ON MODULAR ELIMINATION IN MATROIDS AND ORIENTED MATROIDS

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Abstract. We introduce a new axiomatization of oriented matroids that requires the elimination property only among modular pairs of circuits. This new point of view leads also to a similar strenghtening of the circuit axioms for oriented matroids.

1. Introduction

Matroid Theory was initiated by H. Whitney in [5] as a combinatorial description of linear dependency. Let E denote a finite set of vectors in vector spaces over an arbitrary field $\mathbb{K}$ and consider the collection $\mathcal{C}$ of all (inclusion-) minimal linearly dependent subsets of E. Given any pair of minimal dependent sets $C_1, C_2 \in \mathcal{C}$ that share a common vector e it is easy to form a linear dependency among vectors of C_1 and C_2 not involving e. This new dependent set will then contain a member of $\mathcal{C}$. Let us formalize this so-called "elimination property".

Given a finite set E, a collection $\mathcal{C}$ of subsets of E and $C_1, C_2 \in \mathcal{C}$,

$$\mathscr{E}(C_1, C_2, \mathcal{C}): \text{ for all } e \in C_1 \cap C_2 \text{ there is } C_3 \in \mathcal{C} \text{ with } C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}.$$

Definition 1. Let E be a finite set. A collection $\mathcal{C}$ of incomparable subsets of E is the *set of circuits of a matroid* on the ground set E if $\mathscr{E}(C_1, C_2, \mathcal{C})$ holds for all $C_1, C_2 \in \mathcal{C}$.

This definition truly generalizes the starting situation: there are matroids that do not arise from linear dependencies among vectors. Moreover, matroids bear a significant interest in many fields of mathematics, as they appear to capture some abstract structure that is present in seemingly unrelated contexts. For an up-to-date introduction to this rich and lively theory we point to Oxley's textbook [3].

One important feature of matroid theory is the availability of many axiomatizations, different in spirit but equivalent in substance (or, as one sometimes says in matroid theoretic terms, "cryptomorphic"). For example, notice that the set of all unions of circuits, partially ordered by inclusion, is a geometric lattice (see [3, Chapter 1.7]). [1] Indeed more is true: every geometric lattice defines a matroid. So one alternative axiomatization is "a matroid on the ground set E is a geometric lattice with set of coatoms E". Much of the richness of the theory comes from the ability to transfer concepts from one approach to another - for instance, given the set $\mathcal{C}$ of circuits of a matroid, let us call a pair of circuits $A, B \in \mathcal{C}$ a *modular pair* if $A \vee B$ has rank 2 in the associated geometric lattice.

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[1] For ease of exposition we take a somewhat onurthodox point of view: to a matroid we associate the order dual of the *lattice of flats* usually associated to a matroid.

In the special case where $\mathbb{K} = \mathbb{R}$, the data given by the collection of all linearly dependent sets can be enriched by taking into account the *sign* ($+, -$ or 0) of the coefficient of every vector in any given linear dependency. This gives rise to the theory of oriented matroids which also features different axiomatizations and has found applications in a wide variety of fields. A general reference for an introduction to oriented matroids is [2]. The basic idea is to consider not only subsets of E, but *signed subsets*, i.e., functions $X : E \to \{-1, 0, +1\}$ representing the "signature" of the set given by the support of X. Notice that the set of signs has a natural $\mathbb{Z}_2$ action (switching sign). The ordering of signs defined as $x < y$ if and only if $|x| < |y|$ for $x, y \in \{+1, 0, -1\}$ can be extended componentwise to a partial ordering of $\{-1, 0, +1\}^E$. We define comparability between signed sets according to comparability with respect to this partial ordering. In order to give a definition that exhibits the similarity with the previous one for matroids, let us define a "oriented elimination" property for any $\mathfrak{C} \subseteq \{\pm 1, 0\}^E$ and any $X, Y \in \mathfrak{C}$.

$$\begin{aligned} \mathscr{O}\mathscr{E}(X, Y, \mathfrak{C}) : \ & \text{for all } e, f \text{ with } X(e) = -Y(e) \neq 0, X(f) \neq Y(f) \text{ there is } Z \in \mathfrak{C} \\ & \text{with } Z(e) = 0, Z(f) \neq 0, \text{ and } Z(g) \in \{0, X(g), Y(g)\} \text{ for all } g \in E. \end{aligned}$$

Then, one definition of oriented matroids is the following.

Definition 2 (see [2]). A $\mathbb{Z}_2$-invariant collection $\mathfrak{C}$ of incomparable subset of $\{-1, 0, +1\}^E$ is the set of signed circuits of an oriented matroid if

(1) the collection $\mathcal{C} := \{\mathrm{supp}(X) \mid X \in \mathfrak{C}\}$ is the set of circuits of a matroid on E, and
(2) $\mathscr{O}\mathscr{E}(X, Y, \mathfrak{C})$ holds for all $X, Y \in \mathfrak{C}$ such that $\mathrm{supp}(X), \mathrm{supp}(Y)$ is a modular pair in $\mathcal{C}$.

