F-VECTORS OF SUBDIVIDED SIMPLICIAL COMPLEXES

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ABSTRACT. In [2], it was shown that for any simplicial n-dimensional complex X, the f-vectors of successive barycentric subdivisions of X have roots which converge to fixed values depending only on the dimension of X. We give an alternate and intuitive proof of this fact here, and compute explicit formulas for the values of these roots. In the process, we observe and prove an interesting symmetry of these roots about the real number -2. This symmetry can be seen via a nice realization of barycentric subdivision as a simple map on a formal power series in two variables.

1. Introduction

Throughout this paper, we let X be an arbitrary finite simplicial complex of dimension $d-1$, and we assume that all vectors and matrices will be indexed by rows and columns starting at 0. We are interested in roots of the *f-polynomial* of X, defined as follows. Let f_i^X denote the number of i-dimensional faces of X. We declare that $f_{-1}^X = 1$, where the (-1)-dimensional face corresponds to the empty face, $\emptyset$. The *face vector*, or *f-vector* of X is the vector

$$f^X := (f_{-1}^X, f_0^X, \dots, f_{d-1}^X).$$

Let $\underline{t}$ denote the column vector of powers of t, $(t^d, t^{d-1}, \dots t^0)^T$. The *$f$-polynomial* $f^X(t)$ encodes the f-vector as a polynomial:

$$f^X(t) := \sum_{j=0}^{d} f_{j-1}^X t^{d-j} = f^X \underline{t}$$

Much work has been devoted to the study of f-vectors of simplicial complexes, their close relatives, the g- and h-vectors, and the associated polynomials. As it turns out, the entries of these objects encode many combinatorial and algebraic aspects of the complex to which they are associated (see [1, 3, 9] for background and further references).

In this paper, we focus on a recent result of Brenti and Welker [2], which can initially appear surprising. Let X' denote the barycentric subdivision of X, and more generally let $X^{(n)}$ denote the n^{th} barycentric subdivision of X.

Theorem 1.1. [2] *Let X be a d-dimensional simplicial complex. Then, as n grows, the roots of $f^{X^{(n)}}$ converge to $d-1$ negative real numbers which depend only on d, not on X.*

In this paper, we attempt to provide some geometric intuition and motivation for why this result should hold. We offer an alternate proof of this theorem based on these geometric observations. In the process, we show how to compute the $d-1$ real values for each d. Our first main theorem is:

Theorem A. *Let X be a d-dimensional simplicial complex. Then the roots of $f^{X^{(n)}}$ converge to $d-1$ roots depending only on d, not on X. The roots of $f^{X^{(n)}}$ converge to the roots of a polynomial $p_d(t)$, whose coefficients are listed in the last row of the inverse of a particular matrix, P_d.*

The entries of the matrix P_d, and of its inverse P_d^{-1}, are computed in Section 6. Our calculations allow us to compute the 'limit roots' thus obtained. In the examples, one can observe that these 'limit roots' are symmetrically distributed about the point -2, with respect to the Möbius transformation $x \mapsto \frac{-x}{x+1}$. Our second main theorem proves this symmetry:

Theorem B. *For any dimension d, the $d-1$ 'limit roots' are invariant under the map $x \mapsto \frac{-x}{x+1}$.*

In fact, more may be said. The existence of a 'limit polynomial' and the symmetry result holds for an arbitrary subdivision method, as we show in Theorem 5.5 .

For barycentric subdivision, this symmetry can be seen through a beautiful theorem. Barycentric subdivision, seen as a map on f-polynomials, induces a function $b : \mathbb{Z}[t] \to \mathbb{Z}[t]$, as in Section 4. We list the values of b on monomials as coefficients in the formal power series in the variable x over $\mathbb{Z}[t]$, by defining $B : \mathbb{Z}[t][[x]] \to \mathbb{Z}[t][[x]]$ to be given by $B(\sum_{k\geq 0} g_k(t)x^k) = \sum_{k\geq 0} b\big(g_k(t)\big)x^k$.

Theorem C. *In $\mathbb{Z}[t][[x]]$, barycentric subdivision satisfies the identity*

$$B(e^{tx}) = \frac{1}{1-(e^x-1)t}.$$

This paper is organized as follows. In Section 2, we discuss the geometric intuition and motivation behind Theorem 1.1. In Section 3, we prove Theorem A. In Section 4, we prove the symmetry stated in Theorem B, and prove Theorem C. In Section 5, we extend the symmetry to arbitrary subdivision methods. We end with Section 6, where we compute the entries in the coefficient matrix P_d found in Theorem A and all limit polynomials and roots up to the value $d = 10$.

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2. Geometric Motivation

Theorem 1.1 of Brenti and Welker [2] may be surprising at first: there is no dependence on the initial complex X, only on the dimension d. However, geometrically this makes perfect sense. Barycentrically subdividing a simplicial complex X over and over again causes the resulting complex $X^{(n)}$ to have far more cells than the original X. Because higher-dimensional cells contribute more new cells (in every dimension) upon subdividing than lower-dimensional ones, the top-dimensional cells begin to dominate in their 'number of contributions' to subdivisions. For example, think of geometric realizations so that $X^{(n)}$ is a subset of X. Then a randomly chosen cell of $X^{(n)}$ should, with higher and higher probability as n increases, be contained in the interior of a top-dimensional cell of X, as top-dimensional cells contribute far more cells to $X^{(n)}$ than other cells.

