development here has been limited. Ideally, one would hope for cryptomorphic axiom systems similar to those for oriented matroids, resulting in a Topological Representation Theorem.

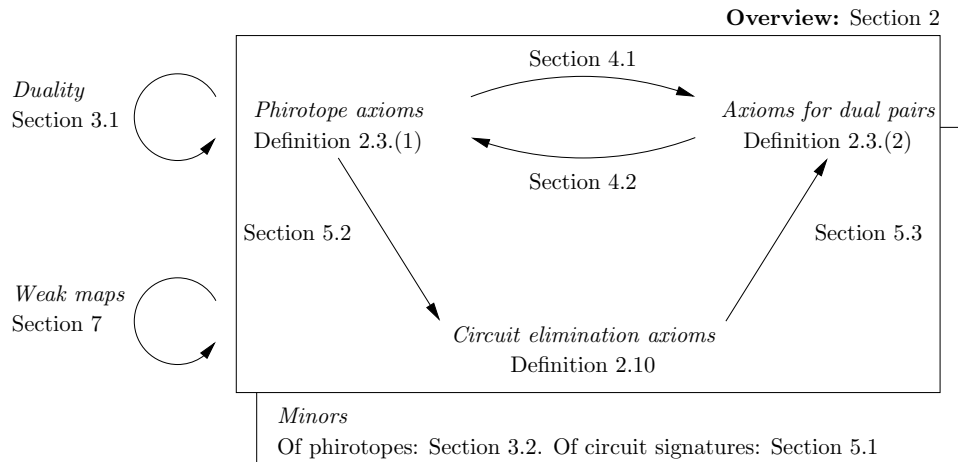
Ziegler [15] defined *complex oriented matroids* with extra structure given by the structure set  $\{0, +, -, i, -i\}$ . However, these objects are only known to have one axiomatization, in terms of covectors [15, Definition 1.3 and 4.1]. That is, where the set of covectors of a matroid realized by a matrix  $M$  over a field  $\mathbb{K}$  says whether various elements of  $\mathbb{K}$  are zero or nonzero, and the corresponding data set for oriented matroids realized over  $\mathbb{R}$  describes whether these elements of  $\mathbb{R}$  are zero, positive, or negative, the corresponding data set for Ziegler complex matroids realized over  $\mathbb{C}$  describes whether these elements of  $\mathbb{C}$  are zero, positive real, negative real, have positive imaginary part, or have negative imaginary part. Ziegler’s complex matroids have a Topological Representation Theorem [15, Theorem 3.5].

Below, Krummeck, and Richter-Gebert [2] developed another notion of complex matroid, with structure set  $S^1 \cup \{0\}$ , where  $S^1$  is the set of unit elements in  $\mathbb{C}$ , and with axiomatization only in terms of bases with structure, or *phirotopes*. That is, where the set of bases of a matroid realized by a matrix  $M$  over a field  $\mathbb{K}$  says whether various maximal minors of  $M$  are zero or nonzero, the corresponding data set for phirotopes realized over  $\mathbb{C}$  additionally describes the phase  $\theta$  of each nonzero maximal minor  $re^{i\theta}$ .

Below, Krummeck, and Richter-Gebert gave an axiomatization for phirotopes and proved various interesting properties in rank 2, in particular about realizability. Delucchi ([5]) developed a notion of orthogonality for this context, leading to dual phirotopes, and defined circuits and cocircuits associated to a phirotope (although he did not find circuit axioms).

Taking the point of view of the theory of *matroids with coefficients* developed by Dress and Wenzel, phirotopes correspond to basis orientations over the fuzzy ring  $\mathbb{C}/\mathbb{R}^+$  [7], of which  $S^1 \cup \{0\}$  is a subset. Within this framework, phirotopes are shown to be cryptomorphic to what can be roughly taken to be an axiomatization for “signed flats” (with coefficients in the full fuzzy ring), and one can prove that duality pairs of matroids with coefficients have “orthogonal” signatures. However, there is no cryptomorphic axiomatization of matroids with coefficients in terms of dual pairs, nor of circuits.

We ask (and, to some extent, answer) how much of the foundations of oriented matroids can be paralleled with the structure set  $S^1 \cup \{0\}$ . We give two different axiomatizations for circuits and cocircuits of a complex matroid and show them to be cryptomorphic to the phirotope axioms. We then turn to the issue of vectors and covectors and show that there is no “good” set of vector axioms. Finally, we briefly discuss weak maps of complex matroids. The structure of the paper is summarized in the following chart.



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## 1. BACKGROUND: MATROIDS AND ORIENTED MATROIDS

One purpose of this section is to get the relevant definitions on paper; another is to illustrate the philosophy of oriented matroids as “matroids with extra structure”. The latter motivation leads us to give a correct but mildly unorthodox presentation. We will give several axiomatizations of matroids, then give the corresponding axiomatizations for oriented matroids. The following section will then present our axiomatizations for complex matroids in a similar form.

For matroid theory we follow the notation of [10] and recommend this text for a reference: the similar text for oriented matroids is [3]. In particular, we will follow the notational convention of writing  $x$  for the singleton set  $\{x\}$  whenever this will not cause confusion.

### 1.1. Matroids.

#### 1.1.1. Axioms: bases, circuits, vectors, duality and cryptomorphisms.

**Definition 1.1.** 1. A family  $\mathbf{B} \subseteq 2^E$  of subsets of  $E$  is the set of *bases* of a matroid  $M$  if and only if  $\mathbf{B} \neq \emptyset$  and

(B1) given  $B_1, B_2 \in \mathbf{B}$  and  $e \in B_1 \setminus B_2$ , there is  $f \in B_2 \setminus B_1$  such that  $(B_1 \setminus e) \cup f \in \mathbf{B}$  (*the Basis Exchange Axiom*).

2. A family  $\mathbf{V} \subseteq 2^E$  is the set of *vectors* of a matroid on the ground set  $E$  if and only if  $E \in \mathbf{V}$  and

(V1) if  $X_1, X_2 \in \mathbf{V}$  then  $X_1 \cup X_2 \in \mathbf{V}$

(V2) if  $X \in \mathbf{V}$  and  $\{Y_1, \dots, Y_k\}$  is the set of maximal elements of  $\mathbf{V}$  properly contained in  $X$ , then the sets  $X - Y_1, \dots, X - Y_k$  partition  $X$ .

3. A family  $\mathbf{C} \subseteq 2^E$  is the set of *circuits* of a matroid on the ground set  $E$  if and only if  $\emptyset \notin \mathbf{C}$  and

(C1) if  $C_1, C_2 \in \mathbf{C}$  and  $C_1 \subseteq C_2$ , then  $C_1 = C_2$  (*Incomparability*)

(C2) if  $C_1, C_2 \in \mathbf{C}$  are distinct and there is an element  $e \in E$  with  $e \in C_1 \cap C_2$ , then there is  $C_3 \in \mathbf{C}$  with  $C_3 \subseteq (C_1 \cup C_2) \setminus e$  (*Elimination*).

To briefly state the cryptomorphisms:

- Given  $\mathbf{B}$  the set of bases of a matroid, we say  $A \subseteq E$  is a vector if for all  $B \in \mathbf{B}$  and  $e \in B \setminus A$ , all elements  $f \in E$  with  $(B \setminus e) \cup f \in \mathbf{B}$  are contained in  $E \setminus A$ . The set  $\mathbf{V}$  of all vectors is the set of vectors of a matroid.
- Given  $\mathbf{V}$  the set of vectors of a matroid,  $\mathbf{C}$  is the set of minimal nonempty elements of  $\mathbf{V}$ .
- Given  $\mathbf{C}$  the set of circuits of a matroid, we say that  $A \subseteq E$  is a *basis* if  $A$  is maximal among sets not containing a circuit. The set  $\mathbf{B}$  of all bases is the set of bases of a matroid.

**Definition 1.2.** For  $\mathbf{S} \subseteq 2^E$ , we define  $\mathbf{S}^\perp = \{A \subseteq E \mid \forall B \in \mathbf{S} |A \cap B| \neq 1\}$ .

It can be shown that for a subspace  $W$  of a vector space  $\mathbb{K}^n$ ,  $\{\text{supp}(x) : x \in W\}^\perp = \{\text{supp}(y) : y \in W^\perp\}$ . In particular, if  $\mathbf{V}$  is the set of vectors of a matroid  $M$  representable over some field  $\mathbb{K}$ , then  $\mathbf{V}^\perp = \mathbf{V}^*$ , the set of covectors of that matroid. This holds for non-representable matroids as well:

**Definition 1.3.** If  $\mathbf{V}$  is the set of vectors of a matroid  $M$ , define the set of *covectors* of  $M$  to be  $\mathbf{V}^* := \mathbf{V}^\perp$  and the set of *cocircuits* of  $M$  to be the set of minimal nonempty elements of  $\mathbf{V}^*$ .

**Theorem 1.4.** (cf. [10]) *If  $M$  is a matroid with ground set  $E$ , basis set  $\mathbf{B}$ , vector set  $\mathbf{V}$ , and circuit set  $\mathbf{C}$ , then there is a matroid  $M^*$  with ground set  $E$ , basis set  $\mathbf{B}^* := \{E \setminus X \mid X \in \mathbf{B}\}$ , vector set  $\mathbf{V}^*$ , and circuit set  $\mathbf{C}^*$ .*

*If  $M$  is realized by a matrix with row space  $W$ , then  $M^*$  is realized by a matrix with row space  $W^\perp$ .*

This  $M^*$  is called the *dual* to  $M$ .

In particular, we will make frequent use of the following basic fact. It follows immediately from our definitions, but we state it here for later reference.

**Lemma 1.5** (Proposition 2.1.20 of [10]). *Let  $C$  be a circuit and  $D$  be a cocircuit of a matroid  $M$ . Then,  $|C \cap D| \neq 1$ . In fact, the set*

$$\min\{D \subseteq E \mid D \neq \emptyset, |D \cap C| \neq 1 \text{ for all } C \in \mathbf{C}\},$$

where  $\min$  denotes inclusion-minimality, is the set of cocircuits of  $M$ .

### 1.1.2. Minors.

**Definition 1.6** (Section 3.1 of [10]). Let  $M$  be a matroid on the ground set  $E$  with set of bases  $\mathbf{B}$ , and let  $A \subseteq E$ . Choose  $a_1, \dots, a_l$  a maximal independent set in  $A$ . We define

- (1) the *contraction*  $M/A$  as the matroid given by the set of bases

$$\mathbf{B}(M/A) := \{B \subset E \mid B \cup \{a_1, \dots, a_l\} \in \mathbf{B}\}$$

- (2) the *deletion*  $M \setminus A$  as the matroid with set of bases

$$\mathbf{B}(M \setminus A) := \max\{B \setminus A \mid B \in \mathbf{B}\},$$

where  $\max$  denotes inclusion-maximality.

For any  $A \subseteq E$ , we let  $M(A)$  denote  $M \setminus (E \setminus A)$ . The matroids  $M/A$ ,  $M \setminus A$ ,  $M(A)$  are called *minors* of  $M$ . In fact, in the representable case they encode data related to the minors of the original matrix.

**Lemma 1.7** (Section 3.1 of [10]). *The contraction and deletion of a matroid  $M$  on the ground set  $E$  can also be defined by means of their set of circuits:*

$$\mathbf{C}(M \setminus A) = \{C \in \mathbf{C}(M) \mid C \cap A = \emptyset\},$$

$$\mathbf{C}(M/A) = \min\{C \setminus A \mid C \in \mathbf{C}(M), C \not\subseteq A\}.$$

Moreover, the operations of contraction and deletion are dual to each other in the sense that

$$(M/A)^* = M^* \setminus A.$$

1.1.3. *Rank.* It is easy to check from the definition that all bases of a matroid have the same size ([10, Lemma 1.2.1]). This fact gives meaning to the following definition.

**Definition 1.8** (Rank). Let  $M$  be a matroid on the ground set  $E$  with set of bases  $\mathbf{B}$ , and let  $A \subseteq E$ . Define the *rank* of  $A$  to be

$$\text{rank}(A) := \max\{|A \cap B| \mid B \in \mathbf{B}\}.$$

The rank of the matroid  $M$  is defined as  $\text{rank}(M) := \text{rank}(E)$ .

The notion of rank defines a closure operator on  $E$ , defines as follows.

**Definition 1.9** (Closure). Let  $M$  be a matroid on the ground set  $E$  with rank function  $\text{rank}$ . Given  $A \subseteq E$  define

$$\text{cl}(A) := \max\{A' \subseteq E \mid A \subseteq A', \text{rank}(A) = \text{rank}(A')\}.$$

The function  $\text{cl} : E \rightarrow E$  is the *closure operator* of  $E$ .

1.1.4. *Flats and modularity.* A *flat* of a matroid  $M$  with set of vectors  $\mathbf{V}$  is any  $A \subseteq E$  of the form  $A = E \setminus X$  for a  $X \in \mathbf{V}$ . An important property of the set of flats of a matroid is that it is closed under intersection and union. Indeed, intersection and union turn the set of flats, partially ordered by inclusion, into a geometric lattice (and this gives rise to yet another cryptomorphic characterization of matroids [10, Theorem 1.7.5]). It is easy to see that flats can be defined as the subsets  $A \subseteq E$  such that  $\text{cl}(A) = A$ .

**Definition 1.10.** Let  $M$  be a matroid with vector set  $\mathbf{V}$ . Two flats  $X, Y \in \mathbf{V}$  are a *modular pair* if

$$\text{rank}(X) + \text{rank}(Y) = \text{rank}(X \cap Y) + \text{rank}(X \cup Y).$$

The maximal proper flats of a matroid  $M$  are called *hyperplanes*. Two hyperplanes  $H_1$  and  $H_2$  are modular if intersect in a flat of maximal rank (i.e., of rank  $r(M) - 2$ ). One can check that the cocircuits of  $M$  are exactly the complements of hyperplanes of  $M$ . This motivates the following definition.

**Definition 1.11.** Let  $\mathbf{S}$  be a collection of incomparable subsets of a ground set  $E$ . Consider the poset obtained by partially ordering  $\{\bigcup \mathbf{K} \mid \mathbf{K} \subseteq \mathbf{S}\}$  by inclusion. This poset is an atomic lattice, so the *meet*  $A \vee B$  is defined for every pair  $A, B$  of its elements (see [11, Chapter 3]).

Two elements of  $\mathbf{S}$  give a *modular pair* if the longest chain from the minimal element of  $U$  to their meet  $A \vee B$  has length 2.

Notice that if  $\mathbf{S}$  satisfies the axioms for the set of circuits of a matroid  $M$ , then two elements  $A, B \in \mathbf{S}$  are a modular pair if and only if  $E \setminus A, E \setminus B$  are a modular pair of hyperplanes in  $M^*$ .

**Lemma 1.12** ([6]). *A collection  $\mathbf{C}$  of incomparable subsets of a ground set  $E$  is the set of circuits of a matroid if the elimination property (C2) of Definition 1.1 holds for all modular pairs  $C_1, C_2$  of elements of  $\mathbf{C}$ .*

1.1.5. *More on bases.* We introduce now a notion that will be a key tool in the proof of the cryptomorphism between the axioms for dual pairs and the phirotope axioms Section 4.2. Basis graphs of matroids were introduced by Maurer [9].

**Definition 1.13** (Basis Graph). The *basis graph* of a matroid  $M$  with set of bases  $\mathbf{B}$  is the simple graph with vertex set

$$V(G) := \mathbf{B}$$

and edge set

$$E(G) := \{\{B_1, B_2\} \mid |B_1 \setminus B_2| = 1\}.$$

The exchange represented by an edge  $\{B_1, B_2\}$  of the basis graph is possible because the set  $B_1 \cup B_2$  contains a circuit

**Lemma 1.14.** *If  $B$  is a basis of a matroid  $M$  on ground set  $E$  and  $e \in E \setminus B$  then there is a unique circuit  $X \subseteq B \cup \{e\}$ , called the basic circuit of  $e$  with respect to  $B$  and denoted by  $C(B, e)$ . In particular, for any pair of bases of the form  $B_1 = A \cup e_1, B_2 = A \cup e_2$  there is a unique circuit supported on  $B_1 \cup B_2$ .*

Maurer gave a thorough treatment of these graphs, giving for instance a complete characterization of which graphs are basis graphs of a matroid. For the scope of this paper we will only need the following result.