The previous definition is usually presented as an interesting "curiosum", whereas the standard definition requires $\mathscr{O}\mathscr{E}(X, Y, \mathfrak{C})$ to hold for every pair of elements $X, Y \in \mathfrak{C}$. However, recent work on cryptomorphic axiom systems for complex matroids by Laura Anderson and the author [1] strongly hints to the fact that modular pairs of circuits truly encode all of the information. For instance, it turns out that in the setting of complex matroids the *only* case of circuit elimination where one can in general control the (complex) signs of the involved circuits is elimination among modular pairs.

We show that indeed modular circuits suffice to characterize any matroid, whether realizable or not. As a corollary, we can remove condition (1) from Definition 2.

2. Main result

For notation and basic facts about posets and lattices we refer to [4, Chapter 3]. Here let us only recall that a *chain* J in a poset $(P, \leq)$ is any totally ordered subset of P; the *length* of the chain J is then $\ell(J) := |J| - 1$. Given $p \in P$ we write $P_{\geq p} = \{p' \in P \mid p' \geq p\}$ and $P_{\leq p} = \{p' \in P \mid p' \leq p\}$. The *length* of P is $\ell(P) := \max\{\ell(J) \mid J \text{ a chain of } P\}$, and for $p \in P$ write $\ell(p) := \ell(P_{\leq p})$.

If for any $p, q \in P$ the poset $P_{\geq p} \cap P_{\geq q}$ has a unique minimal element, this element is denoted $p \vee q$ and called the *meet* of p and q. Analogously we call $p \wedge q$, or *join* of p and q, the unique maximal element of $P_{\leq p} \cap P_{\leq q}$, if it exists. The poset P is called a *lattice* if meet and join are defined for every pair of elements of P. In particular, every lattice has a unique minimal element, called $\widehat{0}$, and a unique maximal element, called $\widehat{1}$.

Definition 3. Let L be a finite lattice. The *atoms* of L are the elements that cover $\widehat{0}$ in L. The lattice L is called *atomic* if every $x \in L$ is $x = \bigvee A$ for some set A of atoms of L. Two atoms a, b of L form a *modular pair* if $\ell(L_{\leq a \vee b}) = 2$.

Given any family $\mathcal{S}$ of subsets of the finite set E, consider the set

$$U(\mathcal{S}) := \{\bigcup \mathcal{T} \mid \mathcal{T} \subseteq \mathcal{S}\}$$

partially ordered by inclusion - so, for $A, B \in U(\mathcal{S})$, $A \leq B$ if $A \subseteq B$.

If the members of $\mathcal{S}$ are incomparable, then $U(\mathcal{S})$ is an atomic lattice. By slight abuse of terminology we will say that two members of $\mathcal{S}$ are a *modular pair* if they are a modular pair in $U(\mathcal{S})$.

Theorem 1. *A collection $\mathcal{C}$ of incomparable subsets of a finite ground set E is the set of circuits of a matroid on E if $\mathscr{E}(A, B, \mathcal{C})$ for all modular pairs $A, B \in \mathcal{C}$.*

Proof. We prove that, under the assumption of the theorem, $\mathscr{E}(A, B, \mathcal{C})$ for all $A, B \in \mathcal{C}$. Take $A \neq B \in \mathcal{C}$, $e \in (A \cup B)$ and let $Z := A \cup B$. We want to show that a $C \in \mathcal{C}$ exists with $e \in C \subseteq Z$. We will proceed by induction on $\ell(Z)$.

If $\ell(Z) = 2$ then A, B is a modular pair and we are done. Suppose now $\ell(Z) = n > 2$ and let J be a chain of maximal cardinality in $U(\mathcal{C})_{\leq Z}$. The chain J contains exactly one element $A' \in \mathcal{C}$ and at least an element Y with $A' \lneq Y \lneq Z$. If $e \in A'$ we are done with $C := A'$. Else, since $U(\mathcal{C})$ is atomic, there is $B' \in \mathcal{C}$ with $A' \vee B' = Y$. Again, if $e \in B'$ then $C := B'$ does it; otherwise we apply the inductive hypothesis to the pair A', B' (because $Y < Z$ implies $\ell(Y) < \ell(Z)$), obtaining C as desired. □

As a straightforward consequence we have a corresponding strenghtening of the axiomatics of oriented matroids.

Corollary 1. *A $\mathbb{Z}_2$-invariant collection $\mathfrak{C}$ of incomparable subset of $\{-1, 0, +1\}^E$ is the set of signed circuits of an oriented matroid if $\mathscr{O}\mathscr{E}(X, Y, \mathfrak{C})$ holds for all $X, Y \in \mathfrak{C}$ such that $\mathrm{supp}(X), \mathrm{supp}(Y)$ is a modular pair in $\mathcal{C}$.*

Proof. By Theorem 1 we know that under the hypotheses of the theorem $\mathcal{C} := \{\mathrm{supp}(X) \mid X \in \mathfrak{C}\}$ is the set of circuits of a matroid (indeed, $\mathscr{O}\mathscr{E}(X, Y, \mathfrak{C})$ implies $\mathscr{E}(\underline{X}, \underline{Y}, \mathcal{C})$).

□

References

[1] L. Anderson and E. Delucchi. Complex matroids. In preparation.

[2] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1999.

[3] J. G. Oxley. *Matroid theory*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1992.

[4] R. P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[5] H. Whitney. On the Abstract Properties of Linear Dependence. *Amer. J. Math.*, 57(3):509–533, 1935.

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