Each of the f^X_{d-1} top-dimensional cells of X contribute the same amount of cells to $X^{(n)}$. Since these cells eventually dominate contributions from smaller-dimensional cells, the f-polynomial for $X^{(n)}$ can approximated by f^X_{d-1} times the f-polynomial for subdividing a single top-dimensional cell n times, $\sigma_d^{(n)}$. Since the roots of a polynomial are unaffected by multiplication by constants, the roots of $f^{X^{(n)}}$ converge to the roots of $f^{\sigma_d^{(n)}}$ as n increases.

By definition, the coefficients of $f^{\sigma_d^{(n)}}$ record the number of cells of each dimension occurring in $\sigma_d^{(n)}$. The number of cells in each dimension is bounded by a constant times the number of top-dimensional cells. Thus, if we normalize $f^{\sigma_d^{(n)}}$ by dividing by the number of top-dimensional cells, we have coefficients which, for each k, record the density of k-cells relative to the number of top-dimensional cells. As this density is positive but strictly decreases upon subdividing, there is a limiting value for the coefficient. Thus, there is a limiting polynomial, with well-defined roots.

We now formalize this intuition.

3. f-POLYNOMIALS OF BARYCENTRIC SUBDIVISIONS

3.1. **Barycentric Subdivision and the Matrix F_d.** To prove Theorem A, we begin by observing the effect on f-vectors of barycentric subdivision. One key observation is that barycentric subdivision multiplies f-vectors by a fixed matrix, F_d:

Definition 3.1. Define $\mathring{f}_i^X$ to be the number of interior i-faces of X for $i \geq 0$. We set $\mathring{f}_{-1}^X = 1$ if the dimension of X is -1, and 0 otherwise. Let σ_d denote the standard $(d-1)$-dimensional simplex. Define F_d to be the $(d+1)\times(d+1)$ matrix determined by the interior $(j-1)$-faces of the subdivided i-simplex:

$$F_d := [\mathring{f}_{j-1}^{\sigma'_i}].$$

Lemma 3.2. *Barycentrically subdividing a $(d-1)$-dimensional simplicial complex X multiplies the f-vector by F_d:*

$$f^{X'} = f^X F_d.$$

Proof. The faces of X' can be partitioned according to the lowest-dimensional faces of X containing them. Each face of X is a simplex of some dimension i, and thus its interior contributes $\mathring{f}_j^{\sigma'_i}$ to the total number of j-cells of X' (or, if $i = 0$, exactly one vertex to X'). The claim then follows by linearity. □

Corollary 3.3. *For any $n \geq 0$,*

$$f^{X^{(n)}} = f^X F_d^n.$$

Thus, to understand barycentric subdivision, we need to understand the matrix F_d. We will compute the entries in F_d more explicitly in the following two sections, but for now we simply observe a formula which follows from Inclusion-Exclusion:

Lemma 3.4. *If $j > i$ then $\mathring{f}_j^{\sigma'_i} = 0$. If $j \leq i$, then*

$$\mathring{f}_j^{\sigma'_i} = \sum_{k=0}^{i} (-1^k) \binom{i}{k} f_j^{\sigma'_{i-k}}. \qquad \square$$

By this lemma, F_d is lower triangular with diagonal entries $\mathring{f}_i^{\sigma'_i} = f_i^{\sigma'_i} = i!$. Thus, the eigenvalues of F_d are $0!, 1!, 2!, 3!, \ldots, d!$.

3.2. **Limit Behavior of the Roots.** We now turn to the roots of the f-polynomials $f^{X^{(n)}}(t)$. By Corollary 3.3,

$$f^{X^{(n)}}(t) = f^X F_d^n \underline{t}.$$

As the greatest eigenvalue of F_d is $d!$, we normalize $f^{X^{(n)}}(t)$ by dividing by $(d!)^n$ - let $p_n^X(t)$ denote the result:

$$p_n^X(t) := \frac{1}{(d!)^n} f^{X^{(n)}}(t).$$

Note this normalization does not alter the roots. It will also often be convenient to reverse the order of the coefficients of $p_n^X(t)$, with the effect of inverting the roots of $p_n^X(t)$ (that is, the roots of $f^{X^{(n)}}(t)$) about the unit circle in the extended complex plane:

$$q_n^X(t) := t^d p_n^X(t^{-1}).$$

We are interested in the behavior of the roots of $p_n^X(t)$ and $q_n^X(t)$ as n goes to infinity, so we are interested in the powers of F_d. To take powers of F_d, we diagonalize,

$$F_d = P_d D_d P_d^{-1},$$

where D_d is the diagonal matrix of eigenvalues $0!, 1!, \ldots, d!$ and P_d is the (lower triangular) diagonalizing matrix of eigenvectors. Thus, $F_d^n = P_d D_d^n P_d^{-1}$.