A sequence of edges  $e_1, \dots, e_k$  in a graph  $G$  is a *path from the vertex  $A$  to the vertex  $B$*  if for all  $j = 1, \dots, k-1$   $e_j, e_{j+1}$  share a vertex, and if  $A$  ( $B$ ) is the vertex of  $e_1$  ( $e_k$ ) that is not shared with  $e_2$  ( $e_{k-1}$ ). We say that an *elementary move* on the given path is the substitution of any subpath  $e_j e_{j+1}$  with another path consisting of at most two edges of  $G$ , and such that the replacement yields again a path. The *trivial path* is the path corresponding to an empty sequence of edges.

**Theorem 1.15** ([9]). *Let  $G$  be the basis graph of a matroid  $M$  and choose a vertex  $A$  of  $G$ . Then every closed path in  $G$  from  $A$  to  $A$  can be reduced to the trivial path by a sequence of elementary moves and of inverses thereof.*

## 1.2. Oriented matroids.

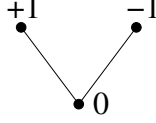
### 1.2.1. Signs.

**Definition 1.16.** Given a finite ground set  $E$ , a *sign vector* (or *signed set*) is any

$$X \in (S^0 \cup \{0\})^E$$

where  $S^0 = \{+1, -1\}$  is the unit sphere in  $\mathbb{R}$ . We will denote by  $X(e)$  the  $e$ -th component of  $X$ . The signed set with value 0 for all components will be denoted by  $\underline{0}$ .

We order  $S^0 \cup \{0\}$  according to the following Hasse diagram.



The *sign*  $\text{sign}(x)$  of  $x \in \mathbb{R}$  is defined to be 0 if  $x = 0$  and  $\frac{x}{|x|}$  otherwise. The sign  $\text{sign}(v)$  of  $v \in \mathbb{R}^E$  is defined to be the sign vector with  $e$ -th component  $\text{sign}(v_e)$ .

**Definition 1.17.** If  $X, Y \in (S^0 \cup \{0\})^E$ , the *composition*  $X \circ Y \in (S^0 \cup \{0\})^E$  is defined by: for every  $e \in E$ ,

$$X \circ Y(e) = \begin{cases} X(e) & \text{if } X(e) \neq 0 \\ Y(e) & \text{otherwise.} \end{cases}$$

We define the *convex hull* of a subset  $P$  of  $S^0 \cup \{0\}$  to be the set of all signs of positive linear combinations of the elements of  $P$ . Thus

- $\text{conv}(\emptyset) = \emptyset$
- $\text{conv}(\{0\}) = \{0\}$
- $\text{conv}(\{0, \epsilon\}) = \epsilon$  for  $\epsilon \in S^0$
- $\text{conv}(P) = (S^0 \cup \{0\})$  if  $S^0 \supseteq P$ .

### 1.2.2. Axioms: chirotopes, circuits, vectors, duality and cryptomorphisms.

**Definition 1.18.** 1. A function  $E^d \rightarrow S^0 \cup \{0\}$  is called a *rank  $d$  chirotope* of an oriented matroid  $M$  if

- ( $\chi$ 1)  $\chi$  is nonzero
- ( $\chi$ 2)  $\chi$  is alternating
- ( $\chi$ 3) For any two subsets  $x_1, \dots, x_{d+1}$  and  $y_1, \dots, y_{d-1}$  of  $E$ , 0 is contained in the convex hull of the numbers

$$(-1)^k \chi(x_1, x_2, \dots, \widehat{x_k}, \dots, x_d) \chi(x_k, y_1, \dots, y_{d-1}).$$

2. A family  $\mathcal{V} \subseteq (S^0 \cup \{0\})^E$  of signed sets is the set of *signed vectors* of an oriented matroid  $\mathcal{M}$  if

- (V0)  $\underline{0} \in \mathcal{V}^*$
- (V1)  $\mathcal{V} = -\mathcal{V}$  (*Symmetry*)
- (V2) if  $X, Y \in \mathcal{V}^*$  then  $X \circ Y \in \mathcal{V}^*$  (*Composition*)
- (V3) for every  $X, Y \in \mathcal{V}$  and  $e \in E$  with  $X(e) = -Y(e)$  there is some  $Z \in \mathcal{V}$  with
  - $Z(f) = \max\{X(f), Y(f)\}$  for all  $f$  for which this maximum exists and
  - $Z(e) = 0$ .
 (*Elimination*)

3. A family  $\mathcal{C} \subseteq (S^0 \cup \{0\})^E \setminus \{\underline{0}\}$  of signed sets is the set of *signed circuits* of an oriented matroid  $\mathcal{M}$  if

- (C0)  $\mathcal{C} = -\mathcal{C}$  (*Symmetry*)
- (C1) if  $X, Y \in \mathcal{C}$  and  $\text{supp}(X) \subseteq \text{supp}(Y)$  then  $X = \pm Y$  (*Incomparability*)
- (C2) for every  $X, Y \in \mathcal{C}$  such that  $X \neq -Y$  and  $e, f \in E$  with  $X(e) = -Y(e)$  and  $X(f) \neq -Y(f)$ , there is some  $Z \in \mathcal{C}$  with  $f \in \text{supp}(Z) \subseteq \text{supp}(X) \cup \text{supp}(Y) \setminus e$  and  $Z(f) = \max\{X(f), Y(f)\}$ . (*Circuit Elimination*)

As for matroids, there are cryptomorphisms allowing us to speak of “the oriented matroid with chirotopes  $\chi$  and  $-\chi$ , vector set  $\mathcal{V}$ , and circuit set  $\mathcal{C}$ ”. The *underlying matroid* of this oriented matroid has basis set  $\text{supp}(\chi)$ , vector set  $\{\text{supp}(X) : X \in \mathcal{V}\}$ , and circuit set  $\{\text{supp}(X) : X \in \mathcal{C}\}$ . We define the *rank* of an oriented matroid to be the rank of its chirotope or, equivalently, the rank of the underlying matroid.

As is well known:

**Proposition 1.19.** *Let  $M$  be a  $d \times n$  matrix of rank  $d$  over the reals. Let  $v_1, \dots, v_n$  denote the columns of  $M$ . Then there is a rank  $d$  oriented matroid with*

- *chirotope the function  $[n]^d \rightarrow S^0 \cup \{0\}$  sending each  $(i_1, \dots, i_d)$  to the determinant of the matrix  $(v_{i_1} \cdots v_{i_d})$*
- *signed vector set  $\{\text{sign}(x) : x \in \ker(M)\}$ , and*
- *signed circuit set the set of all minimal nonzero signed vectors.*

**Definition 1.20.** The oriented matroids arising from matrices over  $\mathbb{R}$  are called *realizable*.

**Definition 1.21.** Two sign vectors  $X, Y \in \{+, 0, -\}^E$  are defined to be *orthogonal* if  $0 \in \text{conv}(\{X(e)Y(e) \mid e \in E\})$ .

This definition is inspired by orthogonality of vectors in  $\mathbb{R}^n$ , and it leads to a definition of orthogonality of oriented matroids that nicely models orthogonality of real vector spaces.

**Theorem 1.22.** *If  $\mathcal{M}$  is an oriented matroid with ordered ground set  $E$ , chirotope  $\chi : E^r \rightarrow S^0 \cup \{0\}$ , circuit set  $\mathcal{C}$ , and vector set  $\mathcal{V}$ , then there is an oriented matroid  $\mathcal{M}^*$  with*

- (0) *ground set  $E$ ,*
- (1) *chirotope  $\chi^* : E^{|E|-r} \rightarrow \{0, +, -\}$  given by*

$$\chi^*(x_1, \dots, x_{n-r}) = \chi(y_1, \dots, y_r) \sigma(x_1, \dots, x_{n-r}, y_1, \dots, y_r)$$

where  $\{y_1, \dots, y_r\} = E \setminus \{x_1, \dots, x_{n-r}\}$  and  $\sigma$  denotes the sign of the indicated permutation of  $E$

- (2) covector set  $\mathcal{V}^* = \mathcal{V}^\perp$ , and
- (3) cocircuit set  $\mathcal{C}^* = \min\mathcal{V}^\perp$ , and

where  $\min$  denotes support minimality. The underlying matroid of  $\mathcal{M}^*$  is the dual of the underlying matroid of  $\mathcal{M}$ . If  $\mathcal{M}$  is realized by a matrix with row space  $W$ , then  $\mathcal{M}^*$  is realized by a matrix with row space  $W^\perp$ .

This  $\mathcal{M}^*$  is called the *dual* to  $\mathcal{M}$ . The vectors of  $\mathcal{M}^*$  are the *covectors* of  $\mathcal{M}$ , and the circuits of  $\mathcal{M}^*$  are called the *cocircuits* of  $\mathcal{M}$ .

1.2.3. *More axiomatizations for circuits.* While the axiomatization of signed circuits given in Definition 1.18 is the standard one, we will have use for two additional characterizations: *Modular Elimination* and *Axioms for Dual Pairs*.

**Proposition 1.23** (Modular Elimination Axiom [3]). *In the definition of signed circuits, if the set  $\{\text{supp}(C) \mid C \in \mathcal{C}\}$  is known to be the set of circuits of a matroid, the Circuit Elimination Axiom can be replaced by the Modular Elimination Axiom:*

- (C2') *for every  $X, Y \in \mathcal{C}$  such that  $\text{supp}(X), \text{supp}(Y)$  is a modular pair of circuits of the matroid underlying  $\mathcal{M}$ ,  $X \neq -Y$  and  $e, f \in E$  with  $X(e) = -Y(e)$  and  $X(f) \neq -Y(f)$ , there is some  $Z \in \mathcal{C}$  with  $f \in \text{supp}(Z) \subseteq \text{supp}(X) \cup \text{supp}(Y) \setminus e$  and, for all  $g$ ,  $Z(g) \in \{0, X(g), Y(g)\}$ .*

**Proposition 1.24** (Axioms for dual pairs [4]). *Let  $\mathcal{C}, \mathcal{C}^* \subseteq \{S^0 \cup \{0\}\}^E$ .  $\mathcal{C}$  and  $\mathcal{C}^*$  are the signed circuits resp. signed cocircuits of an oriented matroid if and only if:*

- (S1)  $\mathcal{C} = -\mathcal{C}$
- (S1\*)  $\mathcal{C}^* = -\mathcal{C}^*$
- (S2) *if  $X, Y \in \mathcal{C}$  and  $\text{supp}(X) = \text{supp}(Y)$  then  $X = \pm Y$*
- (S2\*) *if  $X, Y \in \mathcal{C}^*$  and  $\text{supp}(X) = \text{supp}(Y)$  then  $X = \pm Y$*
- (S3)  $\{\text{supp}(X) \mid X \in \mathcal{C}\}$  and  $\{\text{supp}(X) \mid X \in \mathcal{C}^*\}$  are the set of circuits resp. cocircuits of a matroid
- (S4)  $X \perp Y$  for all  $X \in \mathcal{C}$  and  $Y \in \mathcal{C}^*$ .

1.2.4. *Minors.*

**Definition 1.25.** Let  $\mathcal{M}$  be a rank  $d$  oriented matroid on the ground set  $E$  with chirotope  $\chi$ . and let

- (1) Choose a maximal independent subset  $a_1, \dots, a_l$  in  $A$ . Define the *contraction*  $M/A$  as the matroid given by the chirotope

$$(\chi/A)(x_1, \dots, x_{d-l}) := \chi(x_1, \dots, x_{d-l}, a_1, \dots, a_l)$$

- (2) Choose  $a_1, \dots, a_{d-r}$  in  $A$  be such that  $(E \setminus A) \cup \{y_1, \dots, y_{d-r}\}$  spans  $E$ . Define the *deletion*  $M \setminus A$  as the matroid with chirotope

$$(\chi \setminus A)(x_1, \dots, x_r) := \begin{cases} \chi(x_1, \dots, x_r) & \text{if } r = d, \\ \chi(x_1, \dots, x_r, b_1, \dots, b_{d-r}) & \text{if } r < d. \end{cases}$$

For  $X \in (S^0 \cup \{0\})^E$  and  $A \subseteq E$  let  $X_{\setminus A}$  denote the element of  $(S^0 \cup \{0\})^{E \setminus A}$  defined by the restriction of  $X$  to  $E \setminus A$ .

**Lemma 1.26.** *Let  $\mathcal{M}$  be an oriented matroid on the ground set  $E$ , and let  $A \subseteq E$ .*

- (1) *The deletion  $M \setminus A$  is the matroid with set of signed circuits*

$$\mathcal{C}(M \setminus A) = \{X_{\setminus A} \mid X \in \mathcal{C}, \text{supp}(X) \cap A = \emptyset\}.$$

- (2) The contraction  $\mathcal{M}/A$  is the oriented matroid given by the set of signed circuits

$$\mathcal{C}(\mathcal{M}/A) = \min\{X_{\setminus A} \mid X \in \mathcal{C}\},$$

where  $\min$  denotes support minimality.

As is the case with matroids, deletion and contraction are dual operations:

$$(\mathcal{M}/A)^* = (\mathcal{M}^*) \setminus A.$$

## 2. COMPLEX MATROIDS

This section outlines our main results and should serve the reader as a road map through the remainder of the paper. We start by defining complex phases and putting some notation in place. Then, we present our cryptomorphic axiomatizations for complex matroids. We close by sketching the discussion about covectors, complexification and weakmaps that will take place in the last sections of the paper.

### 2.1. Complex phases.

**Definition 2.1** (Phase vectors). Given a finite ground set  $E$ , a *phase vector* (or “*phased set*”) is any

$$X \in (S^1 \cup \{0\})^E$$

where  $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$  is the unit circle in the Gauss plane of the complex numbers. We will denote by  $X(e)$  the  $e$ -th component of  $X$ . We define a partial order on phases by setting  $0 < \mu$  for all  $\mu \in S^1$ , and declaring any two elements of  $S^1$  as incomparable. This extends to a partial order on phase vectors defined componentwise. The minimal phase vector with respect to this ordering is the zero vector, which has value 0 on every component and will be denoted by  $\underline{0}$ .

The *phase*  $\text{ph}(x)$  of  $x \in \mathbb{C}$  is defined to be 0 if  $x = 0$  and  $\frac{x}{|x|}$  otherwise. For  $v \in \mathbb{C}^E$ ,  $\text{ph}(v)$  is defined to be the vector with components  $(\text{ph}(v))_e = \text{ph}(v_e)$ .

**Definition 2.2.** Define the *phase convex hull*  $\text{pconv}(S)$  of a finite  $S \subset S^1 \cup \{0\}$  to be the set of all phases of positive linear combinations of  $S$ . Thus

- $\text{pconv}(\emptyset) = \emptyset$
- $\text{pconv}(\{\mu\}) = \{\mu\}$  for all  $\mu$
- $\text{pconv}(\{\mu, -\mu\}) = \{0, \mu, -\mu\}$  for all  $\mu$
- if  $S = \{e^{i\alpha_1}, \dots, e^{i\alpha_k}\}$  with  $k \geq 2$  and  $\beta < \alpha_1 < \dots < \alpha_k < \beta + \pi$  for some  $\beta$ , then

$$\text{pconv}(S) = \text{pconv}(S \cup \{0\}) = \{e^{i\gamma} : \alpha_1 < \gamma < \alpha_k\}$$

- if  $S = \{e^{i\alpha_1}, \dots, e^{i\alpha_k}\}$  with  $k \geq 3$  and  $\alpha_1 < \dots < \alpha_k = \alpha_1 + \pi$ , then

$$\text{pconv}(S) = \text{pconv}(S \cup \{0\}) = \{e^{i\gamma} : \alpha_1 < \gamma < \alpha_k\}$$

- otherwise (i.e., if the nonzero elements of  $S$  do not lie in a closed half-circle of  $S^1$ )  $\text{pconv}(S) = S^1 \cup \{0\}$ .

