Math 323
Midterm 3

December 7., 2009

Solutions

1. (20 pts.) Consider the following two vector fields.

- $\mathbf{F}_1 = xy\mathbf{i} - x^2\mathbf{j}$
- $\mathbf{F}_2 = (4x^3y - 4xy^3)\mathbf{i} + (x^4 - 6x^2y^2 + 4y^3)\mathbf{j}$

(a) (10 pts.) One of the above vector fields is conservative. Determine which one it is and find a potential function for it.

(b) (10 pts.) For both above vector fields compute the integral along the circle $x^2+y^2 = 1$ oriented counterclockwise.

SOLUTION.(a) The vector field $\mathbf{F}_1$ is not conservative since

$$\frac{\partial}{\partial x}(-x^2) = -2x \neq x = \frac{\partial}{\partial y}xy$$

The vector field $\mathbf{F}_2$ is conservative since

$$\frac{\partial}{\partial x}\big(x^4 - 6x^2y^2 + 4y^3\big) = 4x^3 - 12xy^2 = \frac{\partial}{\partial y}\big(4x^3y - 4xy^3\big).$$

A scalar potential for $\mathbf{F}$ is is

$$f(x, y, z) = x^4y - 2x^2y^3 + y^4.$$

(b) Let C denote the given circle. We have immediately $\oint_C \mathbf{F}_2 \bullet d\mathbf{r} = 0$ because $\mathbf{F}_2$ is conservative.

For $\mathbf{F}_1$ write $\mathbf{F}_1 = \langle P, Q\rangle$ and let D be the unit disk, so that $C = \partial D$. We know from part (a) that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2x + x = -x$. Using Green's theorem we have then

$$\oint_C \mathbf{F}_1 \bullet d\mathbf{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \iint_D -x dA$$

which we solve using polar coordinates:

$$\iint_D -x dA = \int_0^1 \int_0^{2\pi} -r\cos(\phi) r dr d\phi = \int_0^1 -r^2 \big|\sin(\phi)\big|_0^{2\pi} dr = 0$$

2. (15 pts.) Find the area of the part of the hyperbolic paraboloid $z = y^2 - x^2$ between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

SOLUTION.
The range of the x and y coordinates is an annulus centeres at the origin, with inner radius 1 and outer radius 2. It is therefore appropriate to parametrize the region using polar coordinates, as follows:

$$\mathbf{r}(u,v) = \langle u\cos v, u\sin v, u^2\sin^2 v - u^2\cos^2 v\rangle; \quad 1 \leq u \leq 2,\ 0 \leq v\ leq2\pi.$$

We then have:

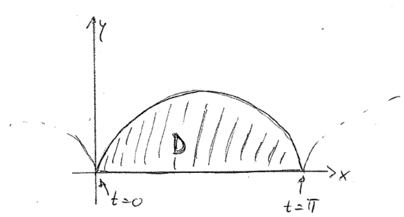
$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial u} &= \langle \cos v, \sin v, 2u\sin^2 v - 2u\cos^2 v\rangle \\
\frac{\partial \mathbf{r}}{\partial v} &= \langle -u\sin v, u\cos v, 4u^2 \sin v\cos v\rangle \\
\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| &= |\langle 2u^2\cos v, -2u^2\sin v, u\rangle| = u\sqrt{4u^2-1}
\end{aligned}$$

Therefore the requested area is computed by

$$\int_0^{2\pi}\int_1^2 u\sqrt{4u^2-1}dudv = \int_0^{2\pi} \left|\frac{1}{12}(4u^2-1)^{\frac{3}{2}}\right|_1^2 dv = \int_0^{2\pi} \frac{17^{\frac{3}{2}} - 5^{\frac{3}{2}}}{12} dv = \frac{17^{\frac{3}{2}} - 5^{\frac{3}{2}}}{6}\pi$$

3. (15 pts.) Find the area under the arch of cycloid

$$x(t) = t - \sin t \quad y(t) = 1 - \cos t \quad 0 \leq t \leq 2\pi$$



Hint: you can use Green's Theorem with the vector field $\mathbf{F}(x,y) = \langle -y, 0\rangle$.

SOLUTION.
Let D be the region of the plane below the given arch of the cycloid.

Writing $\mathbf{F} = \langle P, Q\rangle$ we have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - (-1) = 1$. Thus, with Green's Theorem, the requested area is

$$\iint_D dA = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \oint_{\partial D} \mathbf{F} \bullet d\mathbf{r}$$

First method

The boundary of D with counterclockwise orientation decomposes into the segment

$$\mathbf{r}_1(s) = \langle s, 0\rangle, \quad 0 \le s \le 2\pi$$

and the arc

$$\mathbf{r}_2(t) = \langle x(t), y(t)\rangle, \quad 2\pi \ge t \ge 0.$$

We have

$$\mathbf{r}_1'(s) = \langle 1, 0\rangle, \quad \mathbf{r}_2'(t) = \langle x'(t), y'(t)\rangle = \langle 1 - \cos t, \sin t\rangle$$

and therefore we compute

$$\begin{aligned}
\oint_{\partial D} \mathbf{F} \bullet d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}_1)(s) \bullet \mathbf{r}_1'(s) ds + \int_{2\pi}^0 \mathbf{F}(\mathbf{r}_2(t)) \bullet \mathbf{r}_2(t) dt \\
&= \int_0^{2\pi} 0 ds + \int_{2\pi}^0 -(1-\cos t)(1-\cos t) dt = \int_{2\pi}^0 -1 + 2\cos t - \cos^2 t\, dt \\
&= \int_{2\pi}^0 2\cos t - \frac{1}{2}\cos 2t - \frac{3}{2} dt \\
&= |2\sin t - \frac{1}{4}\sin 2t - \frac{3t}{2}|_{2\pi}^0 = 3\pi
\end{aligned}$$