Let $\widetilde{D}_d := \frac{1}{d!} D_d$. Let $\bar{t}$ denote the column vector $\underline{t}$ in reverse order, $\bar{t} = (t^0, t^1, \ldots t^d)^T$. For any simplicial complex X, we thus have the following equations:

$$\begin{aligned} f^{X^{(n)}}(t) &= f^X P_d D_d^n P_d^{-1} \underline{t} \\ &= (d!)^n \left(f^X P_d\right) \left(\widetilde{D}_d\right)^n \left(P_d^{-1}\right) \underline{t} \end{aligned}$$

$$\begin{aligned} p_n^X(t) &= \left(f^X P_d\right) \left(\widetilde{D}_d\right)^n \left(P_d^{-1}\right) \underline{t} \\ q_n^X(t) &= \left(f^X P_d\right) \left(\widetilde{D}_d\right)^n \left(P_d^{-1}\right) \bar{t} \end{aligned}$$

The goal of Section 6 will be to describe more precisely the matrices P_d and P_d^{-1}. As the eigenvalues of F_d are $0!, 1!, \ldots, d!$, for large n, D_d^n is dominated by its d^{th} diagonal entry, $(d!)^n$. In the limit, the powers of the matrix $\widetilde{D}_d = \frac{1}{d!} D_d$ converge to the matrix

$$M_{d,d} := \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}.$$

Thus, as n grows, the polynomials p_n^X and q_n^X respectively approach the polynomials

$$\begin{aligned} p_\infty^X(t) &:= \left(f^X P_d\right) M_{d,d} \left(P_d^{-1}\right) \underline{t}, \\ q_\infty^X(t) &:= \left(f^X P_d\right) M_{d,d} \left(P_d^{-1}\right) \bar{t} \end{aligned}$$

in the sense that each sequence converges coefficient-wise in the vector space of polynomials of degree at most d.

By Corollary 3.3 and Lemma 3.4, we know the leading and trailing coefficients of $p_n^X(t)$ and $q_n^X(t)$: $p_n^X(t) = (d!)^{-n} t^d + \ldots + f_{d-1}^X$ and $q_n^X(t) = (d!)^{-n} + \ldots + f_{d-1}^X t^d$. Hence, in the limit, $p_\infty^X(t)$ does not have 0 as a root, but has degree less than d (one root of the p_n^X diverges to $-\infty$), while $q_\infty^X(t)$ is of degree d with 0 as a root. Because the polynomials $q_n^X(t)$ converge coefficient-wise to the polynomial $q_\infty^X(t)$ of the same degree, their roots also converge:

Lemma 3.5. [7] *Let $(P_n(t))_n$ be a sequence of monic polynomials of degree d that converges to a monic polynomial $P_\infty(t)$ of the same degree d. Then we can number the roots of $P_n(t)$ as $r_1^n, \ldots, r_d^n$ and the roots of $P_\infty(t)$ as $r_1^\infty, \ldots, r_d^\infty$ in such a way that for all $j = 1, \ldots, d$ the sequence r_j^n converges to r_j^∞ for $n \to \infty$.*

Since the roots of $q_n^X(t)$ converge to the roots of $q_\infty^X(t)$, it follows that the roots of $p_n^X(t)$ converge to the roots of $p_\infty^X(t)$ (with one of the roots 'converging' to $-\infty$).

Because the matrix P_d is lower triangular and $M_{d,d}$ has only one nonzero entry in position (d, d), we have

$$\left(f^X P_d\right) M_{d,d} = c_{X,d} e_d^T,$$

where e_d is the unit vector with a 1 in the d^{th} row, and $c_{X,d}$ is a constant depending on f^X and P_d. As both f^X and P_d do not depend on the amount of subdivision n, the roots of p_∞^X and q_∞^X do not depend on the value of $c_{X,d}$, and thus do not depend on *any* coefficient of f_d^X. This leads us to the following definition:

Definition 3.6. Define the *limit p-polynomial* by

$$p_d(t) := e_d^T P_d^{-1} \underline{t}$$

and the *limit q-polynomial* by

$$q_d(t) := e_d^T P_d^{-1}\bar{t}.$$

In this section we have proven:

Theorem 3.7. *The following facts hold:*

1. *The roots of $f^{X^{(n)}}(t)$ are equal to the roots of $p_n^X(t)$.*
2. *The roots of $q_n^X(t)$ converge to the roots of $q_d(t)$, and depend only on the dimension of X.*
3. *The roots of $p_n^X(t)$ converge to the roots of $p_d(t)$, and depend only on the dimension of X.*
4. *The coefficient of t^i in the polynomial $p_d(t)$ is the $(d-i)^{th}$ entry in last row of P_d^{-1}.*
5. *The coefficient of t^i in the polynomial $q_d(t)$ is the i^{th} entry in the last row of P_d^{-1}.*

This proves Theorem A. Note the first two facts give an alternate proof of Brenti and Welker's theorem, Theorem 1.1, except for the fact that the roots are all real (that the roots are negative would then follow from the fact that all coefficients of these polynomials are positive).

In Section 6, we will explore the final two facts of Theorem 3.7 by computing the coefficients of P_d^{-1}.

4. Symmetry of the roots

Our goal is now to show that the limits of the roots satisfy a certain symmetry, as stated in Theorem B. We will prove this symmetry for the roots of q_d instead of p_d, as it becomes a mirror symmetry instead of a Möbius invariance.

