

	Matroids	Oriented Matroids	Complex Matroids
Basis axioms	$\mathcal{B} \subset 2^E$ is the set of bases of a matroid if <ul style="list-style-type: none"> • $\mathcal{B} \neq \emptyset$ • If $B_1, B_2 \in \mathcal{B}$, for all $e_1 \in B_1$ there is $e_2 \in B_2$ s.t. $(B_1 \setminus e_1) \cup e_2 \in \mathcal{B}$ and $(B_2 \setminus e_2) \cup e_1 \in \mathcal{B}$. 	$\chi : E^d \rightarrow \{-1, 0, +1\} \subset \mathbb{R}$ is a chirotope if <ul style="list-style-type: none"> • $\chi \neq 0$ and alternating; • For all $x_0, \dots, x_d, y_2, \dots, y_d \in E$, with $P := \{\chi(x_1, \dots, \hat{x}_j, \dots, x_d) \chi(x_j, y_2, \dots, y_d) \mid 0 \leq j \leq d\}$, $0 \in \text{relint conv } P$. 	$\varphi : E^n \rightarrow S^1 \cup \{0\} \subset \mathbb{C}$ is a phirotope if <ul style="list-style-type: none"> • $\varphi \neq 0$ and alternating; • For all $x_0, \dots, x_d, y_2, \dots, y_d \in E$, with $P := \{\chi(x_1, \dots, \hat{x}_j, \dots, x_d) \chi(x_j, y_2, \dots, y_d) \mid 0 \leq j \leq d\}$, $0 \in \text{relint conv } P$.
Duality	$\mathcal{B}^* := \{E \setminus B \mid B \in \mathcal{B}\}$ is the set of bases of the dual matroid.	A chirotope for the dual oriented matroid is given by $\chi^*(x_{\sigma(d+1)}, \dots, x_{\sigma(n)}) := 0$ if $\chi(x_{\sigma(0)}, \dots, x_{\sigma(d)}) = 0$, $\chi^*(x_{\sigma(d+1)}, \dots, x_{\sigma(n)}) := \text{sgn}(\sigma) \chi(x_{\sigma(0)}, \dots, x_{\sigma(d)})^{-1}$ else, for every total ordering x_1, \dots, x_n of E and all $\sigma \in S_n$.	A phirotope for the dual complex matroid is given by $\varphi^*(x_{\sigma(d+1)}, \dots, x_{\sigma(n)}) := 0$ if $\varphi(x_{\sigma(0)}, \dots, x_{\sigma(d)}) = 0$, $\varphi^*(x_{\sigma(d+1)}, \dots, x_{\sigma(n)}) := \text{sgn}(\sigma) \varphi(x_{\sigma(0)}, \dots, x_{\sigma(d)})^{-1}$ else, for every total ordering x_1, \dots, x_n of E and all $\sigma \in S_n$.
Circuit axioms	$\mathcal{C} \subset 2^E$ is the set of circuits of a matroid if the following hold. <ul style="list-style-type: none"> • $\emptyset \notin \mathcal{C}$ • If $X, Y \in \mathcal{C}$ and $e, f \in E$ are such that $e \in X \cap Y$ and $f \in (X \setminus Y) \cup (Y \setminus X)$ then there is $Z \in \mathcal{C}$ with $f \in \text{supp}(Z) \subseteq \text{supp}(X) \cup \text{supp}(Y) \setminus e$. 	$\mathcal{C} \subseteq \{-1, 0, +1\}^E$ is the set of signed circuits of an oriented matroid if the following hold. <ul style="list-style-type: none"> • $(0, \dots, 0) \notin \mathcal{C}$ • $\mathcal{C} = -\mathcal{C}$ • If $X, Y \in \mathcal{C}$ and $\text{supp}(X) \subseteq \text{supp}(Y)$, then $X = \pm Y$. • If $X, Y \in \mathcal{C}$ and $e, f \in E$ are such that $X(e) = -Y(e) \neq 0$ and $X(f) \neq -Y(f) \neq 0$, then there is $Z \in \mathcal{C}$ with $f \in \text{supp}(Z) \subseteq \text{supp}(X) \cup \text{supp}(Y) \setminus e$. ▲ Moreover, $Z(g) \in \{0, X(g), Y(g)\}$ for all $g \in E$. 	