Math 323 Section 1, Midterm 3

December 3., 2008, 8:00-9:30

Solutions

1. (14 pts.) Set up and evaluate the iterated integral

$$\iint_R xy^2 \, dA$$

When R is the region enclosed by $y = 1$, $y = 2$, $x = 0$, and $y = x$.

SOLUTION.

$$\begin{aligned}
\iint_R xy^2 \, dA &= \int_1^2 \int_0^y xy^2 \, dx \, dy \\
&= \int_1^2 \left[\frac{1}{2} x^2 y^2 \right]_0^y dy \\
&= \int_1^2 \frac{1}{2} y^4 \, dy \\
&= \left[\frac{1}{10} y^5 \right]_1^2 \\
&= \frac{1}{10}(2^5 - 1^5) \\
&= \frac{31}{10}
\end{aligned}$$

2. (14 pts.) Evaluate the integral

$$\iint_R \frac{1}{(x^2+y^2+1)^2}\,dA,$$

where R is the part of the disk $x^2+y^2 \le 4$ with $y \ge 0$ and $y \le x$.

Solution.

$$\begin{aligned}
\iint_R \frac{1}{1+x^2+y^2}\,dA &= \int_0^{\pi/4}\int_0^2 \frac{r}{(1+r^2)^2}\,dr\,d\theta \\
&= \int_0^{\pi/4} \left[-\frac{1}{2}\frac{1}{(1+r^2)}\right]_0^2 d\theta \\
&= \int_0^{\pi/4} (-\frac{1}{2}\frac{1}{5}+\frac{1}{2})\,d\theta \\
&= \frac{\pi}{4}\cdot\frac{1}{2}\cdot\frac{4}{5} = \frac{\pi}{10}
\end{aligned}$$

3. (12 pts.) One of the following two vector fields is conservative. Determine which one it is and find a potential function for it.

(a) $\mathbf{F}_1 = xy\,\mathbf{i} - x^2\,\mathbf{j}$

(b) $\mathbf{F}_2 = (4x^3y - 4xy^3)\,\mathbf{i} + (x^4 - 6x^2y^2 + 4y^3)\,\mathbf{j}$

SOLUTION.(a) The vector field $\mathbf{F})_1$ is not conservative since

$$\frac{\partial}{\partial x}(-x^2) = -2x \neq x = \frac{\partial}{\partial y}xy$$

(b) The vector field $\mathbf{F}_2$ is conservative since

$$\frac{\partial}{\partial x}\big(x^4 - 6x^2y^2 + 4y^3\big) = 4x^3 - 12xy^2 = \frac{\partial}{\partial y}\big(4x^3y - 4xy^3\big).$$

A scalar potential for $\mathbf{F}$ is is

$$f(x, y, z) = x^4y - 2x^2y^3 + y^4.$$

4. (14 pts.) Find the volume of the solid tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + z = 4$.

SOLUTION.

$$\begin{aligned}
V &= \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\,dy\,dx \\
&= \int_0^2 \int_0^{4-2x} (4 - 2x - y)\,dy\,dx \\
&= \int_0^2 \left[4y - 2xy - \frac{1}{2}y^2\right]_0^{4-2x} dx \\
&= \int_0^2 \left((4 - 2x)^2 - \frac{1}{2}(4 - 2x)^2\right) dx \\
&= \int_0^2 \frac{1}{2}(4 - 2x)^2\,dx \\
&= \int_0^2 2(2 - x)^2\,dx \\
&= \left[-\frac{2}{3}(2 - x)^3\right]_0^2 \\
&= \frac{16}{3}
\end{aligned}$$

5. (17 pts.) Use Green's Theorem to evaluate

$$\oint_C (2x - y + 4)\,dx + (5y + 3x - 6)\,dy,$$

where C is the triangle in the xy-plane with vertices $(0,0)$, $(3,0)$, $(0,2)$ traversed in a counterclockwise direction.

SOLUTION.

$$\begin{aligned}\oint_C (2x - y + 4)\,dx + (5y + 3x - 6)\,dy &= \iint_R \left(\frac{\partial}{\partial x}(5y + 3x - 6) - \frac{\partial}{\partial y}(2x - y + 4)\right) dA \\ &= \iint_R (3 + 1)\,dA \\ &= 4\left(\frac{1}{2}(3)(2)\right) \\ &= 12\end{aligned}$$

6. (11 pts.) Let $f(x, y, z) = x^2z^3 + 6xy - y^2z$. Compute $\int_C (\nabla f) \cdot d\,\mathbf{r}$, where C is the line segment from $(1, -1, 1)$ to $(2, 1, -1)$.

SOLUTION.

$$\int_C (\nabla f) \cdot d\,\mathbf{r} = f(2, 1, -1) - f(1, -1, 1) = (-4 + 12 + 1) - (1 - 6 - 1) = 9 + 6 = 15$$

7. (18 pts.) The coordinate transformation $x = u^2v$, $y = uv^2$ maps the unit square $[0, 1] \times [0, 1]$ in the uv-plane onto the region R in the xy-plane bounded by the parabolas $x = y^2$ and $y = x^2$. Use this coordinate transformation to calculae the integral

$$\iint_R x^{1/3}y^{1/3}\, dA.$$

Solution.
First plug the transformation into the integrand to get

$$x^{1/3}y^{1/3} = u^{2/3}v^{1/3}u^{1/3}v^{2/3} = uv.$$

Next compute the value of the Jacobian of this coordinate transformation:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2uv & u^2 \\ v^2 & 2uv \end{vmatrix} = 4u^2v^2 - u^2v^2 = 3u^2v^2.$$

Combining these and integrating over the square $[0, 1] \times [0, 1]$ in the uv-plane gives

$$\begin{aligned} \iint_R x^{1/3}y^{1/3}\, dA &= \int_0^1 \int_0^1 (uv)(3u^2v^2\, du\, dv) \\ &= \int_0^1 \int_0^1 3u^3v^3\, du\, dv \\ &= 3\int_0^1 u^3\, du \int_0^1 v^3\, dv \\ &= 3 \cdot \frac{1}{4} \cdot \frac{1}{4} \\ &= \frac{3}{16} \end{aligned}$$