Theorem 4.1. *For every dimension d,*

$$q_\infty(t) = (-1)^d q_\infty(-1-t).$$

In particular, the roots of $q_\infty(t)$ are (linearly) symmetric with respect to $-\frac{1}{2}$.

To prove this theorem, we start by examining the subdivision of a single closed simplex.

Lemma 4.2. *Let σ_s be a closed simplex of dimension s, The f-vector of its barycentric subdivision σ_s' is given by*

$$f_j^{\sigma'_s} = \Delta^j\{f^\sigma(l)\}_l = \Delta^j\{(1+l)^s\}.$$

where the operator Δ is the difference operator on a sequence.

Proof. It is easy to see that:

$$\begin{aligned} f^{\sigma'_s}(t) &:= \sum_{j=0}^{s} f_{j-1}^{\sigma'_s} t^j \\ &= \sum_{j=0}^{s} t^j \sum_{i=0}^{s} \binom{s}{i} \sum_{k=0}^{j} (-1)^k \binom{j}{k} (j-k)^i. \end{aligned}$$

Note that the innermost sum is the Stirling number $S(i,j)$ of the second kind (see [6], page 34). Reordering this triple summation, we have:

$$\begin{aligned}
f^{\sigma'_s}(t) &= \sum_{j=0}^{s} t^j \sum_{k=0}^{j} (-1)^k \binom{j}{k} \sum_{i=0}^{s} \binom{s}{i} (j-k)^i \\
&= \sum_{j=0}^{s} t^j \sum_{k=0}^{j} (-1)^k \binom{j}{k} f^{\sigma}(j-k) \\
&= \sum_{j=0}^{s} t^j \sum_{k=0}^{j} (-1)^k \binom{j}{k} f^{\sigma}(k) \\
&= \sum_{j=0}^{s} \Delta^j \{f^{\sigma}(l)\}_l t^j,
\end{aligned}$$

where in the third equality we replace k with $j-k$. ☐

Corollary 4.3. *Let X be a simplicial complex. The f-polynomial of its barycentric subdivision $f^{X'}(t)$ is given by*

$$f^{X'}(t) = \sum_{j=0}^{d} \Delta^j \{f^X(l)\}_l t^{d-j}.$$

The polynomials p_1^X and q_1^X are given by

$$(d!)p_1^X(t) = \sum_{j=0}^{d} \Delta^j \{p_0^X(l)\}_l t^{d-j}, \quad (d!)q_1^X(t) = \sum_{k=0}^{d} \Delta^k \{q_0^X(l)\}_l t^k.$$

Proof. These formulas follow easily from Lemma 4.2 by linearity of the difference operator Δ. ☐

Taking inspiration from the formula for $f^{X'}(t)$ above, we consider barycentric subdivision as a function on polynomials in t defined by

$$b : \mathbb{Z}[t] \to \mathbb{Z}[t], \quad g(t) \mapsto \sum_{k \geq 0} \Delta^k \{g(l)\}_l t^k. \tag{1}$$

(Note that this sum is finite because the iterated finite differences of a polynomial are eventually all zero.)

For a simplicial complex X of dimension d we have then

$$b(q_j^X(t)) = d! q_{j+1}^X(t).$$

The function b is linear, and thus it is given by its values on monomials. It will be convenient to list these values as arranged on the "clothesline" [8] provided by a formal power series in the variable x over the ring $\mathbb{Z}[t]$. We thus consider a function B on the ring $\mathbb{Z}[t][[x]]$ defined as

$$B: \quad \sum_{k \geq 0} g_k(t) x^k \quad \longmapsto \quad \sum_{k \geq 0} b\big(g_k(t)\big) x^k.$$

Theorem 4.4. *(see Theorem C) In $\mathbb{Z}[t][[x]]$ we have*

$$B(e^{tx}) = \frac{1}{1-(e^x-1)t}.$$

Proof. We expand the right hand side as a formal power series over x and compare the coefficient of $\frac{x^n}{n!}$ therein with the value of $B(t^n) = b(t^n)$ as given in (1). We have

$$\begin{aligned}
\frac{1}{1-(e^x-1)t} &= \sum_{j\geq 0}(e^x-1)^j t^j = \sum_{j\geq 0}\bigg(\sum_{m=0}^{j}\binom{j}{m}(-1)^{j-m}e^{mx}\bigg)t^j \\
&= \sum_{j\geq 0} t^j\bigg(\sum_{m=0}^{j}\binom{j}{m}(-1)^{j-m}\sum_{k\geq 0}\frac{m^k x^k}{k!}\bigg) \\
&= \sum_{k\geq 0}\bigg(\sum_{j\geq 0} t^j\sum_{m=0}^{j}\binom{j}{m}(-1)^{j-m}m^k\bigg)\frac{x^k}{k!} \\
&= \sum_{k\geq 0}\bigg(\sum_{j\geq 0}\Delta^j\{m^k\}_m t^j\bigg)\frac{x^k}{k!} \\
&= \sum_{k\geq 0} b\big(t^k\big)\frac{x^k}{k!} = B\bigg(\sum_{k\geq 0}\frac{t^k x^k}{k!}\bigg) = B(e^{tx})
\end{aligned}$$

□

To investigate the stated symmetry, we consider the following map

$$\iota : \mathbb{Z}[t] \to \mathbb{Z}[t], \quad g(t) \mapsto g(-1-t). \tag{2}$$

Lemma 4.5. *The map ι is an involution, and it satisfies*

$$\iota b \iota = b$$

Proof. The map ι is clearly linear, so it will suffice to prove the claim for monomials.