## 2.2. Axioms for complex matroids.

**Definition 2.3** (Complex matroids). (1) (Phirotope axioms, compare [2]) A function  $\varphi : E^d \rightarrow S^1 \cup \{0\}$  is called a *rank  $d$  phirotope* if

( $\varphi$ 1)  $\varphi$  is nonzero

( $\varphi$ 2)  $\varphi$  is alternating

( $\varphi$ 3) For any two subsets  $x_1, \dots, x_{d+1}$  and  $y_1, \dots, y_{d-1}$  of  $E$ ,

$$0 \in \text{pconv}(\{(-1)^k \varphi(x_1, x_2, \dots, \hat{x}_k, x_d) \varphi(x_k, y_1, \dots, y_{d-1})\})$$

(2) (Elimination axioms) A set  $\mathcal{C} \subseteq (S^1 \cup \{0\})^E$  is the *set of circuits of a complex matroid* if and only if it satisfies

(C0) for all  $X \in \mathcal{C}$  and all  $\alpha \in S^1$ ,  $\alpha X \in \mathcal{C}$

(C1) for all  $X, Y \in \mathcal{C}$  with  $\text{supp}(X) = \text{supp}(Y)$ ,  $X = \alpha Y$  for some  $\alpha \in S^1$

(ME) for all  $X, Y \in \mathcal{C}$  such that  $X \neq \mu Y$  for all  $\mu \in S^1$  and such that  $\text{supp}(X), \text{supp}(Y)$  is a modular pair of circuits of  $M$ , and given  $e, f \in E$  with  $X(e) = -Y(e) \neq 0$  and  $X(f) \neq -Y(f)$ , there is  $Z \in \mathcal{C}$  with

- $f \in \text{supp}(Z) \subseteq (\text{supp}(X) \cup \text{supp}(Y)) \setminus e$ , and
- $\begin{cases} Z(f) \in \text{pconv}(\{X(f), Y(f)\}) & \text{if } f \in \text{supp}(X) \cap \text{supp}(Y) \\ Z(f) \leq \max\{X(f), Y(f)\} & \text{else.} \end{cases}$

**Remark 2.4.**

- Property (C0), (C1) and (ME) show that the set  $\{\text{supp}(X) \mid X \in \mathcal{C}\}$  is the set of circuits of a matroid  $M_{\mathcal{C}}$  (see Lemma 1.12)
- The phirotope axioms imply that the set  $\{\{b_1, \dots, b_d\} \subset E \mid \varphi(b_1, \dots, b_d) \neq 0\}$  is the set of bases of a matroid  $M_{\varphi}$ .
- Our feeling is that any “general” elimination axiom that is weak enough to hold for all complex matroids will not be strong enough to define the corresponding cocircuit signature (i.e., to prove Proposition 5.8).
- It is easily seen that, if  $M$  is a rank  $d$  matrix with columns indexed by  $E$ , the function  $E^d \rightarrow S^1 \cup \{0\}$  taking each  $d$ -tuple to the sign of the determinant of the corresponding submatrix of  $M$  is a phirotope. In this case Property ( $\varphi$ 3) follows from the Grassmann-Plücker relations.

**Definition 2.5.** For a phirotope  $\varphi$  on  $E$ , we say  $\{e_1, \dots, e_k\} \subseteq E$  is  $\varphi$ -*independent* if  $\varphi(a_1, \dots, a_k, b_{k+1}, \dots, b_d) \neq 0$  for some  $\{b_{k+1}, \dots, b_d\}$ . A set  $B$  is  $\varphi$ -*dependent* if it is not  $\varphi$ -independent. By a  $\varphi$ -*completion* of any  $\varphi$ -independent set  $\{a_1, \dots, a_k\}$  we mean a set  $\{a'_1, \dots, a'_{d-k}\}$  such that  $\varphi(a_1, \dots, a_k, a'_1, \dots, a'_{d-k}) \neq 0$ . If  $k = 0$ , the only completion is the empty set.

**Theorem A.** *There is a bijection between the set of all phirotopes on a set  $E$  and the set of all sets of circuits of complex matroids on  $E$ , determined as follows. For a phirotope  $\varphi$  and the corresponding set  $\mathcal{C}$  of phased circuits,*

- *The set of all supports of elements of  $\mathcal{C}$  is the set of minimal  $\varphi$ -dependent sets, and*
- *The phases of  $X \in \mathcal{C}$  are determined by the rule*

$$\frac{X(x_i)}{X(x_0)} = (-1)^{i-1} \frac{\varphi(x_0, \dots, \hat{x}_i, \dots, x_d)}{\varphi(x_1, \dots, x_d)}$$

*for all  $i = 0, \dots, k$ , where we have written  $\{x_0, \dots, x_k\} = \text{supp}(X)$  and  $\{x_{k+1}, \dots, x_d\}$  is any  $\varphi$ -completion of  $\text{supp}(X) \setminus \{x_0\}$ .*

Thus we can refer to “the complex matroid with phirotope  $\varphi$  and circuit set  $\mathcal{C}$ ”.

**Corollary 2.6.** *With the notation introduced in Remark 2.4, if  $\mathcal{M}$  is a complex matroid with phirotope  $\varphi$  and phased circuit set  $\mathcal{C}$ , then  $M_\varphi = M_{\mathcal{C}}$ .*

We call this matroid the *underlying matroid* of  $\mathcal{M}$ . The rank of  $\mathcal{M}$  is the rank of its underlying matroid.

**Definition 2.7.** Until we prove Theorem A, we will refer to any  $\mathcal{C}$  satisfying all of the conditions of the theorem as a *complex circuit orientation* of the underlying matroid.

Consider two vectors  $v, w \in \mathbb{C}^E$ . By definition, they are orthogonal if their (Hermitian) scalar product equals zero:  $\langle v, w \rangle = \sum v_e \overline{w_e} = 0$ . Now,  $\text{ph}(v_e \overline{w_e}) = \text{ph}(v_e) \text{ph}(w_e)^{-1}$  and if complex numbers with such phases must add up to zero, then the point 0 in the complex plane must be contained in  $\text{pconv}(\{\text{ph}(v_e) \text{ph}(w_e)^{-1} : e \in E\})$ . Thus we make the following definition.

**Definition 2.8** (Orthogonality). Let  $S, T \in (S^1 \cup \{0\})^E$  be two phased sets for some finite set  $E$ . Let

$$P_{S,T} := \left\{ \frac{S(e)}{\overline{T(e)}} \mid e \in \text{supp}(S) \cap \text{supp}(T) \right\}.$$

We say  $S$  and  $T$  are *orthogonal*, written  $S \perp T$ , if

$$0 \in \text{pconv}(P_{S,T}).$$

Two sets  $\mathcal{S}, \mathcal{T} \subseteq (S^1 \cup \{0\})^E$  are called orthogonal, written  $\mathcal{S} \perp \mathcal{T}$ , if  $S \perp T$  for all  $S \in \mathcal{S}$  and all  $T \in \mathcal{T}$ . The set of all phased sets orthogonal to  $S$  is denoted  $S^\perp$ .

The notion of orthogonality introduced above behaves naturally with respect to duality.

**Theorem B.** *If  $\mathcal{M}$  is a complex matroid with ordered ground set  $E$ , phirotope  $\varphi : E^d \rightarrow S^1 \cup \{0\}$ , and circuit set  $\mathcal{C}$ , then there is a complex matroid  $\mathcal{M}^*$  with ground set  $E$  and*

- (1) *chirotope  $\varphi^* : E^{|E|-d} \rightarrow S^1 \cup \{0\}$  given by*

$$\varphi^*(x_1, \dots, x_{n-d}) = \varphi(y_1, \dots, y_d) \sigma(x_1, \dots, x_{n-d}, y_1, \dots, y_d)$$

*where  $\{y_1, \dots, y_d\} = E \setminus \{x_1, \dots, x_{n-d}\}$  and  $\sigma$  denotes the sign of the indicated permutation of  $E$*

- (2) *circuit set  $\mathcal{C}^* = \min(\mathcal{C}^\perp \setminus \{0\})$ ,*

*where  $\min$  denotes support inclusion minimality. The underlying matroid of  $\mathcal{M}^*$  is the dual of the underlying matroid of  $\mathcal{M}$ . If  $\mathcal{M}$  is realized by a vector space  $W \subset \mathbb{C}^E$  then  $\mathcal{M}^*$  is realized by  $W^\perp$ .*

**Remark 2.9.** The reader will perhaps notice a “missing item” in the statement of Theorem B as compared to its counterpart for oriented matroids, Theorem 1.22. We will show in Section 6 that there can be no axiomatic description of the signs of the row space of a matrix with complex coefficients that is cryptomorphic to the other axiomatizations.

**Definition 2.10** (Axioms for dual pairs). Let  $E$  be a finite set. Two subsets  $\mathcal{C}, \mathcal{D}$  of  $(S^1 \cup \{0\})^E$  are the *dual pair of complex circuit signatures* of a complex matroid on the ground set  $E$  if

- (S1) for all  $X \in \mathcal{C}$  and all  $\alpha \in S^1$ ,  $\alpha X \in \mathcal{C}$

- (S1\*) for all  $X \in \mathcal{D}$  and all  $\alpha \in S^1$ ,  $\alpha X \in \mathcal{D}$
- (S2) for all  $X, Y \in \mathcal{C}$  with  $\text{supp}(X) = \text{supp}(Y)$ ,  $X = \alpha Y$  for some  $\alpha \in S^1$ .
- (S2\*) for all  $X, Y \in \mathcal{D}$  with  $\text{supp}(X) = \text{supp}(Y)$ ,  $X = \alpha Y$  for some  $\alpha \in S^1$ .
- (S3) For some matroid  $M$  on  $E$ , the set  $\{\text{supp}(X) \mid X \in \mathcal{C}\}$  is the set of circuits of  $M$  and the set  $\{\text{supp}(X) \mid X \in \mathcal{D}\}$  is the set of cocircuits of  $M$ .
- (S4)  $\mathcal{C} \perp \mathcal{D}$ .

**Definition 2.11.** If  $\mathcal{C} \subset (S^1 \cup \{0\})^E$  satisfies S1 and S2 and  $\{\text{supp}(X) \mid X \in \mathcal{C}\}$  is the set of circuits of a matroid  $M$ , we say  $\mathcal{C}$  is a *complex circuit signature* of  $M$ . Similarly, if  $\mathcal{D} \subset (S^1 \cup \{0\})^E$  satisfies (S1\*) and (S2\*) and  $\{\text{supp}(X) \mid X \in \mathcal{D}\}$  is the set of cocircuits of  $M$ , we say  $\mathcal{D}$  is a *complex cocircuit signature* of  $M$ .

**Theorem C.** *Let  $\mathcal{C}$  be a complex circuit signature,  $\mathcal{D}$  be a complex cocircuit signature of a matroid  $M$ . Then  $\mathcal{C}$  and  $\mathcal{D}$  are the set of phased circuits and cocircuits of a complex matroid with underlying matroid  $M$  if and only if*

$$\mathcal{C} \perp \mathcal{D}.$$

### 2.3. Minors and maps of complex matroids.

2.3.1. *Minors.* A very nice and extremely useful fact about complex matroids is that deletion and contraction of subsets of the ground set are defined in a very natural way.

**Definition 2.12.** For  $X \in (S^1 \cup \{0\})^E$  and  $A \subseteq E$  let  $X_{\setminus A}$  denote the element of  $(S^1 \cup \{0\})^{E \setminus A}$  defined by the restriction of  $X$  to  $E \setminus A$ . For  $\mathcal{U} \subseteq (S^1 \cup \{0\})^E$  define

- (1) the *deletion* of  $A$  from  $\mathcal{U}$  as

$$\mathcal{U} \setminus A = \{X_{\setminus A} \mid X \in \mathcal{U}, \text{supp}(X) \cap A = \emptyset\}.$$

- (2) the *contraction* of  $A$  from  $\mathcal{U}$  as

$$\mathcal{U}/A := \min\{X_{\setminus A} \mid X \in \mathcal{U}\},$$

where  $\min$  denotes support minimality.

**Theorem D.** *Let  $\mathcal{C}$  be the set of phased circuits of a complex matroid  $\mathcal{M}$  on the ground set  $E$  with underlying matroid  $M$ . If  $A \subseteq E$ , then  $\mathcal{C} \setminus A$  is the set of phased circuits of a complex matroid with underlying matroid  $M \setminus A$ ,  $\mathcal{C}/A$  is the set of phased circuits of a complex matroid with underlying matroid  $M/A$  and, with the notation of Definition 2.12 and Theorem B,*

$$\mathcal{C}^*/A = (\mathcal{C} \setminus A)^*.$$

*Moreover, after replacing real signs with complex signs, Definition 1.25 gives the phirotopes associated to  $\mathcal{C} \setminus A$  and  $\mathcal{C}/A$  in terms of the phirotope associated to  $\mathcal{C}$ .*

The complex matroids associated to  $\mathcal{C} \setminus A$  and  $\mathcal{C}/A$  are denoted  $M \setminus A$  and  $M/A$  and called respectively *deletion* of  $A$  from  $\mathcal{M}$  and *contraction* of  $A$  in  $\mathcal{M}$ .

2.3.2. *Weak and Strong maps.* In Section 7 we will discuss at length the notion of weak and strong maps of complex matroids. To briefly summarize, there is a natural notion of weak maps of complex matroids that parallels the one for oriented matroids. Roughly speaking, a weak map between realizable oriented matroids corresponds to “moving subspaces into more special position” with respect to the coordinate hyperplanes. The same is true for realizable complex matroids, as we prove in Theorem 7.5.

The notion of strong maps of oriented matroids is best defined on covectors or, equivalently, via the Topological Representation Theorem. Thus, the already discussed lack of a good notion of “covectors” for complex matroids makes the notion of a strong map of complex matroids rather mysterious, and deserves further study.

### 3. PHIROTOPES, DUALITY AND MINORS

This section deals with phirotopes as defined in Definition 2.3 and its goal is to establish some basic facts about duality and minors in terms of phirotopes.

**3.1. Duality.** Recall from Section 2 that given a phirotope  $\varphi$  on the ground set  $E$ , the set  $\mathbf{B}_\varphi := \{\{b_1, \dots, b_d\} \mid \varphi(b_1, \dots, b_d) \neq 0\}$  is the set of bases of the underlying matroid  $M_\varphi$ .

**Definition 3.1.** Given a rank  $d$  phirotope  $\varphi$ , choose a total ordering of  $E$ , and for all  $(x_1, x_2, \dots, x_{n-d}) \in E^{n-d}$  let  $(x'_1, \dots, x'_d)$  be a permutation of  $E \setminus \{x_1, \dots, x_{n-d}\}$ . Define the *dual* of  $\varphi$  as

$$\varphi^*(x_1, \dots, x_{n-d}) := \varphi(x'_1, \dots, x'_d)^{-1} \text{sign}(x_1, \dots, x_{n-d}, x'_1, \dots, x'_d)$$

Notice that, up to a global change of sign,  $\varphi^*$  is independent of the choice of orderings on  $E$  and  $\{x'_1, \dots, x'_d\}$ .

**Lemma 3.2.**  $\varphi^*$  is a rank  $(n-d)$  phirotope, and the underlying matroid  $M_{\varphi^*}$  is the dual  $(M_\varphi)^*$  to  $M_\varphi$ .

*Proof.* By definition,  $\mathbf{B}_{\varphi^*} = \{E \setminus B \mid B \in \mathbf{B}_\varphi\}$  which, by Theorem 1.4, is the set of bases of  $(M_\varphi)^*$ . Thus, to prove the lemma it suffices to prove that  $\varphi^*$  is indeed a phirotope.

