Math 304 **Test 3** **November 24, 2008**

1. (5 points) Suppose a 4×8 matrix A has rank 3. Fill in the boxes:

(a) the dimension of the column space of A is
Solution: 3

(b) the dimension of the row space of A is
Solution: 3

(c) the dimension of the null space of A is
Solution: 5

(d) the dimension of the row space of A^T is
Solution: 3

(e) the dimension of the null space of A^T is
Solution: 1

2. (5 points) **Circle** the correct answer in each of the following.

(a) | True | False | If the rows of a square matrix are linearly independent then the columns of the matrix are linearly independent.
Solution: True

(b) | True | False | $\det(A + B) = \det(A) + \det(B)$ for any 7×7 matrices A and B.
Solution: False

(c) | True | False | $\det(cA) = c^n \det(A)$ for any $n \times n$ matrix A.
Solution: True

(d) If a square matrix A is invertible then 0 | must | may or may not | cannot | be an eigenvalue of A.
Solution: 0 cannot be an eigenvalue of A

(e) If X, Y and Z are bases for $\mathbb{R}^5$ then the product $({}_X I_Y)({}_Y I_Z)$ equals | I | ${}_X I_Z$ | ${}_Z I_X$ | ${}_Y I_Y$ | .
Solution: ${}_X I_Z$

3. (5 points) Suppose A and B are invertible $n \times n$ matrices. Use the properties of determinants to calculate $\det(ABA^{-1}B^{-1})$.

Solution:

$$\begin{aligned}
\det(ABA^{-1}B^{-1}) &= \det(A)\det(B)\det(A^{-1})\det(B^{-1}) && \text{since } \det(XY) = \det(X)\det(Y) \\
&= \det(A)\det(B)\frac{1}{\det(A)}\frac{1}{\det(B)} && \text{since} \det(X) = \frac{1}{\det(X)} \\
&= \frac{\det(A)}{\det(A)} \cdot \frac{\det(B)}{\det(B)} && \text{algebra} \\
&= 1 \cdot 1 = 1 && \text{algebra}
\end{aligned}$$

4. (20 points)

(a) $\det \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & -5 \\ 1 & 1 & 0 \end{bmatrix} =$

Solution: By formula:

$$\det \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & -5 \\ 1 & 1 & 0 \end{bmatrix} = 1 \cdot 0 \cdot 0 + (-2) \cdot (-5) \cdot 1 + (-1) \cdot 2 \cdot 1 - 1 \cdot 0 \cdot (-1) - 1 \cdot (-5) \cdot 1 - 0 \cdot 2 \cdot (-2) = 13$$

(b) $\det \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 4 \\ 2 & 1 & 1 & -3 \\ 0 & 5 & 0 & -5 \end{bmatrix} =$

Solution: By row reduction:

$$\det \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 4 \\ 2 & 1 & 1 & -3 \\ 0 & 5 & 0 & -5 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & 5 & 0 & -5 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & -13 \\ 0 & 5 & 0 & -5 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & -13 \\ 0 & 0 & 0 & -25 \end{bmatrix} =$$

$1 \cdot 1 \cdot (-1) \cdot (-25) = 25$

5. (5 points) Let A be a 4×4 matrix with rows $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$, in that order. If $\det(A) = 3$, find $\det \begin{bmatrix} \mathbf{r}_2 + 4\mathbf{r}_3 \\ \mathbf{r}_1 \\ \mathbf{r}_3 \\ 5\mathbf{r}_4 \end{bmatrix}$.

Solution: Subtract 4 times the third row from the first: $\det \begin{bmatrix} \mathbf{r}_2 + 4\mathbf{r}_3 \\ \mathbf{r}_1 \\ \mathbf{r}_3 \\ 5\mathbf{r}_4 \end{bmatrix} = \det \begin{bmatrix} \mathbf{r}_2 \\ \mathbf{r}_1 \\ \mathbf{r}_3 \\ 5\mathbf{r}_4 \end{bmatrix}$. Swap the first and second rows to get $-\det \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ 5\mathbf{r}_4 \end{bmatrix}$. Factor out 5 from the fourth row to get $-5 \det \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{bmatrix}$. Now $\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{bmatrix} = A$ so the answer is $-5 \det(A) = -5 \cdot 3 = -15$.

6. (15 points) Let $E = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ be the standard basis for $\mathbb{R}^2$. Another basis is $X = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$.

(a) Find the change of basis matrix ${}_E I_X$.

Solution: ${}_E I_X$ is the matrix with columns given by the basis X, so ${}_E I_X = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

(b) Find the change of basis matrix ${}_X I_E$.