It is easy to see that ι is an involution. Moreover, $\iota b = b\iota$, as

$$\iota B(e^{tx}) = \iota\bigg(\frac{1}{1-(e^x-1)t}\bigg) = \frac{1}{1-(e^x-1)(t-1)} = B(e^{(-1-t)x}) = B\iota(e^{tx}).$$

The claim follows with term-by-term comparison. □

We are now ready to prove the theorem.

Proof of Theorem 4.1. Barycentric subdivision has the effect on each p- and q-polynomial of multiplying on the right by F before the $\underline{t}$ and $\overline{t}$, respectively, and rescaling by dividing by $d!$. In the limit, the limit p- and q-polynomials are invariant under barycentric subdivision up to this scaling, so that

$$b\big(q_\infty(t)\big) = d!q_\infty(t).$$

Since the eigenvalues of F are all distinct, q_∞ is characterized by this identity, and by having leading coefficient f^X_{d-1}.

Applying Lemma 4.5, we have

$$b\big(q_\infty(-1-t)\big) = b\big(\iota(q_\infty(t))\big) = \iota\big(b(q_\infty(t))\big) = \iota\big((d!)q_\infty(t)\big) = d!\big(q_\infty(-1-t)\big)$$

and since the lead coefficient of $q_\infty(-1-t)$ is $(-1)^d f^X_{d-1}$, the claim follows.

□

5. Symmetry for Other Subdivision Methods

In general, given any polynomial $g(t) \in \mathbb{Z}[t]$, we can consider the polynomial $\iota g(t) = g(-1-t)$. The coefficient of t^k in $g(t)$ contributes $(-1)^k\binom{k}{j}$ times itself to the coefficient of t^j in $\iota g(t)$: this contribution is exactly the number of $(j-1)$-dimensional faces of the $(k-1)$-dimensional simplex. Thus, we can interpret ι as a map on formal sums of simplices, as follows.

Let S be the set of simplices of a given simplicial complex X with vertex set VX. We will think of every simplex $\sigma \in S$ as a subset of VX. Now we can write

$$\iota : \mathbb{Z}[S] \to \mathbb{Z}[S], \quad \sigma \mapsto (-1)^{\dim \sigma + 1} \sum_{\tau \subseteq \sigma} \tau.$$

We represent the simplicial complex X as the formal sum of all its simplices, each with "weight" 1. Applying ι to this sum we have

$$\begin{aligned}
\iota\Big(\sum_{\sigma \in S} \sigma\Big) &= \sum_{\sigma \in S} (-1)^{\dim \sigma + 1} \sum_{\tau \subseteq \sigma} \tau \\
&= \sum_{\tau \in S} \Bigg(\sum_{\substack{\sigma \in S \\ \sigma \supseteq \tau}} (-1)^{\dim \sigma - \dim \tau} \Bigg) (-1)^{\dim \tau + 1} \tau \\
&= \sum_{\tau \in S} (-1)^{\dim \tau} \big(\chi(\operatorname{link} \tau) - 1\big) \tau.
\end{aligned}$$

Let us recall some basics about subdivisions of simplicial complexes, pointing to [4] as a reference for a more detailed discussion. In the following we will write $|X|$ for the geometric realization of a given simplicial complex X [4, Section 3.1].

Definition 5.1 (Compare Section 3.3 of [4])**.** A *subdivision* (not necessarily barycentric) of X is a simplicial complex $\widetilde{X}$ whose vertices are points of $|X|$ and such that

1. For every simplex $\widetilde{\sigma}$ of $\widetilde{X}$ there is a simplex σ of X such that $\widetilde{\sigma} \subseteq |\sigma|$. The simplex σ is called the *support* of $\widetilde{\sigma}$.
2. The linear map $|\widetilde{X}| \to |X|$ mapping each vertex of $\widetilde{X}$ to the corresponding point of $|X|$ is a homeomorphism.

We will identify a subdivision of X by the triple $(X, \widetilde{X}, \phi)$, where $\phi : \widetilde{S} \to S$ is the function associating to each $\widetilde{\sigma}$ its support in X.

Now, a subdivision $(X, \widetilde{X}, \phi)$ induces a linear map

$$b_\phi : \mathbb{Z}[S] \to \mathbb{Z}[\widetilde{S}], \quad \sigma \mapsto \sum_{\phi(\widetilde{\sigma}) = \sigma} \widetilde{\sigma}.$$

Every finite simplicial complex X has finitely many (possibly 0) n-simplices for each n. Each such n-simplex can be thought of as a map $i_n : \sigma_n \to X$ from the standard n-simplex in exactly $(n+1)!$ ways, depending on how the vertices of σ_n are identified in X.