Axioms  $(\varphi 1)$  and  $(\varphi 2)$  are clear from the definition. For  $(\varphi 3)$ , consider two sets  $X := \{x_0, \dots, x_{n-d}\}$  and  $Y := \{y_1, \dots, y_{n-d-1}\}$ , numbered such that  $X \cap Y = \{x_{n-d-l}, \dots, x_{n-d}\} = \{y_1, \dots, y_l\}$ . Without loss of generality we can assume that the total ordering of  $E$  is given by:

$$x_0, \dots, x_{n-d}, y_{l+1}, \dots, y_{n-d-1}, A$$

where  $A$  is any total ordering of  $E \setminus (X \cap Y)$ .

Then we have

$$\begin{aligned} & \varphi^*(x_0 \dots, \hat{x}_k, \dots, x_{n-d}) \varphi^*(x_k, y_1, \dots, y_{n-d-1}) = \\ & \varphi(x_k, y_{l+1}, \dots, y_{n-d-1}, A)^{-1} \underbrace{\text{sign}(x_0 \dots, \hat{x}_k, \dots, x_{n-d}, x_k, y_{l+1}, \dots, y_{n-d-1}, A)}_{\sigma_1} \\ & \varphi(x_0, \dots, \hat{x}_k, \dots, x_{n-d-l}, A)^{-1} \underbrace{\text{sign}(x_k, y_1, \dots, y_{n-d-1}, x_0, \dots, \hat{x}_k, \dots, x_{n-d-l}, A)}_{\sigma_2} \end{aligned}$$

where the sign

$$\begin{aligned} & \sigma_1 \sigma_2 = \\ & (-1)^{n-d-k} \text{sign}(x_0, \dots, x_{n-d}, y_{l+1}, \dots, y_{n-d-1}, A) \\ & (-1)^{n-d+k} \text{sign}(y_1, \dots, y_{n-d-1}, x_0, \dots, x_{n-d-l}, A) \\ & = \text{sign}(x_0, \dots, x_{n-d}, y_{l+1}, \dots, y_{n-d-1}, A) \text{sign}(y_1, \dots, y_{n-d-1}, x_0, \dots, x_{n-d-l}, A) \end{aligned}$$

does not depend on  $k$ . Then,

$$\{(-1)^k \varphi^*(x_0 \dots, \hat{x}_k, \dots, x_{n-d}) \varphi^*(x_k, y_1, \dots, y_{n-d-1}) \mid x_k \in X \setminus Y\} =$$

$\sigma_1\sigma_2\{(-1)^k\varphi(x_k, y_{l+1}, \dots, y_{n-d-1}, A)^{-1}\varphi(x_0, \dots, \hat{x}_k, \dots, x_{n-d-l}, A)^{-1} \mid x_k \in X \setminus Y\}$

We have now to prove that 0 is in the relative interior of the convex hull of the latter set. Equivalently, we show that there are positive real numbers  $\lambda_k$  such that

$$(1) \quad \sum_k \lambda_k (-1)^k \varphi(x_k, y_{l+1}, \dots, y_{n-d-1}, A)^{-1} \varphi(x_0, \dots, \hat{x}_k, \dots, x_{n-d-l}, A)^{-1} = 0.$$

Because  $\varphi$  is a phirotope, we know that there are positive real numbers  $\lambda_k$  with

$$(2) \quad \sum_k \lambda_k (-1)^k \varphi(x_k, y_{l+1}, \dots, y_{n-d-1}, A) \varphi(x_0, \dots, \hat{x}_k, \dots, x_{n-d-l}, A) = 0.$$

Since Equation 1 is the complex conjugate of Equation 2, the claim follows.  $\square$

**3.2. Deletion and contraction.** The following two lemmas prove the last part of Theorem D.

**Lemma 3.3.** *Let  $A \subset E$  be given, and choose a maximal  $\varphi$ -independent subset  $\{a_1, a_2, \dots, a_l\}$  of  $A$ . Then*

$$(\varphi/A)(x_1, \dots, x_{d-l}) := \varphi(x_1, \dots, x_{d-l}, a_1, \dots, a_l)$$

*is a phirotope, and  $M_{\varphi/A} = M_\varphi/A$ . Up to global multiplication by a constant  $c \in S^1$ ,  $\varphi/A$  is independent of the choice of  $\{a_1, a_2, \dots, a_l\}$ .*

*Proof.* The phirotope axioms for  $\varphi/A$  are easy to check. That  $M_{\varphi/A} = M_\varphi/A$  follows by Definition 1.6.(1) because

$$\begin{aligned} \mathbf{B}_{\varphi/A} &= \{\{x_1, \dots, x_{d-l}\} \mid \varphi(x_1, \dots, x_{d-l}, a_1, \dots, a_l) \neq 0\} \\ &= \{B \subseteq E \mid B \cup \{a_1, \dots, a_l\} \in \mathbf{B}_\varphi\}. \end{aligned}$$

$\square$

**Lemma 3.4.** *Let  $A \subset E$  be given, and let  $r$  be the rank of  $E \setminus A$  in  $M_\varphi$ . If  $r < d$ , choose  $\{a_1, \dots, a_{d-r}\} \subseteq A$  such that  $(E \setminus A) \cup \{a_1, \dots, a_{d-r}\}$  spans  $M_\varphi$ . Define a function  $\varphi \setminus A : E \setminus A \rightarrow S^1 \cup \{0\}$  as follows:*

$$(\varphi \setminus A)(x_1, \dots, x_r) := \begin{cases} \varphi(x_1, \dots, x_r) & \text{If } r = d \\ \varphi(x_1, \dots, x_r, a_1, \dots, a_{d-r}), & \text{if } r < d. \end{cases}$$

*Then, up to global multiplication by a constant,  $\varphi \setminus A$  is independent of the choice of  $a_1, \dots, a_{d-r}$  and  $(\varphi \setminus A)^* = \varphi^*/A$  - in particular, it is a phirotope - and  $M_{\varphi \setminus A} = M_\varphi \setminus A$ .*

*Proof.* We prove the case where  $A = \{a\}$  and we fix a linear ordering of  $E$  where  $a$  is the biggest element.

If  $r < d$ , then  $a$  is in every basis of  $M_\varphi$ , thus

$$\varphi^*(x_1, \dots, x_t) \neq 0 \text{ only if } a \notin \{x_1, \dots, x_t\}$$

Hence

$$\begin{aligned} (\varphi^*/a)(x_1, \dots, x_t) &= \varphi^*(x_1, \dots, x_t) \\ &= \varphi(x_{t+1}, \dots, x_{n-1}, a)^{-1} \text{sign}(x_1, \dots, x_{n-1}, a) \\ &= (\varphi \setminus a)(x_{t+1}, \dots, x_{n-1})^{-1} \text{sign}(x_1, \dots, x_{n-1}) \\ &= (\varphi \setminus a)^*(x_1, \dots, x_t). \end{aligned}$$

If on the other hand  $r = d$ , then

$$\begin{aligned}
 (\varphi^*/a)(x_1, \dots, x_t) &= \varphi^*(x_1, \dots, x_t, a) \\
 &= \varphi(x_{t+1}, \dots, x_{n-1})^{-1} \text{sign}(x_1, \dots, x_t, a, x_{t+1}, \dots, x_{n-1}) \\
 &= \varphi(x_{t+1}, \dots, x_{n-1})^{-1} (-1)^{n-t-2} \text{sign}(x_1, \dots, x_{n-1}, a) \\
 &= (\varphi \setminus a)(x_{t+1}, \dots, x_{n-1})^{-1} (-1)^{n-t-2} \text{sign}(x_1, \dots, x_{n-1}, a) \\
 &= (-1)^{n-t-2} (\varphi \setminus a)^*(x_1, \dots, x_t).
 \end{aligned}$$

□

#### 4. CRYTOMORPHISM FROM PHIROTYPES TO DUAL PAIRS

**4.1. Dual pairs from phirotopes.** The point of this section is to prove Proposition 4.3, asserting that every phirotope  $\varphi$  induces a dual pair of complex circuit and cocircuit signatures over  $M_\varphi$ .

**Lemma 4.1.** *Let  $\varphi$  be a phirotope,  $M_\varphi$  its underlying matroid, and  $\mathbf{B}_\varphi$  the set of bases of  $M_\varphi$ . Let  $C = \{e, f, x_2, \dots, x_k\}$  be a circuit of  $M_\varphi$ , and choose  $x_{k+1}, \dots, x_d$  so that  $\{f, x_2, \dots, x_d\} \in \mathbf{B}_\varphi$ . Then the number*

$$\frac{\varphi(e, x_2, \dots, x_d)}{\varphi(f, x_2, \dots, x_d)}$$

*does not depend on the choice of  $x_{k+1}, \dots, x_d$ .*

*Proof.* Let  $x'_d$  be such that  $\{f, x_2, \dots, x_{d-1}, x'_d\} \in \mathbf{B}_\varphi$ . Then axiom  $(\varphi 3)$  for  $\varphi$  applied to  $\{e, f, x_2, \dots, x_d\}$  and  $\{x_2, \dots, x_{d-1}, x'_d\}$  reduces to

$$\varphi(f, x_2, \dots, x_d) \varphi(e, x_2, \dots, x_{d-1}, x'_d) - \varphi(e, x_2, \dots, x_d) \varphi(f, x_2, \dots, x_{d-1}, x'_d) = 0$$

and proves the claim for pairs of choices of completions of  $C \setminus e$  that differ by one element. The full claim follows by induction on the number of elements by which any two choices of completion differ. □

**Definition 4.2.** Given a phirotope  $\varphi$ , let  $\mathcal{C}_\varphi$  be the family of all phased sets  $X$  such that  $C := \text{supp}(X)$  is a circuit of  $M_\varphi$  and for all  $e, f \in \text{supp}(X)$  we have

$$\frac{X(f)}{X(e)} := - \frac{\varphi(e, x_2, \dots, x_d)}{\varphi(f, x_2, \dots, x_d)}.$$

Notice that for any  $c \in S^1$  we have  $\mathcal{C}_{c\varphi} = \mathcal{C}_\varphi$ . Thus, it makes sense to talk about  $\mathcal{C}_{\varphi^*}$ ,  $\mathcal{C}_{\varphi \setminus e}$ , and  $\mathcal{C}_{\varphi/e}$ . Let  $\mathcal{D}_\varphi := \mathcal{C}_{\varphi^*}$ .

**Proposition 4.3.** *For every phirotope  $\varphi$  the sets  $\mathcal{C}_\varphi$  and  $\mathcal{D}_\varphi$  satisfy Definition 2.10 and are thus a dual pair of complex circuit signatures of a complex matroid with underlying matroid  $M_\varphi$ . Moreover, given an element  $e$  of the ground set we have*

- (1)  $\mathcal{C}_{\varphi \setminus e} = \mathcal{C}_\varphi \setminus e$
- (2)  $\mathcal{C}_{\varphi/e} = \mathcal{C}_\varphi/e$

*Proof.* All of the properties in the definition of phased circuits and cocircuits are clear except (S4). To see (S4), let  $X \in \mathcal{C}$  and  $Y \in \mathcal{D}$ . If  $\text{supp}(X) \cap \text{supp}(Y) = \emptyset$ , then  $X \perp Y$  by definition. Otherwise, let  $\text{supp}X = \{x_1, \dots, x_k\}$  and  $\text{supp}Y = \{y_1, \dots, y_l\}$ , with the elements of  $\text{supp}(X) \cap \text{supp}(Y)$  written first. Thus,  $x_i = y_i$  for all  $i$  less than some value  $m$ .

We can extend  $\text{supp}(X)$  to  $\{x_1, \dots, x_{d+1}\}$  so that every  $\{x_1, \dots, \hat{x}_k, \dots, x_{d+1}\}$  with  $x_k \in \text{supp}(X)$  is a basis for  $M_\varphi$ . Similarly, we extend  $\text{supp}(Y)$  to  $\{y_1, \dots, y_{n-d+1}\}$  so that every  $\{y_1, \dots, \hat{y}_k, \dots, y_{n-d+1}\}$  with  $y_k \in \text{supp}(Y)$  is a basis for  $M_\varphi^*$ . Let  $\{z_1, \dots, z_{d-1}\} = E \setminus \{y_1, \dots, y_{n-d+1}\}$ .

The Grassmann-Plücker relations tell us that 0 is in the phase convex hull of

$$\{(-1)^k \varphi(x_1, \dots, \hat{x}_k, \dots, x_{d+1}) \varphi(x_k, z_1, \dots, z_{d-1}) \mid k = 1, \dots, d+1\}.$$

Note that one of the factors of  $\varphi(x_1, \dots, \hat{x}_k, \dots, x_{d+1}) \varphi(x_k, z_1, \dots, z_{d-1})$  will be 0 unless  $x_k \in \text{supp}(X) \cap \text{supp}(Y)$ . Applying the definition of  $\varphi^*$ , we see that the above set can be written

$$\left\{ \frac{(-1)^k \varphi(x_1, \dots, \hat{x}_k, \dots, x_{d+1}) \varphi^*(y_1, \dots, \hat{y}_k, \dots, y_{n-d+1})^{-1}}{\text{sign}(x_k, z_1, \dots, z_{d-1}, y_1, \dots, \hat{y}_k, \dots, y_{n-d+1})} \mid \begin{array}{l} x_k = y_k, \text{ both in} \\ \text{supp}(X) \cap \text{supp}(Y) \end{array} \right\}.$$

Now note that

$$\begin{aligned} & \text{sign}(x_k, z_1, \dots, z_{d-1}, y_1, \dots, \hat{y}_k, \dots, y_{n-d+1}) \\ &= (-1)^{d-1+k} \text{sign}(z_1, \dots, y_1, \dots, y_{n-d+1}) \end{aligned}$$

and that if 0 is in the phase convex hull of a set  $A$  of complex numbers then 0 is in the phase convex hull of  $cA$  for any complex number  $c$ .

So, multiplying all elements of our set by

$$(-1)^{d-1} \text{sign}(z_1, \dots, y_1, \dots, y_{n-d+1}) \varphi(x_2, \dots, x_{d+1})^{-1} \varphi^*(y_2, \dots, y_{n-d+1}),$$

we see that 0 is in the phase convex hull of

$$\left\{ \frac{X(x_k)Y(x_k)}{X(x_1)Y(y_1)} \mid x_k \in \text{supp}(X) \cap \text{supp}(Y) \right\}.$$

Multiplying all elements of this set by  $X(x_1)Y(y_1)$ , we see that  $X \perp Y$ .

That  $\mathcal{C}_{\varphi \setminus e} = \mathcal{C}_\varphi \setminus e$  and  $\mathcal{C}_{\varphi/e} = \mathcal{C}_\varphi/e$  follows immediately from the definition of  $\mathcal{C}$ .  $\square$

**Corollary 4.4.** *Given a phirotope  $\varphi$ , consider  $X \in \mathcal{C}_\varphi$  and  $Y \in \mathcal{D}_\varphi$  such that  $\text{supp}(X) = \{x_0, \dots, x_l\}$ ,  $\text{supp}(Y) = \{y_1, \dots, y_h\}$ . Choose elements  $x_{l+1}, \dots, x_d$  such that  $\{x_1, \dots, x_d\} \in \mathbf{B}_\varphi$  and elements  $z_2, \dots, z_d$  that span the hyperplane  $E \setminus \text{supp}(Y)$  of  $M_\varphi$ . Then,*

(1) *For every  $x_i, x_j \in \text{supp}(X)$ ,*

$$\frac{X(x_i)}{X(x_j)} = (-1)^{i-j+1} \frac{\varphi(x_0, \dots, \hat{x}_i, \dots, x_d)}{\varphi(x_0, \dots, \hat{x}_j, \dots, x_d)}.$$

(2) *For every  $y_i, y_j \in \text{supp}(Y)$ ,*

$$\frac{Y(y_i)}{Y(y_j)} = \frac{\varphi(y_i, z_2, \dots, z_d)}{\varphi(y_j, z_2, \dots, z_d)}.$$

*In particular,  $\mathcal{D}_\varphi$  can be defined alternatively as the family of all phased sets  $Y \subset (S^1 \cup \{0\})^E$  satisfying (2).*

*Proof.* The claim (1) follows because  $\varphi$  is alternating, and thus it is enough to keep track of the permutations involved.