4. (15 pts.) Evaluate the surface integral $\iint_S \mathbf{F} \bullet d\mathbf{S}$ for the vector field

$$\mathbf{F}(x, y, z) = \langle xze^y, -xze^y, z\rangle$$

and where S is the part of the plane $x + y + z = 1$ in the first octant (i.e., $x \ge 0$, $y \ge 0$, $z \ge 0$), with downward orientation.

Solution.
A parametrization for S is given by

$$\mathbf{r}(u, v) = \langle u, v, 1 - u - v\rangle, \quad 0 \le u \le 1,\ 0 \le v \le 1 - u$$

We compute $\frac{\partial \mathbf{r}}{\partial u} = \langle 1, 0, -1\rangle$ and $\frac{\partial \mathbf{r}}{\partial v} = \langle 0, 1, -1\rangle$ and thus

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \langle -1, -1, -1\rangle.$$

Other than already having the correct orientation, this tells us that since the first two components of $\mathbf{F}$ are opposite to each other they will canvel in the scalar product $\mathbf{F} \bullet d\mathbf{S}$. Thus we compute

$$\begin{aligned}
\iint_S \mathbf{F} \bullet d\mathbf{S} &= \int_0^1 \int_0^{1-u} \mathbf{F}(\mathbf{r}(u, v)) \bullet \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) dv\, du \\
&= \int_0^1 \int_0^{1-u} (1 - u - v) \cdot (-1)\, dv\, du \\
&= \int_0^1 \left|(u-1)v + \frac{v^2}{2}\right|_0^{1-u} du = \int_0^1 -(1-u)^2 + \frac{(1-u)^2}{2} du \\
&= -\frac{1}{2}\int_0^1 (1-u)^2\, du = -\frac{1}{2}\left| -\frac{(1-u)^3}{3}\right|_0^1 = -\frac{1}{2}\frac{1}{3} = -\frac{1}{6}
\end{aligned}$$

5. (15 pts.) Use Green's Theorem to evaluate

$$\oint_C (2x - y + 4)\,dx + (5y + 3x - 6)\,dy,$$

where C is the triangle in the xy-plane with vertices $(0,0)$, $(3,0)$, $(0,2)$ traversed in a counterclockwise direction.

SOLUTION.

$$\begin{aligned}\oint_C (2x - y + 4)\,dx + (5y + 3x - 6)\,dy &= \iint_R \left(\frac{\partial}{\partial x}(5y + 3x - 6) - \frac{\partial}{\partial y}(2x - y + 4)\right) dA \\ &= \iint_R (3 + 1)\,dA \\ &= 4\left(\frac{1}{2}(3)(2)\right) \\ &= 12\end{aligned}$$

6. (15 pts.) Use Stoke's theorem to compute $\oint_C \mathbf{F} \bullet \mathbf{T}\,ds$ where $\mathbf{F}(x,y,z) = x^2 z\mathbf{i} + x\mathbf{j} + \frac{y^3}{3}\mathbf{k}$ and where C goes once counterclockwise around the intersection of the plane $x+y+2z = 20$ and the cylinder $x^2 + y^2 = 1$.

SOLUTION.
First we compute $\operatorname{curl} \mathbf{F}(x,y,z) = \langle y^2, x^2, 1\rangle$ and then we apply Stoke's Theorem, obtaining

$$\oint_C \mathbf{F} \bullet \mathbf{T}\,ds = \iint_S \operatorname{curl} F\,d\mathbf{S}$$

where we choose S to be the ellipse bounded by C inside the plane $x + y + 2z = 20$. A parametrization of S is

$$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 10 - \frac{r}{2}(\cos\theta + \sin\theta)\rangle,\ 0 \le \theta < 2\pi,\ - \le r \le 1.$$

The partial derivatives of $\mathbf{r}$ are

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial r} &= \langle \cos\theta, \sin\theta, -\frac{1}{2}(\cos\theta + \sin\theta)\rangle \\ \frac{\partial \mathbf{r}}{\partial \theta} &= \langle -r\sin\theta, r\cos\theta, \frac{r}{2}(\sin\theta - \cos\theta)\rangle\end{aligned}$$

and thus

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \langle \frac{r}{2}, \frac{r}{2}, r\rangle.$$

We can then compute

$$\iint_S \operatorname{curl} F \, d\mathbf{S} = \int_0^{2\pi} \int_0^1 \langle r^2 \sin^2\theta, r^2\cos^2\theta, 1\rangle \bullet \langle \frac{r}{2}, \frac{r}{2}, r\rangle \, dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 \frac{r^3}{2} + r \, dr d\theta$$

$$\int_0^{2\pi} \left| \frac{r^4}{8} + \frac{r^2}{2} \right|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{5}{8} \, d\theta = \frac{5\pi}{4}.$$