Definition 5.2. A *subdivision method* Φ is a collection of subdivisions $\Phi := \{(\sigma_n, \widetilde{\sigma}_n, \phi_n)\}_{n \geq 0}$ such that for every map $i_k : \sigma_k \to \sigma_m$ identifying a k-face of the standard m-simplex, the map ϕ_k is the restriction of ϕ_m to $i_k\sigma_k$. This ensures that, given any simplicial complex X, the complex $\Phi(X)$, called *subdivision of X according to the rule* Φ is uniquely defined by requiring that every n-simplex of X is subdivided as $(\sigma_n, \widetilde{\sigma}_n, \phi_n) \in \Phi$. More precisely, the complex $\Phi(X)$ is such that there exists $\phi : \Phi(X) \to X$ so $(X, \Phi(X), \phi)$ is a subdivision and for every k-simplex τ of $\Phi(X)$, $\phi(\tau) = i_{\dim \phi(\tau)} \phi_k i_k^{-1}(\tau)$. A subdivision method is *nontrivial in dimension n* if ϕ_k is not the identity map for some $k \leq n$. Clearly if a subdivision is nontrivial in dimension n, then ϕ_n is not the identity map.

Barycentric subdivision is the subdivision method where $\widetilde{\sigma}_n = 2^{\sigma_n}$ and $\phi_n j = \{j\}$ for every vertex $j \in \sigma_n$.

Given a subdivision method Φ, in view of the linearity of b_ϕ for each subdivision, it makes sense to write

$$b_\Phi(\sum_{\sigma\in X} \sigma) = \sum_{\sigma\in X} b_\Phi \sigma.$$

As with the map b given by barycentric subdivision, for any subdivision method the induced map b_Φ always commutes with the map ι:

Lemma 5.3. *For any subdivision method Φ, $\iota b_\Phi = b_\Phi \iota$.*

Before we prove this lemma, we need to characterize how ι acts on simplices. We do so by looking at how ι affects an arbitrary manifold. Let M be a d-dimensional manifold with boundary ∂M, and let $[M]$ denote the formal sum of all simplices of a finite simplicial complex X so that M is (PL-homeomorphic to) the geometric realization of X.

Lemma 5.4. *We have*

$$\iota[M] = (-1)^{d+1}\big([M] - [\partial M]\big).$$

Proof. The link of every simplex $\sigma \in [M]$ is of dimension $d - \dim\sigma - 1$, and is a ball or a sphere according to whether σ is on the boundary ∂M or not. When $\operatorname{link}\sigma$ is a ball, $\chi(\operatorname{link}\sigma) - 1 = 0$, and when $\operatorname{link}\sigma$ is a sphere, $\chi(\operatorname{link}\sigma) - 1 = (-1)^{d-\dim\sigma-1}$. Thus,

$$\begin{aligned}\iota[M] &= \sum_{\sigma\notin\partial M} (-1)^{\dim\sigma}\Big((-1)^{d-\dim\sigma-1}\Big)\sigma + \sum_{\sigma\in\partial M} (-1)^{\dim\sigma}\cdot 0\cdot\sigma \\ &= \sum_{\sigma\notin\partial M} (-1)^{d-1}\sigma = (-1)^{d+1}\big([M] - [\partial M]\big).\end{aligned}$$

□

Proof of Lemma 5.3. By linearity, it suffices to prove that $\iota b_\Phi([\sigma]) = b_\Phi\iota([\sigma])$ for any simplex σ, where $[\sigma] = \sum_{\tau\subseteq\sigma}\tau$ is the sum of the simplices contained in the manifold σ. Since Φ is a subdivision method, $b_\Phi([\sigma])$ will also be the sum of the simplices contained in some manifold $\Phi\sigma$ of dimension $\dim\sigma$. Also, $b_\Phi([\partial\sigma]) = [\partial\Phi\sigma]$.

The result now follows from Lemma 5.4:

$$\begin{aligned}b_\Phi\iota([\sigma]) &= b_\Phi\big((-1)^{\dim\sigma+1}([\sigma] - [\partial\sigma])\big) \\ &= (-1)^{\dim\sigma+1}(b_\Phi([\sigma]) - b_\Phi([\partial\sigma])) \\ &= (-1)^{\dim\sigma+1}([\Phi\sigma] - [\partial\Phi\sigma]) \\ &= \iota([\Phi\sigma]) \\ &= \iota b_\Phi([\sigma]).\end{aligned}$$

□

Theorem 5.5. *For any dimension n and any subdivision method Φ which is nontrivial in dimension n, there exists a unique 'limit polynomial' $p_{n,\Phi}(t)$, such that, for any d-dimensional simplicial complex X, the roots of $f^{\Phi^k(X)}(t)$ converge to the roots of $p_{n,\Phi}(t)$ as k increases. The roots of $p_{n,\Phi}(t)$ are invariant under the Möbius transformation $x \mapsto \frac{-x}{x+1}$.*

Proof. The proof of this theorem is exactly the same as the proof of Theorem 5.5 in the case of barycentric subdivision. The key observations there were that $\iota b_\Phi = b_\Phi \iota$ and that there exists a unique eigenvector for the maximal eigenvalue of the matrix realizing the effect of subdivision on f-vectors. That this eigenvector is unique in general follows from Φ being nontrivial in dimension n, and is left as an exercise for the reader. ☐

Remark 5.6. Since the above interpretation is on the level of formal sums of simplices, the most natural context in which to study it seems to be the Stanley-Reisner ring $\mathbb{K}[X]$, defined by any simplicial complex X and any field $\mathbb{K}$. A good introduction to these rings can be found in [5], where some properties of the Stanley-Reisner ring of a subdivision of a simplicial complex are explored. This brings us to ask the following questions.