For claim (2), consider  $Z \in \mathcal{C}_\varphi$  such that  $\text{supp}(Z)$  is the basic circuit of  $y_i$  with respect to  $\{y_j, z_2, \dots, z_d\}$ . Then,  $\text{supp}(Z) \cap \text{supp}(Y) = \{y_i, y_j\}$ , and thus since  $Z \perp Y$  we must have

$$\frac{Y(y_i)}{Y(y_j)} = -\frac{Z(y_i)}{Z(y_j)} = \sigma^{y_j, y_i} = \frac{\varphi(y_i, z_2, \dots, z_d)}{\varphi(y_j, z_2, \dots, z_d)}$$

□

**4.2. Phirotopes from dual pairs.** Recall from Definition 1.13 the notion of *basis graph* of a matroid. By a slight abuse of notation we will sometimes picture the edges of the basis graph as arrows even if the basis graph itself is undirected, to underline “in which direction” the basis exchange is performed. Also, recall from Lemma 1.14 that if  $B$  is a basis of a matroid  $M$  on the ground set  $E$ , and  $x \in E \setminus B$ , then there is a unique circuit  $C(B, x)$  contained in  $B \cup \{x\}$ , and it is called the basic circuit of  $x$  with respect to  $B$ .

To construct a phirotope from a dual pair of circuit orientations we will follow the strategy of [3, Proposition 3.5.2 (2. proof)], which proves a similar result for oriented matroids. That is, we will first construct an appropriate labeling of the edges of the basis graph of the underlying matroid; then, the values of the phirotope will be obtained by multiplying the labels along certain paths.

We first need a preliminary lemma that investigates the values of the signatures of the circuits involved in the basis exchanges of “triangles” and “squares” of basis graphs.

**Lemma 4.5.** *Let  $\mathcal{C}, \mathcal{D}$  be the set of phased circuits resp. cocircuits of a complex matroid with underlying matroid  $M$ .*

- (1) *Given three distinct elements  $e, f, g \in E$  with bases  $B_e, B_f, B_g$  of  $M$  and  $A \subset E$  such that  $B_e = A \cup e$ ,  $B_f = A \cup f$ ,  $B_g = A \cup g$ , and for all  $x, y \in \{e, f, g\}$  consider  $X_{x,y} \in \mathcal{C}$  with  $\text{supp}(X_{x,y}) = C(A \cup x, y)$ ,*

$$\frac{X_{e,f}(e) X_{f,g}(f)}{X_{e,f}(f) X_{f,g}(g)} = -\frac{X_{e,g}(e)}{X_{e,g}(g)}$$

- (2) *Given three distinct elements  $e, f, g \in E$  with bases*

$$B_{e,f} = A \cup \{e, f\}, B_{f,g} := A \cup \{f, g\}, B_{e,g} := A \cup \{e, g\}$$

*of  $M$  for some  $A \subset E$ , choose any  $X \in \mathcal{C}$  with  $\text{supp}(X) = C(B_{e,f}, g)$ . Then,*

$$\frac{X(g) X(e)}{X(e) X(f)} = \frac{X(g)}{X(f)}$$

- (3) *Consider an independent set  $A \subset E$  and distinct elements  $e, f, g, h \in E$  such that*

$$B_1 := A \cup \{f, g\}, B_2 := A \cup \{e, g\}, B'_1 := A \cup \{f, h\}, B'_2 := A \cup \{e, h\}$$

*are bases of  $M$ , with*

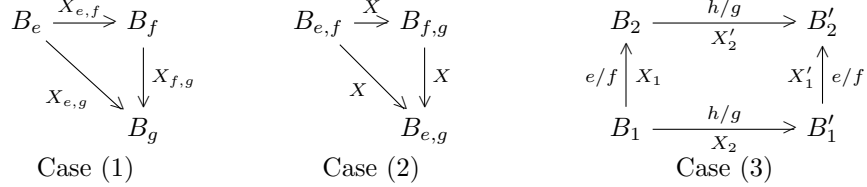
$$f \in C_1 := C(B_1, e), f \in C'_1 := C(B'_1, e),$$

$$g \in C_2 := C(B_1, h), g \in C'_2 := C(B_2, h).$$

Then for any  $X_1, X_2, X'_1, X'_2 \in \mathcal{C}$  with  $\text{supp}(X_1) = C_1$ ,  $\text{supp}(X_2) = C_2$ ,  $\text{supp}(X'_1) = C'_1$ ,  $\text{supp}(X'_2) = C'_2$ ,

$$\frac{X_1(e) X_2(h)}{X_1(f) X_2(g)} = \frac{X'_1(e) X'_2(h)}{X'_1(f) X'_2(g)}$$

The following diagrams illustrate the three cases of the lemma.

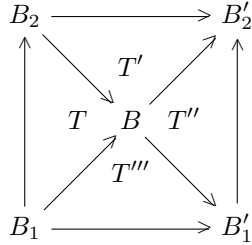


*Proof.* (1) For the cocircuit  $D := E \setminus \text{cl}(A)$ , we have  $D \cap C(A \cup x, y) = \{x, y\}$  for all  $x, y \in \{e, f, g\}$ . therefore, for any  $Y \in \mathcal{D}$  with  $\text{supp}(Y) = D$  we have  $Y \perp X_{x,y}$  for all  $x, y \in \{e, f, g\}$  and thus

$$\frac{X_{e,f}(e) X_{f,g}(f)}{X_{e,f}(f) X_{f,g}(g)} = \left( -\frac{Y(e)}{Y(f)} \right) \left( -\frac{Y(f)}{Y(g)} \right) = \frac{Y(e)}{Y(g)} = -\frac{X_{e,g}(e)}{X_{e,g}(g)}$$

(2) is evident.

(3) The claim is trivial when  $C_1 = C'_1$  and  $C_2 = C'_2$ . If this is not the case, then w.l.o.g. suppose that  $g \in C_1$ . Then we can use  $C_1$  to eliminate  $g$  from  $B_1$  (or from  $B_2$ ), and we obtain that  $B := A \cup \{e, f\}$  is a basis. Since  $g \in C_1$  implies  $h \in C'_1$  (for else one could eliminate  $e$  and obtain a circuit contained in  $B_1$ ), we can use  $C'_1$  to eliminate  $h$  from  $B'_1$  (or from  $B'_2$ ). Then, the basis graph of the matroid contains



and we can apply part (1) to the “triangles”  $T, T', T'', T'''$  to conclude.  $\square$

**Proposition 4.6.** *If  $\mathcal{C}$  and  $\mathcal{D}$  are the phased circuits resp. phased cocircuits of a complex matroid, then  $\mathcal{C} = \mathcal{C}_\varphi$  and  $\mathcal{D} = \mathcal{D}_\varphi$  for a phirotope  $\varphi$ . Moreover,  $\varphi$  is uniquely determined up to a constant.*

*Proof.* In this proof we fix a total ordering  $>$  of the ground set  $E$  of the underlying matroid  $M$ . We will often identify a subset  $A \subseteq E$  with the corresponding sequence ordered by  $>$ .

1. *Labeling the basis graph.* Consider the basis graph  $G$  of  $M$ . We define a function  $\gamma$  on pairs of adjacent vertices of  $G$ . Given two bases  $B_1, B_2$  of  $M$  corresponding to a pair of adjacent vertices of  $G$  we define

$$\gamma(B_1, B_2) := (-1)^{i-j+1} \frac{X(x_i)}{X(x_j)},$$

where  $B_1 \cup B_2 = \{x_0, \dots, x_d\}$ ,  $B_1 \setminus B_2 = \{x_i\}$ ,  $B_2 \setminus B_1 = \{x_j\}$ , the  $x_l$  are numbered in increasing order with respect to  $>$ , and  $X \in \mathcal{C}$  is any phased circuit with  $\text{supp}(X) = C(B_1, x_j)$ . Clearly,  $\gamma(B_1, B_2) = \gamma(B_2, B_1)^{-1}$ .

Our labeling has the property that, given any closed path  $A = B_0, B_1, B_2, \dots, B_k = A$  in  $G$ ,

$$\prod_{i=0}^{k-1} \gamma(B_i, B_{i+1}) = 0.$$

To see this note that by Theorem 1.15 it is enough to check the cases  $k = 3, 4$ , which is easy to do using Lemma 4.5 and keeping track of the signs.

2. *Construction of the phirotope associated with  $\mathcal{C}, \mathcal{D}$ .* If we fix a “basepoint”  $B \in V(G)$ , Step 1 above tells us there is a well-defined quantity associated to every  $B' \in V(G)$  and given by

$$\overline{\varphi}_{\mathcal{C}}(B') := \prod_{i=1}^{k-1} \gamma(B_i, B_{i+1})$$

where  $B = B_0, B_1, \dots, B_k = B'$  is any path from  $B$  to  $B'$  in  $G$ , and the empty product equals 1.

Now we are ready to define a function  $\varphi_{\mathcal{C}} : E^d \rightarrow S^1 \cup \{0\}$  as follows. Given  $x_1 < x_2 < \dots < x_d \in E$ , let

$$\varphi'_{\mathcal{C}}(x_1, \dots, x_d) := \begin{cases} 0 & \text{if } \{x_1, \dots, x_d\} \notin V(G), \\ \overline{\varphi}_{\mathcal{C}}(\{x_1, \dots, x_d\}) & \text{else.} \end{cases}$$

This function can be extended to any ordered  $d$ -tuple of elements of  $E$  by setting

$$\varphi_{\mathcal{C}}(x_1, \dots, x_d) := \text{sign}(\sigma) \varphi'_{\mathcal{C}}(x_{\sigma(1)}, \dots, x_{\sigma(d)})$$

where  $\sigma$  is a permutation such that  $x_{\sigma(i)} < x_{\sigma(j)}$  if  $i < j$ . For every  $X \in \mathcal{C}$  let  $\text{supp}(X) = \{x_0, x_1, \dots, x_l\}$  be numbered, as usual, in increasing order with respect to  $>$ . For all  $0 \leq i, j \leq l$  we can complete  $\text{supp}(X) \setminus x_i$  to a basis of  $M$  by a set  $\{a_1, \dots, a_m\}$ . Then  $\text{supp}(X) = C(A_i, x_i)$ , where  $A_i := \{x_0, \dots, \hat{x}_i, \dots, x_l, a_1, \dots, a_m\}$ . We have

$$\begin{aligned} (3) \quad \frac{X(x_i)}{X(x_j)} &= (-1)^{i-j+1} \gamma(A_j, A_i) = (-1)^{i-j+1} \overline{\varphi}_{\mathcal{C}}(A_i)^{-1} \overline{\varphi}_{\mathcal{C}}(A_j) \\ &= (-1)^{i-j+1} \frac{\varphi_{\mathcal{C}}(x_0, \dots, \hat{x}_i, \dots, x_l, a_1, \dots, a_m)}{\varphi_{\mathcal{C}}(x_0, \dots, \hat{x}_j, \dots, x_l, a_1, \dots, a_m)} \end{aligned}$$

Since  $\mathcal{C} \perp \mathcal{D}$ , for any pair of adjacent vertices  $B_1, B_2 \in V(G)$  with  $\{e\} = B_1 \setminus B_2$ ,  $\{f\} = B_2 \setminus B_1$ , the basic circuit  $C = C(B_1, f)$  of  $M$  intersects the basic circuit  $D = C^*(E \setminus B_2, e)$  of  $M^*$  in the set  $\{e, f\}$ . Choose  $X \in \mathcal{C}, Y \in \mathcal{D}$  with  $\text{supp}(X) = C$ ,  $\text{supp}(Y) = D$ . Again by  $\mathcal{C} \perp \mathcal{D}$  we have

$$\frac{X(e)}{Y(e)} = -\frac{X(f)}{Y(f)}$$

and so

$$\frac{Y(e)}{Y(f)} = -\frac{X(e)}{X(f)} = \gamma(B_1, B_2).$$

For every  $Y \in \mathcal{D}$  and  $e, f \in \text{supp}(Y)$ , choose a basis  $T$  of the hyperplane  $H$  of  $M$  defined by  $H := E \setminus D$ . Then,  $T \cup \{e, f\}$  contains a circuit  $C$  with  $C \cap \text{supp}(Y) =$

$\{e, f\}$ . Writing  $T_e = T \cup e$ ,  $T_f = T \cup f$  we have, as above,

$$(4) \quad \frac{Y(e)}{Y(f)} = (-1)^{i-j+2} \gamma(T_e, T_f) = (-1)^{i-j} \overline{\varphi_C}(T_e)^{-1} \overline{\varphi_C}(T_f) \\ = \frac{\varphi_C(f, t_2, \dots, t_d)}{\varphi_C(e, t_2, \dots, t_d)}$$

where  $e$  and  $f$  are respectively  $i$ -th and  $j$ -th in the  $>$ -ordering of  $T \cup \{e, f\}$ , and  $t_2, \dots, t_d$  is any total ordering of  $T$ . In view of Corollary 4.4, equations (3) and (4) show that  $\mathcal{C} = \mathcal{C}_{\varphi_C}$ ,  $\mathcal{D} = \mathcal{D}_{\varphi_C}$ .

*3. Verification of the axioms for phirotopes* The function  $\varphi_C$  we constructed so far is an alternating, nonzero function  $E^d \rightarrow (S^1 \cup 0)$ . We now prove that  $\varphi_C$  satisfies  $(\varphi 3)$ . To this end, consider any two subsets  $S := \{x_0, \dots, x_d\} \subset E$ ,  $T := \{y_2, \dots, y_d\} \subset E$ . If for some  $j$  the set  $S \setminus x_j$  is a basis of the underlying matroid  $M$ , then  $S \setminus x_i$  is a basis of  $M$  only if  $x_i$  is in the basic circuit  $C_S$  of  $x_j$  with respect to  $S \setminus x_j$ . Also,  $T \cup x_j$  is a basis only if  $T$  is an independent set and  $x_j$  is in the cocircuit  $D_T$  given by the complement of the hyperplane spanned by  $T$ .

We may from now suppose that  $T$  is independent and  $S \setminus x_j$  is a basis of  $M$  for some  $j$ . Then, the product

$$\varphi_C(x_0, \dots, \hat{x}_i, \dots, x_d) \varphi_C(x_i, y_2, \dots, y_d)$$

is nonzero if and only if  $x_i \in C_S \cap D_T$ .

We thus have to consider the set

$$Q := \{(-1)^i \varphi_C(x_0, \dots, \hat{x}_i, \dots, x_d) \varphi_C(x_i, y_2, \dots, y_d) \mid x_i \in C_S \cap D_T\}$$

and show that  $0 \in \text{relint conv } Q$ .

Let us suppose without loss of generality that  $x_0 \in C_S \cap D_T$ . Take  $X \in \mathcal{C}$  such that  $\text{supp}(X) = C_S$  and  $X(x_0) = 1$ ,  $Y \in \mathcal{D}$  such that  $\text{supp}(Y) = D_T$  and  $Y(x_0) = 1$ .