$\mathcal{C} \subseteq (S^1 \cup \{0\})^E$ is the set of phased circuits of a complex matroid if the following hold. <ul style="list-style-type: none"> • $(0, \dots, 0) \notin \mathcal{C}$ • For all $\mu \in S^1$, $\mathcal{C} = \mu \mathcal{C}$. • If $X, Y \in \mathcal{C}$ and $\text{supp}(X) \subseteq \text{supp}(Y)$, then $X = \mu Y$ for some $\mu \in S^1$. • If $X, Y \in \mathcal{C}$ and $e, f \in E$ are such that $X(e) = -Y(e) \neq 0$ and $X(f) \neq -Y(f) \neq 0$, then there is $Z \in \mathcal{C}$ with $f \in \text{supp}(Z) \subseteq \text{supp}(X) \cup \text{supp}(Y) \setminus e$. ▲ Moreover, if X, Y are comodular, $Z(g) \in \{0, [[X(g), Y(g)]]\}$ for all $g \in E$.
Orthog.	For $A, B \subseteq E$, define $A \perp B$ if $ A \cap B \neq 1$. Then, $\mathcal{C}^* = \min_{\subseteq} \mathcal{C}^\perp$.	For $X, Y \subseteq \{-1, 0, +1\}^E$, define $X \perp Y$ if $0 \in \text{relint conv}\{X(e)/Y(e) \mid e \in \text{supp}(X) \cap \text{supp}(Y)\}$. Then, $\mathcal{C}^* = \min_{\text{supp}} \mathcal{C}^\perp$.	For $X, Y \subseteq (S^1 \cup \{0\})^E$, define $X \perp Y$ if $0 \in \text{relint conv}\{X(e)/Y(e) \mid e \in \text{supp}(X) \cap \text{supp}(Y)\}$. Then, $\mathcal{C}^* = \min_{\text{supp}} \mathcal{C}^\perp$.
Dual pairs	Two sets $\mathcal{C}, \mathcal{D} \subseteq 2^E$ are the sets of circuits of a pair of dual matroids if and only if <ul style="list-style-type: none"> • \mathcal{C} and \mathcal{D} satisfy the circuit axioms for matroids; • $\mathcal{C} \perp \mathcal{D}$. 	Two sets $\mathcal{C}, \mathcal{D} \subseteq \{-1, 0, +\}^E$ are the sets of signed circuits of a pair of dual oriented matroids if and only if <ul style="list-style-type: none"> • $\{\text{supp}(X) \mid X \in \mathcal{C}\}$ and $\{\text{supp}(Y) \mid Y \in \mathcal{D}\}$ satisfy the circuit axioms for matroids; • $\mathcal{C} \perp \mathcal{D}$; • $\mathcal{C} = -\mathcal{C}$, $\mathcal{D} = -\mathcal{D}$; • If $X, Y \in \mathcal{C}$ and $\text{supp}(X) = \text{supp}(Y)$, then $X = \pm Y$ (and if $X, Y \in \mathcal{D}$ with $\text{supp}(X) = \text{supp}(Y)$, then $X = \pm Y$).	Two sets $\mathcal{C}, \mathcal{D} \subseteq (S^1 \cup \{0\})^E$ are the sets of signed circuits of a pair of dual oriented matroids if and only if <ul style="list-style-type: none"> • $\{\text{supp}(X) \mid X \in \mathcal{C}\}$ and $\{\text{supp}(Y) \mid Y \in \mathcal{D}\}$ satisfy the circuit axioms for matroids; • $\mathcal{C} \perp \mathcal{D}$; • For all $\mu \in S^1$, $\mathcal{C} = \mu \mathcal{C}$, $\mathcal{D} = \mu \mathcal{D}$; • If $X, Y \in \mathcal{C}$ and $\text{supp}(X) = \text{supp}(Y)$, then $X = \mu Y$ for some $\mu \in S^1$ (and if $X, Y \in \mathcal{D}$ with $\text{supp}(X) = \text{supp}(Y)$, then $X = \mu Y$ for some $\mu \in S^1$).

COMPLEX MATROIDS

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Handout

(Table of the axiom systems)