Question 5.7. *Is there a (multi-)complex in each dimension whose f-polynomial is related to the limit polynomials $p_\infty^X(t)$ or $q_\infty^X(t)$?*

More generally,

Question 5.8. *Is there a geometric interpretation of the coefficients or the roots of $p_\infty^X(t)$ (equivalently, $q_\infty^X(t)$)?*

6. Computations

We finish this paper by computing explicit values for the limit roots up to $d = 10$. As observed in Theorem 3.7, to compute p_d we need to compute the matrix P_d^{-1}.

6.1. **The Coefficients of P_d^{-1}.** We now compute the coefficients of the matrix P_d^{-1}. To do so, we first compute a more explicit expression for F_d, then compute the matrix P_d, and finally compute the matrix P_d^{-1}.

Recall that $F_d = [\mathring{f}_j^{\sigma'_i}]$. We begin by finding a more explicit expression for the entries of F_d.

Lemma 6.1. *We have*

$$\mathring{f}_j^{\sigma'_i} = \sum_{l=0}^{j} (-1)^l \binom{j}{l} l^i.$$

Proof. We use Lemma 4.2 as a starting point for this calculation. It follows that:

$$\begin{aligned}
\mathring{f}_j^{\sigma'_i} &= \sum_{k=0}^{i}(-1^k)\binom{i}{k} f_j^{\sigma'_{i-k}} \\
&= \sum_{k=0}^{i}(-1)^k\binom{i}{k}\Delta^j\{(1+l)^{i-k}\}_l \\
&= \sum_{k=0}^{i}(-1)^k\binom{i}{k}\sum_{l=0}^{j}(-1)^l\binom{j}{l}(1+l)^{i-k} \\
&= \sum_{l=0}^{j}(-1)^l\binom{j}{l}\sum_{k=0}^{i}(-1)^k\binom{i}{k}(1+l)^{i-k} \\
&= \sum_{l=0}^{j}(-1)^l\binom{j}{l}(1+l-1)^i \\
&= \sum_{l=0}^{j}(-1)^l\binom{j}{l}l^i.
\end{aligned}$$

□

Now that we have the coefficients of F_d, we may compute the coefficients of the diagonalizing matrix P_d.

We compute P by performing Gauss-Jordan elimination. Let us write $P_d = [c_{i,j}]$ and $P_n^{-1} = [\gamma_{i,j}]$. For P_d, we have:

Lemma 6.2 (Gauss-Jordan Elimination)**.** *Let $B = [b_{i,j}]_{i,j=0,\dots,d}$ be a diagonalizable lower triangular matrix, C a matrix whose columns are given by the eigenvectors of B. Then C is also lower triangular, with $c_{i,i} = 1$ for all i and, for $i > j$:*

$$c_{i,j} = -\frac{b_{i,j}}{b_{i,i}}c_{i,i} + \sum_{k=j-1}^{i-1}\frac{b_{k,j}}{b_{k,k}}c_{i,k}$$

Applying this to the case $B = F_d$, we use $b_{i,j} = \mathring{f}_j^{\sigma'_i}$, and in particular $b_{i,i} = i!$. Thus,

$$\begin{aligned}
c_{i,j} &= -\sum_{l=0}^{j}(-1)^l\binom{j}{l}\frac{l^i}{i!}c_{i,i} + \sum_{k=j-1}^{i-1}\sum_{l=0}^{j}(-1)^l\binom{j}{l}\frac{l^k}{k!}c_{i,k} \\
&= \sum_{l=0}^{j}(-1)^l\binom{j}{l}\left(-\frac{l^i}{i!} + \sum_{k=j-1}^{i-1}\frac{c_{i,k}l^k}{k!}\right)
\end{aligned}$$

Knowing the coefficients of P_d allows us to compute the coefficients of P_d^{-1}, via the following lemma:

Lemma 6.3 (Inverse Matrix)**.** *Let $C := [c_{i,j}]$ be a square, lower triangular matrix with $c_{i,i} = 1$. Then, the (i,j)-entry of its inverse C^{-1} is 0 if $j > i$, 1 if $i = j$, and otherwise given by*

$$\gamma_{i,j} = \sum_{m=1}^{i-j}(-1)^m \sum_{j=\alpha_0<\dots<\alpha_m=i}\prod_{l=1}^{m}c_{\alpha_l,\alpha_{l-1}}$$

Alternatively, let A be the weighted incidence matrix of the weighted, directed graph on the vertex set $\{0...d\}$ with edge set $\{(j,i) \mid i > j\}$ and weights $w(j,i) = -c_{i,j}$. Then $A = (1-P)$ and $\gamma_{i,j} = (\sum_{m=0}^{d} A^m)_{i,j}$.