Then we may consider the rotated set  $\mu Q$  for  $\mu = \varphi_C(x_1, \dots, x_d)^{-1} \varphi_C(x_0, y_2, \dots, y_d)^{-1}$ . By equations (3) and (4)

$$\mu Q = \left\{ (-1)^i \frac{\varphi_C(x_0, \dots, \hat{x}_i, \dots, x_d)}{\varphi_C(x_1, \dots, x_d)} \frac{\varphi_C(x_i, y_2, \dots, y_d)}{\varphi_C(x_0, y_2, \dots, y_d)} \mid x_i \in C_S \cap D_T \right\} \\ = \left\{ \frac{\varphi_C(x_0, x_1, \dots, \hat{x}_i, \dots, x_d)}{\varphi_C(x_i, x_1, \dots, \hat{x}_i, \dots, x_d)} \frac{\varphi_C(x_i, y_2, \dots, y_d)}{\varphi_C(x_0, y_2, \dots, y_d)} \mid x_i \in C_S \cap D_T \right\} \\ = \left\{ \frac{X(x_0)Y(x_i)}{X(x_i)Y(x_0)} \mid x_i \in C_S \cap D_T \right\} = \left\{ \frac{Y(x_i)}{X(x_i)} \mid x_i \in C_S \cap D_T \right\},$$

thus  $0 \in \text{relint conv } Q$  if and only if  $0 \in (\text{relint conv } \mu Q) = P_{X,Y}$  - but the latter is the case because by assumption  $X \perp Y$ .  $\square$

## 5. ELIMINATION AXIOMS

Our next goal is the statement of a set of axioms governing the behavior of complex phases in circuit elimination. We start by a rather discouraging example, which shows that one cannot hope for a general elimination axiom, as is the case for oriented matroids.

**Example 5.1.** Let  $v_1, \dots, v_7$  denote the column of the following matrix:

$$M := \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & i & 1-i \\ 2 & -1 & 0 & -1 & 0 & -i & 3+i \\ -i & 0 & -i & 0 & 2i & -i & -2i \\ 1 & 0 & 0 & -i & i+1 & 0 & -2 \end{pmatrix}$$

The vectors  $(1, 1, 1, 1, 1, 0, 0)$  and  $(-1, 0, 0, 1, 1, 1, 1)$  are both elements of  $\ker M$  of minimal support, giving rise to two phased circuits  $X := (1, 1, 1, 1, 1, 0, 0)$  and  $Y := (-1, 0, 0, 1, 1, 1, 1)$ . Now, a “general” elimination axiom should describe the phases of the circuit obtained eliminating  $v_1$  from  $X$  and  $Y$  in terms of the phases of  $X$  and  $Y$ .

$$\begin{aligned} v_4 + iv_3 - (1+i)v_2 + \left(\frac{1}{2} + \frac{i}{2}\right)v_5 + v_6 &= 0, \\ v_4 + (1-i)v_3 + (2+i)v_2 + \left(\frac{3}{2} - \frac{1}{2}\right)v_5 + v_7 &= 0, \\ v_4 + \left(-1 + \frac{1}{2}\right)v_3 - \frac{5i}{2}v_2 + \left(1 + \frac{i}{2}\right)v_6 - \frac{i}{2}v_7 &= 0, \\ v_4 + \left(\frac{7}{13} + \frac{4}{13}i\right)v_3 + \left(\frac{25}{26} + \frac{5}{26}i\right)v_5 + \left(\frac{8}{13} - \frac{1}{13}i\right)v_6 + \left(\frac{5}{13} + \frac{i}{13}\right)v_7 &= 0, \\ v_4 + \left(\frac{3}{5} - \frac{4}{5}i\right)v_2 + \left(\frac{7}{10} - \frac{i}{10}\right)v_5 + \left(\frac{3}{5} + \frac{i}{5}\right)v_6 + \left(\frac{2}{5} - \frac{i}{5}\right)v_7 &= 0, \\ v_5 + \left(\frac{3}{2} - \frac{i}{2}\right)v_3 + \left(\frac{1}{2} + \frac{5}{2}i\right)v_2 - \left(\frac{1}{2} + \frac{i}{2}\right)v_6 + \left(\frac{1}{2} + \frac{i}{2}\right)v_7 &= 0. \end{aligned}$$

**5.1. Deletion and contraction.** In the following we will often argue by induction on the size of the ground set of the complex matroid. As a preparation, we prove that our notion of complex circuit orientation (Definition 2.7) behaves well with respect to the operations of deletion and contraction as introduced in Definition 2.12.

**Lemma 5.2.** *Let  $M$  be a matroid on the ground set  $E$ , and let  $e \in E$ . Then*

- (1) *if  $C_1, C_2$  is a modular pair of circuits of  $M \setminus e$ , then it is a modular pair of circuits of  $M$ .*
- (2) *if  $C_1, C_2$  is a modular pair of circuits of  $M/e$ , then  $C_1 \cup e, C_2 \cup e$  is a modular pair of circuits of  $M$ .*

*Proof.* In view of Definition 1.11 we show the equivalent statements about the dual  $M^*$ . In what follows,  $r^*, r_{\setminus e}^*, r_{/e}^*$  are the rank functions of  $M^*, M^* \setminus e, M^*/e$  respectively.

(1) Let  $H_1, H_2$  be a modular pair of hyperplanes of  $M^*/e$ . Then for  $i = 1, 2$ ,  $H'_i := H_i \cup e$  is a hyperplane of  $M^*$ ,

$$r_{/e}^*(H_1 \cap H_2) = r^*((H_1 \cap H_2) \cup e) - r^*(e) = r^*(H'_1 \cap H'_2) - 1$$

and since by assumption  $r_{/e}^*(H_1 \cap H_2) = r_{/e}^*(E \setminus e) - 2 = r^*(E) - 3$ , we have  $r^*(H'_1 \cap H'_2) = r^*(E) - 2$ . So  $H_1, H_2$  is a modular pair.

(2) Let  $H_1, H_2$  be a modular pair of hyperplanes of  $M^* \setminus e$ . For  $i = 1, 2$  let  $H'_i \subset H_i \cup e$  denote a hyperplane of  $M^*$  containing  $H_i$ . If  $r_{\setminus e}^*(E \setminus e) = r^*(E)$ , then

$$r_{\setminus e}^*(E \setminus e) - 2 = r_{\setminus e}^*(H_1 \cap H_2) = r^*(H_1 \cap H_2) \leq r^*(H'_1 \cap H'_2) \leq r^*(E) - 2$$

and  $H'_1, H'_2$  are a modular pair. If however  $r_{\setminus e}^*(E \setminus e) < r^*(E)$ , then  $e$  is in every basis of  $M^*$ , and  $e \in H'_1 \cap H'_2$ . Then

$$r^*(E) - 3 = r_{\setminus e}^*(E \setminus e) - 2 = r^*(H_1 \cap H_2) = r^*(H'_1 \cap H'_2) - 1$$

and  $H'_1, H'_2$  are a modular pair in  $M^*$ .  $\square$

**Proposition 5.3.** *If  $\mathcal{C}$  is a complex circuit orientation of a matroid  $M$  on  $E$ , then for all  $e \in E$*

- (1)  $\mathcal{C}/e$  is a complex circuit orientation of the matroid  $M/e$  and
- (2)  $\mathcal{C} \setminus e$  is a complex circuit orientation of the matroid  $M \setminus e$ .

*Proof.* Let  $\mathcal{C}$  be as in the statement. For (1) note that the elements of  $\mathcal{C} \setminus e$  are all oriented circuits in  $\mathcal{C}$  not containing  $e$  in their support, and so (ME) holds in  $\mathcal{C} \setminus e$  because, by Lemma 5.2.(1), a modular pair of circuits in  $M \setminus e$  is modular in  $M$  too, and the result of modular elimination between them in  $M$  is again an element of  $M \setminus e$ .

For (2), remember first that every element of  $X \in \mathcal{C}/e$  is a subset of some element  $X' \in \mathcal{C}$  with  $\text{supp}(X') = \text{supp}(X) \cup e$ , and in particular  $X'(x) = X(x)$  for all  $x \in \text{supp}(X)$ . Lemma 5.2.(2) ensures that for every modular pair  $X, Y$  in  $\mathcal{C}/e$  the corresponding  $X', Y' \in \mathcal{C}$  defined as above also define a modular pair. As above, the element  $Z'$  obtained by modular elimination of  $f$  between  $X'$  and  $Y'$  contains  $Z \in \mathcal{C}/e$  with  $f \in \text{supp}(Z) \subset \text{supp}(X) \cup \text{supp}(Y)$ . By the uniqueness of modular elimination we are done.  $\square$

**5.2. From phirotopes to circuit orientations.** In this section we prove that the set  $\mathcal{C}_\varphi$  of circuits induced by a phirotope  $\varphi$  satisfies the conditions of Definition 2.3.(2) a complex circuit orientation. Conditions (C0) and (C1) are clear; we have to prove that (ME) holds in  $\mathcal{C}_\varphi$ , and as a stepping stone we prove the following result.

**Lemma 5.4 (SE).** *Let  $\varphi$  be a phirotope on the ground set  $E$ . For all  $X, Y \in \mathcal{C}_\varphi$  and  $e, f \in \text{supp}(X) \cap \text{supp}(Y)$  such that  $X(e) = Y(e)$  and  $X(f) \neq Y(f)$ , there is  $Z \in \mathcal{C}$  with  $f \in \text{supp}(Z) \subseteq \text{supp}(X) \cup \text{supp}(Y)$ .*

*Proof.* Suppose by way of contradiction that there are  $X, Y \in \mathcal{C}_\varphi$ ,  $e, f \in E$  so that the claim does not hold and let  $A := \text{supp}(X) \setminus \{e, f\}$ ,  $B := \text{supp}(Y) \setminus \{e, f\}$ . Then  $f \notin \text{cl}(A \cup B)$  and we can extend  $A$  to  $A'$  and  $B$  to  $B'$ , where  $A'$  and  $B'$  are bases of the hyperplane  $H$  containing  $\text{cl}(A \cup B)$  but not  $e$  (and thus not  $f$  either). Then let  $D := E \setminus H$ . It follows that  $D \cap \text{supp}(X) = D \cap \text{supp}(Y) = \{e, f\}$ . If we fix a total ordering of the ground set  $E$  we can think of any subset of  $E$  as representing an ordered tuple of elements. With  $D' := D \setminus \{e, f\}$  we can write

$$\frac{X(f)}{X(e)} = -\frac{\varphi(e, A')}{\varphi(f, A')} = \frac{\varphi^*(e, D', H \setminus A')}{\varphi^*(f, D', H \setminus A')}.$$

But since this value does not depend on how we complete the set  $D'$  to a complement of a basis of  $M_\varphi$ , we have

$$\frac{X(f)}{X(e)} = \frac{\varphi^*(e, D', H \setminus B')}{\varphi^*(f, D', H \setminus B')} = -\frac{\varphi(e, B')}{\varphi(f, B')} = \frac{Y(f)}{Y(e)},$$

contradicting the assumption.  $\square$

**Proposition 5.5** (ME). *Let  $\varphi$  be a phirotope. For all  $X, Y \in \mathcal{C}_\varphi$  with  $X \neq \mu Y$  for all  $\mu \in S^1$  and such that  $\text{supp}(X), \text{supp}(Y)$  is a modular pair of circuits of  $M_\varphi$ , given  $e, f \in E$  with  $X(e) = -Y(e) \neq 0$  and  $X(f) \neq Y(f)$ , there is  $Z \in \mathcal{C}_\varphi$  with  $f \in \text{supp}(Z) \subseteq \text{supp}(X) \cup \text{supp}(Y) \setminus \{e\}$ , and*

$$\begin{cases} Z(f) \in \text{pconv}(\{X(f), Y(f)\}) & \text{if } f \in \text{supp}(X) \cap \text{supp}(Y) \\ Z(f) \leq \max\{X(f), Y(f)\} & \text{else.} \end{cases}$$

*Proof.* For ease of notation and terminology, let us prove this for the dual matroid - so when  $X, Y \in \mathcal{C}_{\varphi^*}$  are cocircuits of the complex matroid defined by  $\varphi$ .

Since the supports of  $X, Y$  form a modular pair, we have  $x, y \in E$  and  $A \subset E$  such that  $\text{supp}(X) = E \setminus \text{cl}(A \cup \{x\})$ ,  $\text{supp}(Y) = E \setminus \text{cl}(A \cup \{y\})$ . Then it follows that  $x \in \text{supp}(Y)$  and  $y \in \text{supp}(X)$ , for else  $\text{supp}(X) = \text{supp}(Y)$  and  $X = \mu Y$  for some  $\mu \in S^1$ , which cannot be. From now on we fix a total ordering  $a_2, \dots, a_d$  of  $A$  and, when appropriate, write  $A$  for  $a_2, \dots, a_d$ .

Let then  $D$  be the (unique) cocircuit complementary to the hyperplane  $E \setminus \text{cl}(A \cup \{e\})$ . By definition, the sign vector defined by  $Z(x) := Y(x)$  and

$$\frac{Z(f)}{Z(x)} := \frac{\varphi(f, e, A)}{\varphi(x, e, A)} \text{ for all } f \neq x$$

is a signature of  $D$ . We will prove that it satisfies the requirements.

First of all, consider the element  $y \in \text{supp}(X) \cap \text{supp}(Z)$ . We have

$$\frac{Z(y)}{Z(x)} := \frac{\varphi(y, e, A)}{\varphi(x, e, A)} = \frac{\varphi(y, e, A) \varphi(x, y, A)}{\varphi(x, y, A) \varphi(x, e, A)} = -\frac{Y(e) X(y)}{Y(x) X(e)} = \frac{X(y)}{Y(x)}$$

and therefore, since we set  $Z(x) = Y(x)$ , we obtain  $Z(y) = X(y)$ .

Now let us consider an element  $f \in \text{supp}(Z) \setminus \text{supp}(X)$ . Then  $f \notin \text{supp}(X)$ , and since  $f \notin \text{cl}(A)$  (for else  $f \notin \text{supp}(Z)$ ) we conclude that we can exchange  $f$  for  $x$  in the base  $A \cup \{x\}$  of the hyperplane  $E \setminus \text{supp}(X) = \text{cl}(A \cup \{x\}) = \text{cl}(A \cup \{f\})$ . Therefore we can compute

$$\frac{Z(f) Y(x)}{Z(x) Y(f)} = \frac{\varphi(f, e, A) \varphi(x, y, A)}{\varphi(x, e, A) \varphi(f, y, A)} = \frac{\varphi(e, f, A) \varphi(y, x, A)}{\varphi(y, f, A) \varphi(e, x, A)} = \frac{X(e) X(y)}{X(y) X(e)} = 1$$

hence  $Z(f) = Y(f)$ . By a similar argument we obtain  $Z(f) = X(f)$  for every  $f \in \text{supp}(Z) \setminus \text{supp}(Y)$ .

As the last case we consider an element  $f \in \text{supp}(Z) \cap \text{supp}(X) \cap \text{supp}(Y)$ . Because the set  $B := \{e, y\} \cup A$  is a basis of  $M_\varphi$ , and  $f$  is not an element of  $\text{cl}(A \cup \{y\})$  nor of  $\text{cl}(A \cup \{e\})$ , the basic circuit  $C(f, B)$  of  $f$  with respect to  $B$  contains  $e, y, f$  and thus  $C(f, B) \cap \text{supp}(X) = \{e, y, f\}$ . In order to compute  $Z(f)$  we apply the axiom (b) for chirotopes to the tuples of elements  $y, f, e, A$  and  $x, A$  and conclude that 0 must be in the relative interior of the phase convex hull of

$$\{\varphi(f, e, A)\varphi(y, x, A), -\varphi(y, e, A)\varphi(f, x, A), \varphi(y, f, A)\varphi(e, x, A)\}$$

This condition does not depend upon rotation - i.e., multiplication by an element of  $S^1$ . Thus, after multiplication by  $(\varphi(y, e, A)\varphi(y, x, A))^{-1}$ , equivalently we may say

$$0 \in \text{pconv} \left\{ \begin{array}{l} \varphi(f, e, A) \\ \varphi(y, e, A) \end{array}, -\frac{\varphi(f, x, A)}{\varphi(y, x, A)}, \frac{\varphi(y, f, A) \varphi(e, x, A)}{\varphi(y, x, A) \underbrace{\varphi(y, e, A)}_{=-\frac{Z(x)}{Z(y)}}} \right\}$$

Which, by Corollary 4.4, can be rewritten as

$$0 \in \text{pconv} \left\{ \frac{Z(f)}{Z(y)}, -\frac{X(f)}{X(y)}, -\frac{Y(f)Y(x)}{Y(x)X(y)} \right\}$$

We already established that  $Z(y) = X(y)$ , and thus multiplying everything by this number we conclude that

$$0 \in \text{pconv} \{Z(f), -X(f), -Y(f)\}$$

or, equivalently,  $Z(f) \in \text{pconv}(\{X(f), Y(f)\})$ . □

**5.3. From circuit orientations to dual pairs.** The goal of this section is to “close the circle” and show that the axiomatization in terms of circuit elimination given in Definition 2.3 is equivalent to the axiomatization for dual pairs of Definition 2.10. We will do so by showing that the set of circuits of a complex matroid induces a (unique) orthogonal complex signature of the cocircuits of the underlying matroid.

We first need a fact from matroid theory that we summarize in the following lemma.

**Lemma 5.6.** *Let  $M$  be a matroid on the ground set  $E$ . Consider a circuit  $C$  and a cocircuit  $D$  of  $M$  such that  $|C \cap D| \geq 3$ . Then there are elements  $e, f \in D \cap C$  and a cocircuit  $D'$  of  $M$  such that*

- (1)  $D$  and  $D'$  are a modular pair,
- (2)  $e \in (D' \cap C) \subseteq (D \cap C) \setminus f$ .

*Proof.* Let  $D$  and  $C$  be as above, and let  $r$  be the rank of  $M$ . Then  $C \setminus D$  is an independent set of rank at most  $r - 2$  and can be completed to a basis  $B$  of the hyperplane  $H := E \setminus D$ .

For every  $e \in C \cap D$ , the set  $B \cup e$  is a basis of  $M$ . The basic circuit of  $f$  with respect to this basis cannot be contained fully in  $(C \setminus D) \cup e$ , and thus contains an element  $x \in B \setminus (C \setminus D)$ . Let  $A := B \setminus x$ . Then we have  $H = \text{cl}(A \cup x)$  and we can define

$$H' := \text{cl}(A \cup f), \quad D' := E \setminus H'$$

Clearly,  $(D' \cap C) \subseteq (D \cap C) \setminus f$ . To prove  $e \in D' \cap C$ , it is enough to show  $e \notin H'$ . But were it the case, then there would be a circuit contained in the set  $A \cup \{e, f\}$ , and by the uniqueness of basic circuits, this would be also the basic circuit of  $f$  with respect to  $B \cup e$  - contradicting the definition of  $x$ . □

As a first step, we prove the analog of Lemma 5.4.

**Lemma 5.7.** *Let  $\mathcal{C}$  be a circuit orientation of a complex matroid. Then*

- (SE) *for all  $X, Y \in \mathcal{C}$ ,  $e, f \in E$  with  $X(e) = -Y(e) \neq 0$  and  $Y(f) \neq X(f)$ , there is  $Z \in \mathcal{C}$  with  $f \in \text{supp}(Z) \subseteq \text{supp}(X) \cup \text{supp}(Y) \setminus e$ .*

*Proof.* By Lemma 1.12 the set  $\mathbf{C} := \{\text{supp}(X) \mid X \in \mathcal{C}\}$  is the set of circuits of a matroid  $M$ .

We argue by induction on the rank of the  $M$ . The claim is trivial in rank 0 and 1, and every pair of circuits is modular in rank 2. So let  $\mathcal{C}$  be a circuit orientation of a complex matroid of rank  $d > 2$  and suppose the claim holds for all complex matroids of smaller rank.

By way of contradiction, let  $X, Y \in \mathcal{C}$  and  $e, f \in E$  be such that for all  $C \in \mathbf{C}$  with  $C \subseteq \text{supp}(X) \cup \text{supp}(Y)$ ,  $f \notin C$ . The case where  $X(f)Y(f) = 0$  is covered by matroid elimination (Definition 1.1.(C2)). So suppose  $e, f \in \text{supp}(X) \cap \text{supp}(Y)$  and choose  $a \in \text{supp}(Y) \setminus \text{supp}(X)$ . By Proposition 5.3  $\mathcal{C}/a$  is again a complex orientation of the circuits of the rank  $d - 1$  matroid  $M/a$ . By definition there are  $X', Y' \in \mathcal{C}/a$  with  $X'(g) \leq X(g)$ ,  $Y'(g) \leq Y(g)$  for all  $g \in E \setminus a$ , and with  $f \in \text{supp}(X') \cap \text{supp}(Y')$ . With the notation of Definition 2.12,  $Y' = Y_{\setminus a}$  and thus  $e \in \text{supp}(Y')$ .

Now, if  $e \notin \text{supp}(X')$  we reach a contradiction by taking  $C := \text{supp}(X') \cup a$ . Otherwise  $e, f \in \text{supp}(X') \cap \text{supp}(Y')$  so  $X'(e) = X(e) = -Y(e) = -Y'(e)$  and  $X'(f) = X(f) \neq Y(f) = Y'(f)$ . We apply induction hypothesis to the rank- $(d - 1)$  complex matroid  $\mathcal{C}/a$  and find  $Z' \in \mathcal{C}/a$  with  $f \in \text{supp}(Z') \subseteq \text{supp}(X') \cup \text{supp}(Y') \setminus e$ . Then we reach a contradiction by taking  $C := \text{supp}(Z') \cup a \in \mathbf{C}$ .  $\square$

**Proposition 5.8.** *Any complex circuit orientation  $\mathcal{C}$  with underlying matroid  $M$  induces a unique complex circuit signature  $\mathcal{D}$  of  $M^*$  such that  $\mathcal{D} \perp \mathcal{C}$ .*

*Proof.* Let  $\mathcal{C}$  be a complex circuit orientation with underlying matroid  $M$ .

*Definition of  $\mathcal{D}$ :* For every cocircuit  $D$  of  $M$ , choose a maximal independent subset  $A$  of the hyperplane  $D^c$ . Then for every  $e, f \in D$ , there is a unique circuit  $C_{D,e,f}$  of  $M$  with support contained in  $A \cup \{e, f\}$ . (Namely,  $C_{D,e,f}$  is the basic circuit of  $f$  with respect to  $A \cup e$ .) Choose  $X_{D,e,f} \in \mathcal{C}$  with  $\text{supp}(X_{D,e,f}) = C_{D,e,f}$ .

$$\mathcal{D} := \left\{ W \in (S^1 \cup \{0\})^E \mid \begin{array}{l} D := \text{supp}(W) \in \mathbf{C}(M^*), \\ \forall e, f \in \text{supp}(W), \frac{W(e)}{W(f)} = -\frac{X_{D,e,f}(e)}{X_{D,e,f}(f)} \end{array} \right\}$$

**Claim 1.**  $\mathcal{D}$  is well-defined and independent on the choice of the  $X_{D,e,f}$ .

*Proof.* First we prove independence on the choice of the  $X_{D,e,f}$ . Given  $D \in \mathbf{C}^*(M)$  and  $e, f \in D$ , let  $Y$  and  $Y'$  be two candidates for  $X_{D,e,f}$ . Multiplying  $Y$  by an element of  $S^1$ , we may assume  $Y(e) = -Y'(e)$ . If  $Y(e)/Y(f) \neq Y'(e)/Y'(f)$ , then by Lemma 5.7 there is  $Z \in \mathcal{C}$  with  $\text{supp}(Z) \cap D = \{f\}$ , contradicting Lemma 1.5.

To conclude that  $\mathcal{D}$  is well-defined, it is enough to prove that, given  $D \in \mathbf{C}^*(M)$  and  $e, f, g \in D$ ,

$$-\frac{X_{D,f,g}(f)}{X_{D,f,g}(g)} = \left( -\frac{X_{D,e,f}(f)}{X_{D,e,f}(e)} \right) \left( -\frac{X_{D,e,g}(e)}{X_{D,e,g}(g)} \right).$$

The circuits  $C_{D,e,f}$  and  $C_{D,e,g}$  form a modular pair, because their complements both contain the corank 2 coflat  $\text{cl}(E \setminus (A \cup \{f, g\}))$ . Then (modular) elimination of  $e$  from  $X_{D,e,f}$  and  $\frac{-X_{D,e,f}(e)}{X_{D,e,g}(e)} X_{D,e,g}$  gives  $Y \in \mathcal{C}$  with  $f, g \in \text{supp}(Y)$  and  $\frac{Y(f)}{Y(g)} = \frac{X_{D,e,f}(f)}{-X_{D,e,f}(e)} \frac{X_{D,e,g}(e)}{X_{D,e,g}(g)}$ . So

$$\frac{X_{D,f,g}(f)}{X_{D,f,g}(g)} = \frac{Y(f)}{Y(g)} = -\frac{X_{D,e,f}(f)}{X_{D,e,f}(e)} \frac{X_{D,e,g}(e)}{X_{D,e,g}(g)}$$

and the claim follows.

**Claim 2.** Fix  $W \in \mathcal{D}$ . For all  $X \in \mathcal{C}$  with  $|\text{supp}(W) \cap \text{supp}(X)| \leq 3$ ,  $W \perp X$ .

*Proof.* The claim is either trivial or clear by definition if  $|\text{supp}(W) \cap \text{supp}(X)| \leq 2$ . So consider  $X \in \mathcal{C}$  with  $|\text{supp}(X) \cap \text{supp}(W)| = 3$ , and by way of contradiction let  $\text{supp}(W) \cap \text{supp}(X) = \{e, f, g\}$  so that  $P_{X,W}$  is contained in a closed half-circle and includes a point in the interior of this half-circle.

By Lemma 5.6 applied to  $M^*$ , there is a circuit  $X' \in \mathcal{C}$  and two elements of  $\{e, f, g\}$  (say,  $e, f$ ) such that  $\text{supp}(X')$  and  $\text{supp}(X)$  are a modular pair in  $M$ , and  $e \in \text{supp}(X') \cap \text{supp}(W) \subseteq \text{supp}(X) \cap \text{supp}(W) \setminus f$ , and since  $|\text{supp}(X') \cap \text{supp}(W)| \geq 2$ , we know  $\text{supp}(X') \cap \text{supp}(W) = \{e, g\}$ . Multiplying by an element of  $S^1$ , we may assume  $X'(e) = -X(e)$ . Thus

$$\frac{X'(g)}{W(g)} = -\frac{X'(e)}{W(e)} = \frac{X(e)}{W(e)},$$

and

$$P_{X,W} = \left\{ \frac{X'(g)}{W(g)}, \frac{X(f)}{W(f)}, \frac{X(g)}{W(g)} \right\}.$$

In particular, these three points lie in the unit circle as described before. Modular elimination of  $e$  between  $X'$  and  $X$  gives a circuit  $Y \in \mathcal{C}$  with  $\text{supp}(Y) \cap \text{supp}(W) = \{f, g\}$ ,  $Y(f) = X(f)$ , and  $Y(g) \in \text{pconv}(\{X(g), X'(g)\})$ . Thus  $P_{Y,W}$  lies in a half-open half-circle of  $S^1$ , contradicting  $Y \perp W$ .

**Claim 3.**  $\mathcal{D} \perp \mathcal{C}$ .

*Proof.* Induction on the rank of  $M$ . If  $M$  has rank 2, then all circuits have size 3, and we conclude with Claim 2. Assume that  $M$  has rank  $r > 2$  and the claim holds for all matroids of rank  $r - 1$  or less.

Suppose by way of contradiction that there is  $X \in \mathcal{C}$  and  $W \in \mathcal{D}$  with  $X \not\perp W$ . Choose  $e \in E \setminus \text{supp}(W)$ . Then,  $\mathcal{C}/e$  is a complex circuit orientation of the matroid  $M/e$  and  $\mathcal{D}$  is a circuit signature of the matroid  $M^* \setminus e$  satisfying  $X' \perp W'$  for all  $X' \in \mathcal{C}/e$  and  $W' \in \mathcal{D} \setminus e$  with  $|\text{supp}(X') \cap \text{supp}(W')| \leq 2$ . Since the rank of  $M/e$  is  $r - 1$ , by induction hypothesis  $X' \perp W'$  for all  $X' \in \mathcal{C}/e$ ,  $W' \in \mathcal{D} \setminus e$ .

Now look at our  $X, W$  and choose  $f \in \text{supp}(X) \cap \text{supp}(W)$ . By the definition of contraction and deletion,  $W \in \mathcal{D} \setminus e$  and there is  $X' \in \mathcal{C}/e$  with  $X' \subseteq X$  and  $f \in \text{supp}(X')$ . The vertices of  $P_{X',W}$  are a subset of the vertices of  $P_{X,W}$  - thus  $X \not\perp W$  forces  $X' \not\perp W$ , contradicting the induction hypothesis.  $\square$

At last, we can justify Theorem A and Theorem C.

**Corollary 5.9.** *The definition of complex matroids in terms of their oriented circuits obtained from axioms (C0), (C1), (ME) is equivalent to the definition in terms of phirotopes (and, in turn, with the one in terms of dual pairs).*

*Proof.* This is just a combination of Proposition 5.5, Proposition 5.8 and Proposition 4.6.  $\square$

**5.4. Duality.** Given a complex orientation  $\mathcal{C}$  of the circuits of a complex oriented matroid, the corresponding set of oriented cocircuits can be defined by orthogonality.

**Proposition 5.10.** *Let  $\mathcal{C} \subseteq (S^1 \cup \{0\})^E$  be a complex circuit orientation of  $M$ . Then the set of elements of  $\mathcal{C}^\perp \setminus \{0\}$  of minimal support is exactly the complex signature  $\mathcal{D}$  of  $M^*$  given by Proposition 5.8. This is the unique complex orientation of  $M^*$  whose circuits are all orthogonal to  $\mathcal{C}$ .*

*Proof.* Recall

$$\mathcal{C}^\perp := \{W \in (S^1 \cup \{0\})^E \mid W \perp X \text{ for some } X \in \mathcal{C}\}$$

and write  $\text{min}\mathcal{C}^\perp$  for the elements of  $\mathcal{C}^\perp \setminus \{0\}$  with minimal support.

By Proposition 5.8, we have  $\mathcal{D} \subset \mathcal{C}^\perp$ . Since  $\text{supp}(\mathcal{D}) := \{\text{supp}(X) \mid X \in \mathcal{D}\}$  is the set of circuits of the underlying matroid, by [10, Thm] it can be written as  $\text{supp}(\mathcal{D}) = \min\mathcal{S}$ , where

$$\mathcal{S} := \{A \subseteq E \mid |A \cap \text{supp}(X)| \neq 1 \forall X \in \mathcal{C}\}$$

Now,  $\text{supp}(\mathcal{C}^\perp) \subset \mathcal{S}$  (since  $X \perp W$  forbids  $|\text{supp}(X) \cap \text{supp}(W)| = 1$ ), and so  
 (1)  $\mathcal{D} \subseteq \min\mathcal{C}^\perp$  because for every  $W \in \mathcal{C}^\perp$  there is  $Y \in \mathcal{D}$  with  $\text{supp}(Y) \subseteq \text{supp}(W)$ ,  
 (2)  $\mathcal{D} \supseteq \min\mathcal{C}^\perp$  because every  $W \in \min\mathcal{C}^\perp$  has the same support as some  $Y_W \in \mathcal{D}$ , and one sees as in the proof of Proposition 5.8 that for any  $X \in (S^1 \cup \{0\})$  with  $\text{supp}(W) \in \text{supp}(\mathcal{D})$  the condition  $W \perp \mathcal{C}$  determines the ratios  $W(f)/W(e)$  uniquely for every pair  $e, f \in \text{supp}(W)$  - thus,  $Y_W = W$ .  $\square$

## 6. VECTORS

Sadly, there is no vector axiomatization for complex matroids that is cryptomorphic to the other axiomatizations and has the property that, for complex subspaces  $W$  of  $\mathbb{C}^n$ , the complex matroid of  $W$  has vector set  $\{\text{ph}(v) : v \in W\}$ . In this section we give an example to show that the circuits of a complex matroid with realization  $W$  do not determine  $\{\text{ph}(v) : v \in W\}$ .