6.2. **Limit polynomial and roots when $d \leq 10$.** Our method allows us to explicitly compute the coefficients of $q_d(t)$. We do so here for $d \leq 10$ (computations which take less than 1 second of processor time). We have run our calculations for $d \leq 40$, but we omit these further calculations due to space and rounding errors.

Note that, for any $k \geq d$, the d^{th} row of P_k^{-1} does not depend of k, and gives the coefficients of $q_d(t)$. Thus, we present here the matrix P_{10}^{-1}, as obtained by computer:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{3}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{11} & \frac{13}{11} & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{19} & \frac{25}{38} & \frac{40}{19} & \frac{5}{2} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{132}{10411} & \frac{3004}{10411} & \frac{45}{29} & \frac{95}{29} & 3 & 1 & 0 & 0 & 0 & 0 \\
0 & \frac{90}{34399} & \frac{3626}{34399} & \frac{61607}{68798} & \frac{245}{82} & \frac{385}{82} & \frac{7}{2} & 1 & 0 & 0 & 0 \\
0 & \frac{15984}{33846961} & \frac{12351860}{372316571} & \frac{7924}{18469} & \frac{39221}{18469} & \frac{56}{11} & \frac{70}{11} & 4 & 1 & 0 & 0 \\
0 & \frac{983304}{12980789207} & \frac{119432466}{12980789207} & \frac{2296176994}{12980789207} & \frac{536193}{429266} & \frac{919821}{214633} & \frac{567}{71} & \frac{588}{71} & \frac{9}{2} & 1 & 0 \\
0 & \frac{1345248918720}{123031432784730871} & \frac{281136722386176}{123031432784730871} & \frac{4358731100}{67808366729} & \frac{42780833020}{67808366729} & \frac{1335075}{448471} & \frac{3478503}{448471} & \frac{1050}{89} & \frac{930}{89} & 5 & 1
\end{bmatrix}$$

The fractions in the above matrix may be approximated by the following decimal expansions:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 1.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.18182 & 1.1818 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.052632 & 0.65780 & 2.1053 & 2.5 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.012679 & 0.28854 & 1.5517 & 3.2759 & 3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0.0026164 & 0.10541 & 0.89548 & 2.9878 & 4.6951 & 3.5 & 1 & 0 & 0 & 0 \\
0 & 0.00047224 & 0.033176 & 0.42904 & 2.1236 & 5.0909 & 6.3636 & 4 & 1 & 0 & 0 \\
0 & 0.000075751 & 0.0092007 & 0.17689 & 1.2491 & 4.2856 & 7.9859 & 8.2817 & 4.5 & 1 & 0 \\
0 & 0.000010934 & 0.0022851 & 0.064280 & 0.63091 & 2.9770 & 7.7564 & 11.798 & 10.449 & 5 & 1
\end{bmatrix}$$

The roots of $p_d(t)$ are, for $d \leq 10$, approximated by:

$$\left\{\begin{array}{rccccccccc}
d=2: & -1 \\
d=3: & -2 & -1 \\
d=4: & -4.1861 & -1.3139 & -1 \\
d=5: & -8.3642 & -2 & -1.1358 & -1 \\
d=6: & -16.096 & -1.4706 & -3.1252 & -1.0662 & -1 \\
d=7: & -30.121 & -4.8761 & -2 & -1.2570 & -1.0343 & -1 \\
d=8: & -55.208 & -7.5398 & -2.7664 & -1.5661 & -1.1529 & -1.0185 & -1 \\
d=9: & -99.626 & -11.537 & -3.8404 & -2 & -1.3521 & -1.0949 & -1.0101 & -1 \\
d=10: & -177.68 & -17.474 & -5.3206 & -2.5830 & -1.6317 & -1.2315 & -1.0607 & -1.0057 & -1
\end{array}\right.$$

We see that these roots are symmetric about the point -2, with respect to the Möbius transformation $x \mapsto \frac{-x}{x+1}$. The symmetry is more apparent in the (linear) symmetry about $-\frac{1}{2}$ exhibited by the roots of $q_d(t)$, which are the reciprocals of the roots of $p_d(t)$. The roots of $q_d(t)$ are, for $d \leq 10$, approximated by:

$$\left\{\begin{array}{rcccccccccc}
d=2: & -1 & 0 \\
d=3: & -1 & -.5 & 0 \\
d=4: & -1 & -.76112 & -.23888 & 0 \\
d=5: & -1 & -.88044 & -.5 & -.11956 & 0 \\
d=6: & -1 & -.93787 & -.68002 & -.31998 & -.06213 & 0 \\
d=7: & -1 & -.9668 & -.79492 & -.5 & -.20508 & -.0332 & 0 \\
d=8: & -1 & -.98189 & -.86737 & -.63852 & -.36148 & -.13263 & -.01811 & 0 \\
d=9: & -1 & -.98996 & -.91332 & -.73961 & -.5 & -.26039 & -.08668 & -.01004 & 0 \\
d=10: & -1 & -.99437 & -.94277 & -.81205 & -.61285 & -.38715 & -.18795 & -.05723 & -.00563 & 0
\end{array}\right.$$

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