Let  $W_1$  be the row space of

$$\begin{pmatrix} 1 & 1+i & 1 & 0 \\ 1+i & 3i & 0 & 1 \end{pmatrix}$$

and let  $W_2$  be the row space of

$$\begin{pmatrix} 1 & 1+i & 1 & 0 \\ 1+i & 4i & 0 & 1 \end{pmatrix}.$$

We shall verify that  $W_1$  and  $W_2$  have the same complex matroid, but that there is a  $v \in W_1$  such that  $\text{ph}(v) \neq \text{ph}(w)$  for every  $w \in W_2$ .

For each  $W_i$ , the underlying matroid is uniform, of rank 2, with 4 elements, so has 4 (unphased) circuits. Thus each complex matroid has circuit set consisting of four  $S^1$  orbits. We can read two of the orbits for each  $W_i$  directly from the presentation above: each of the two complex matroids has  $\text{ph}(1, 1+i, 1, 0)$  and  $\text{ph}(1+i, 3i, 0, 1) = \text{ph}(1+i, 4i, 0, 1)$  as circuits. To see the remaining two orbits, we perform Gauss-Jordan elimination on the two matrices:

$$\begin{pmatrix} 1 & 1+i & 1 & 0 \\ 1+i & 3i & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & -1+i \\ 0 & 1 & i-1 & -i \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1+i & 1 & 0 \\ 1+i & 4i & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & \frac{1}{2}(-1+i) \\ 0 & 1 & \frac{1}{2}(i-1) & \frac{-i}{2} \end{pmatrix}.$$

So, the two  $W_i$  give the same complex matroid.

On the other hand, note that  $(2+i, 1+4i, 1, 1) \in W_1$ . Assume by way of contradiction that some  $w \in W_2$  has  $\text{ph}(w) = \text{ph}(2+i, 1+4i, 1, 1)$ . Then  $w = k(1, 1+i, 1, 0) + l(1+i, 4i, 0, 1)$  for some  $k$  and  $l$ . To have the correct signs on the last two components,  $k$  and  $l$  must both be positive real numbers. However, one easily checks that no such  $k$  and  $l$  give the correct sign on the first two components.

## 7. WEAK MAPS AND STRONG MAPS

Intuitively, a weak map of matroids is the combinatorial analog to moving a subspace of a vector space  $\mathbb{K}^n$  into more special position with respect to the coordinate hyperplanes. The same intuition motivates the definition of weak maps for oriented matroids, although the intuition is known to be somewhat problematic in this case: there are weak maps of realizable oriented matroids which do not arise from geometrically “close” realizations.

A strong map of (oriented) matroids is the combinatorial analog to taking a subspace of a vector space. In the case of oriented matroids, this analogy has a beautifully straightforward interpretation via the Topological Representation Theorem. The covectors of a rank  $r$  oriented matroid  $\mathcal{M}$  label the cells in a regular cell decomposition of  $S^{r-1}$ , and the covectors of any rank  $k$  strong map image of  $\mathcal{M}$  label the cells in the intersection of this cell complex with a  $(k-1)$ -dimensional “pseudoequator”. For details of this, see [3, Section 7.7]. The big point is that strong maps of oriented matroids have an extremely straightforward definition in terms of covectors (and hence also in terms of vectors), but it is not so clear how to see strong maps directly in terms of circuits, cocircuits, or chirotopes. As far as we know there is no definition of strong maps of oriented matroids in terms of circuits, cocircuits, or chirotopes without involving composition somehow. From the perspective of the Topological Representation Theorem, such a definition seems unlikely: the cocircuits of an oriented matroid represent only the vertices in the cell decomposition of  $S^{r-1}$ , and without referring to composition it’s not clear how to describe how arbitrary pseudoequators intersect the entire cell decomposition.

On the other hand, this section will develop a notion of weak maps of complex matroids that behaves much like weak maps of oriented matroids.

Recall the partial order on  $(S^1 \cup \{0\})^E$ : we order  $S^1 \cup \{0\}$  to have unique minimum 0 and all other elements maximal, and then order  $(S^1 \cup \{0\})^E$  componentwise. Also recall [10, Proposition 7.3.11] that for matroids  $M_1$  and  $M_2$  on the same ground set  $E$ , there is a weak map from  $M_1$  to  $M_2$  if and only if every circuit in  $M_1$  contains a circuit of  $M_2$ .

**Definition 7.1.** Let  $\mathcal{M}_1$  and  $\mathcal{M}_2$  be complex matroids on the set  $E$  with circuit sets  $\mathcal{C}_1$  resp.  $\mathcal{C}_2$ . We say there is a *weak map* from  $\mathcal{M}_1$  to  $\mathcal{M}_2$ , and write  $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$ , if for every  $X \in \mathcal{C}_1$  there exists  $Y \in \mathcal{C}_2$  such that  $X \geq Y$ .

**Proposition 7.2.** Let  $\mathcal{M}_1$  and  $\mathcal{M}_2$ .

- (1) If  $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$  then  $\underline{\mathcal{M}}_1 \rightsquigarrow \underline{\mathcal{M}}_2$ .
- (2) If  $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$  then  $\text{rank}(\underline{\mathcal{M}}_1) \geq \text{rank}(\underline{\mathcal{M}}_2)$ .

*Proof.* The first statement is clear from the definition of weak maps, and the second statement follows from the first.  $\square$

**Proposition 7.3.** Let  $\mathcal{M}_1$  and  $\mathcal{M}_2$  be complex matroids of the same rank and on the same ground set. Let  $\varphi_1$  and  $\varphi_2$  be chirotopes for  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , and let  $\varphi_1^*$  and  $\varphi_2^*$  be their duals. The following are equivalent.

- (1)  $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$ .
- (2) For some  $c \in S^1$ ,  $\varphi_1 \geq c\varphi_2$ .
- (3) For some  $c \in S^1$ ,  $\varphi_1^* \geq c\varphi_2^*$ .

*Proof.* The equivalence of the latter two statements is clear from Theorem B.(1). Let  $M_1$  and  $M_2$  denote the underlying matroids of  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , respectively.

If  $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$ , by Lemma 7.2.1 we know that every basis of  $M_2$  is also a basis of  $M_1$ . In particular,

- (1) There exists  $B_0$  an ordered basis of both  $M_1$  and  $M_2$ . Without loss of generality assume  $\varphi_1(B_0) = \varphi_2(B_0)$ .
- (2) The basis graph of  $M_2$  is a subgraph of the basis graph of  $M_1$ .

For any ordered sequence  $S$ , let  $\underline{S}$  denote the set of elements of  $S$ .

We will induct on distance from  $\underline{B_0}$  in the basis graph to see that  $\varphi_1$  and  $\varphi_2$  coincide on every ordered basis  $B$  of  $M_2$ . If  $\underline{B} \neq \underline{B_0}$ , by basis exchange we can find  $B_1$  a basis closer to  $\underline{B_0}$  such that  $\underline{B} = \{e, x_2, \dots, x_r\}$  and  $B_1 = \{f, x_2, \dots, x_r\}$  for some  $e, f, x_2, \dots, x_r$ . Then by Theorem A, any signature  $X \in \mathcal{C}_{\varphi_1}$  on the basic circuit of  $f$  with respect to  $B$  satisfies

$$\frac{X(e)}{X(f)} = -\frac{\varphi_1(f, x_2, \dots, x_r)}{\varphi_1(e, x_2, \dots, x_r)} = -\frac{\varphi_1(f, x_2, \dots, x_r)}{\varphi_2(e, x_2, \dots, x_r)}.$$

But  $X \geq Y$  for some  $Y \in \mathcal{C}_{\varphi_2}$ , and  $Y$  is a circuit signature in  $\mathcal{M}_2$  on the basic circuit of  $f$  with respect to  $B$  in  $M_2$ . So

$$-\frac{\varphi_2(f, x_2, \dots, x_r)}{\varphi_2(e, x_2, \dots, x_r)} = \frac{Y(e)}{Y(f)} = \frac{X(e)}{X(f)}$$

and thus  $\varphi_2(f, x_2, \dots, x_r) = \varphi_1(f, x_2, \dots, x_r)$ .

Our proof that the second statement implies the first is adapted from [3, ] and is by induction on  $|E|$ .

Recall that a loop of a matroid is an element  $e$  such that  $\{e\}$  is a circuit, and a coloop is an element  $e$  such that  $\{e\}$  is a cocircuit. Loops and coloops of complex matroids are loops or coloops of the underlying matroid. Write  $\mathcal{C}_1 := \mathcal{C}_{\varphi_1}$  and  $\mathcal{C}_2 := \mathcal{C}_{\varphi_2}$  for the sets of circuits of  $\mathcal{M}_1$  and  $\mathcal{M}_2$  respectively. First note:

- If  $\mathcal{M}_1$  has a loop  $e_0$ , then  $e_0$  is also a loop of  $\mathcal{M}_2$ , and the induction hypothesis tells us that  $\mathcal{M}_1 \setminus e_0 \rightsquigarrow \mathcal{M}_2 \setminus e_0$ , hence  $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$ .
- If  $\mathcal{M}_1$  has no loops but  $\mathcal{M}_2$  has a coloop  $e_0$ , then  $\varphi_1/e_0 \rightsquigarrow \varphi_2/e_0$  and  $\{e_0\} \notin \mathcal{C}_1$ , so for every  $X \in \mathcal{C}_1$  there is a  $Y \in \mathcal{C}_2$  such that  $X \setminus e_0 Y \setminus e_0$ . Since  $e_0$  is a coloop, this implies  $Y(e_0) = 0$ , so  $X \rightsquigarrow Y$ .

So consider the case when  $\varphi_1 \geq \varphi_2$  and  $\mathcal{M}_2$  has no coloops. Let  $X \in \mathcal{C}_1$ . Let  $A$  be a maximal subset of  $\text{supp}(X)$  that's independent in  $\mathcal{M}_2$ , and extend  $A$  to a basis  $B$  of  $M_2$ . Let  $\tilde{\varphi}_1, \tilde{\varphi}_2$  be the restrictions of  $\varphi_1$  and  $\varphi_2$  to  $(\text{supp}(X) \cup B)^r$ . Then  $\tilde{\varphi}_1 \geq \tilde{\varphi}_2$ .

If  $A := E \setminus (\text{supp}(X) \cup B) \neq \emptyset$  then, since  $X \in \mathcal{C}_1 \setminus A$ , the induction hypothesis tells us that there is a  $Y \in \mathcal{C}_2 \setminus A \subseteq \mathcal{C}_2$  such that  $X \geq Y$ .

If  $\text{supp}(X) \cup B = E$ , we can see that  $B \subsetneq \text{supp}(X)$ . Otherwise, any  $b \in B \setminus \text{supp}(X)$  satisfies  $\text{rank}_{\mathcal{M}_2}(\text{supp}(X) \cup (B \setminus b)) < \text{rank}_{\mathcal{M}_2}(\text{supp}(X) \cup B)$ . Thus  $b$  is a coloop of  $\mathcal{M}_2$ , but  $\mathcal{M}_2$  has no coloops. Thus  $\text{supp}(X)$  is a circuit of  $\underline{M_2}$ .

An easy induction on the rank shows that whenever  $M_1$  and  $M_2$  are matroids of the same rank such that every circuit of  $M_1$  is a circuit of  $M_2$ , then  $M_1 = M_2$ .

We conclude  $\underline{M_1} = \underline{M_2}$ , and so  $\varphi_1 \geq \varphi_2$  implies  $\varphi_1 = \varphi_2$ . Thus  $\mathcal{M}_1 = \mathcal{M}_2$ .  $\square$

As with realizable oriented matroids, weak maps of realizable complex matroids arise from moving subspaces into more special position with respect to the coordinate hyperplanes. To make this precise, we give here the complex version of the same argument for oriented matroids (cf. [1]). Consider the complex Grassmannian

$G(r, \mathbb{C}^n)$ , the topological space of all rank  $r$  subspaces of  $\mathbb{C}^n$ . For any  $W \in G(r, \mathbb{C}^n)$ , let  $\mu(W)$  be the corresponding rank  $r$  complex matroid (the complex matroid with cocircuit set the set of elements of  $\{\text{ph}(w) : w \in W\}$  of minimal support).

Recall from Remark 2.4 that if  $W = \text{row}(M) \in G(r, \mathbb{C}^n)$  where  $M$  is a complex matrix, then the function  $\varphi_M : [n]^r \rightarrow S^1 \cup \{0\}$  taking each  $(e_1, \dots, e_r)$  to the sign of the minor of  $M$  with columns indexed by  $(e_1, \dots, e_r)$  is a phirotope. The following lemma follows easily by straightforward computation.

**Lemma 7.4.** *For every matrix  $M$  with complex coefficients,  $\varphi_M$  is a phirotope for  $\mu(W)$ .*

The following is our central result on the realizable interpretation of weak maps:

**Theorem 7.5.** *Let  $\mathcal{M}_1$  and  $\mathcal{M}_2$  be rank  $r$  complex matroids on the ground set  $[n]$ . If  $\mu^{-1}(\mathcal{M}_1) \cap \mu^{-1}(\mathcal{M}_2) \neq \emptyset$  then  $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$ .*

*Proof.* For any  $r$ -subset  $B$  of  $[n]$ , let  $U_B \subset G(r, \mathbb{C}^n)$  be the set of all row spaces of  $r \times n$  complex matrices such that the square submatrix with column set indexed by  $B$  is the identity. Then  $U_B \cong \mathbb{C}^{r \times (n-r)}$ , and the set of all  $U_B$  is an atlas on  $G(r, \mathbb{C}^n)$ . Thus  $U_B \cap \overline{\mu^{-1}(\mathcal{M}_1)} \cap \mu^{-1}(\mathcal{M}_2) \neq \emptyset$  for some  $B$ . Without loss of generality assume  $B = [r]$ . Thus we can (and will) identify  $U_B$  with the set of  $r \times n$  matrices  $M$  of the form  $(I|M')$ , where  $I$  is the  $r \times r$  identity matrix, and  $M'$  is a  $r \times (n-r)$  matrix.

Now consider the two maps

$$U_B \xrightarrow{d} \mathbb{C}^{[n]^r} \xrightarrow{\text{ph}} (S^1 \cup \{0\})^{[n]^r}$$

where  $d(M)(e_1, \dots, e_r)$  is the  $(e_1, \dots, e_r)$  minor of  $M$  (that is, the determinant of the submatrix of  $M$  with columns indexed by  $(e_1, \dots, e_r)$ , in that order). By Lemma 7.4, the composition of these two maps takes each  $W$  to the phirotope for  $\mu(W)$  with value 1 on  $(1, 2, \dots, r)$ .

The map  $d$  is continuous, hence the hypothesis gives

$$\overline{d(\mu^{-1}(\mathcal{M}_1))} \cap d(\mu^{-1}(\mathcal{M}_2)) \neq \emptyset.$$

But for each  $i$ ,  $d(\mu^{-1}(\mathcal{M}_i)) \subseteq \text{ph}^{-1}(\varphi_{\mathcal{M}_i})$ , so  $\overline{\text{ph}^{-1}(\varphi_{\mathcal{M}_1})} \cap \text{ph}^{-1}(\varphi_{\mathcal{M}_2}) \neq \emptyset$ . In particular, for every  $X \in [n]^r$ , we have

$$\overline{\text{ph}^{-1}(\varphi_{\mathcal{M}_1}(X))} \cap \text{ph}^{-1}(\varphi_{\mathcal{M}_2}(X)) \neq \emptyset.$$

Notice that, for every  $c \in S^1 \cup \{0\}$ ,  $\text{ph}^{-1}(c) = \mathbb{R}_+ c$ . Thus, for any  $c_1, c_2 \in S^1 \cup \{0\}$ ,

$$\overline{\text{ph}^{-1}(c_1)} \cap \text{ph}^{-1}(c_2) \neq \emptyset \text{ if and only if } c_1 \geq c_2.$$

So  $\varphi_{\mathcal{M}_1} \geq \varphi_{\mathcal{M}_2}$ , and by Proposition 7.3 this means  $\mathcal{M}_1 \rightsquigarrow \mathcal{M}_2$ .  $\square$

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