Solution: ${}_X I_E = ({}_E I_X)^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

(c) Find the coordinate vector $K_X \left(\begin{bmatrix} 2 \\ -3 \end{bmatrix} \right)$.

Solution: $K_X \left(\begin{bmatrix} 2 \\ -3 \end{bmatrix} \right) = {}_X I_E \cdot K_E \left(\begin{bmatrix} 2 \\ -3 \end{bmatrix} \right) = {}_X I_E \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -8 \\ 5 \end{bmatrix}$

7. (15 points) Let $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$. This matrix has only one eigenvalue.

(a) Find the characteristic polynomial of A.

Solution: $\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = (3-\lambda)\cdot(1-\lambda) - (-1)\cdot 1 = \lambda^2 - 4\lambda + 4$

(b) Find the eigenvalue of A.

Solution: $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, so the only root of the characteristic polynomial is 2. This is the eigenvalue.

(c) What is the multiplicity of this eigenvalue?
Solution: 2.

(d) Find a basis for the corresponding eigenspace of A.

Solution: Row reduce: $A - 2I = \det \begin{bmatrix} 3-2 & -1 \\ 1 & 1-2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. So $A - 2I$ has rank 1, so its null space has dimension $2 - 1 = 1$. A basis for the null space is obtained by setting the free variable equal to 1. This gives $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a basis.

(e) Is A diagonalizable? | Yes | No |

Solution: No.

8. (5 points) Show that $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & -2 & 1 \\ 0 & -3 & 1 & 1 \\ -3 & 0 & 0 & 0 \end{bmatrix}$. What is the corresponding eigenvalue?

Solution: $A\mathbf{v} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & -2 & 1 \\ 0 & -3 & 1 & 1 \\ -3 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 3 \\ -3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 3\mathbf{v}$ So $\mathbf{v}$ is an eigenvalue of A with eigenvalue 3.

9. (10 points) Let W be the span of the set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \right\}$. Find a subset of S which is a basis for W. What is the dimension of W?

Solution: Form a matrix A with the elements of S as columns, and row reduce:

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & 1 & -1 \\ 0 & 2 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

W is the column space of the matrix A. Since the rank of A is 2, W has dimension 2. The pivot columns of A give a basis of the column space of A, so $\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right)$ is a basis of W which is a subset of S.

10. (15 points) Remember that P_2 is the vector space of polynomials in the variable x of degree at most 2. The rule $L(p(x)) = xp'(x)$ defines a linear transformation from P_2 to P_2 ($p'(x)$ is the derivative of $p(x)$). The standard basis of P_2 is $S = \left(1,\ x,\ x^2\right)$ and another basis is $X = \left(1 + x,\ x + x^2,\ 1 - x^2\right)$.
Find the matrix representation ${}_S L_X$.

Solution:

First, an apology: This problem statement is incorrect. We did not notice the error until the end of the test, when it was too late to correct it. The students who followed the standard procedure for calculating ${}_SL_X$ got full credit for producing the matrix that we describe below. A few students ran into difficulties because they were actually trying to calculate K_X or because they were incorrectly trying to calculate ${}_XL_S$. We took our mistake into account when we graded the problem. If you have questions you should talk to your instructor.

The mistake in the problem is that the set X is not a basis for P_2. For example, $(1+x)-(x+x^2)=1-x^2$, so X is not linearly independent. Therefore it is not possible to define the coordinate function K_X, since this relies on the unique representation of each element of P_2 as a linear combination of the elements of X. And therefore the matrix representation ${}_SL_X$ does not exist, since it is defined as the matrix corresponding to $K_S \circ L \circ K_X^{-1}$.

However, it is still possible to interpret any vector $\mathbf{c}$ in $\mathbb{R}^3$ as a list of coefficients to use in forming a linear combination of the elements of X. This defines a linear transformation J_X by the rule $J_X(\mathbf{c}) = c_1(1+x)+c_2(x+x^2)+c_3(1-x^2)$. This is just what K_X^{-1} would do, *if it actually existed.*

The solution below actually computes the matrix representation of $K_S \circ L \circ J_X$ rather than ${}_SL_X$.

To find each column of the matrix representation of $K_S \circ L \circ J_X$ just calculate the result of applying $K_S \circ L \circ J_X$ to the corresponding unit vector:

- $J_X(\mathbf{e}_1) = 1+x$, the first element of the set X. Then $L(1+x) = x \cdot 1 = x$, and $x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2$, so the coordinate vector of x in the standard basis is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.
- $J_X(\mathbf{e}_2) = x+x^2$, the second element of the set X. Then $L(x+x^2) = x \cdot (1+2x) = x+2x^2$, and $x+2x^2 = 0 \cdot 1 + 1 \cdot x + 2 \cdot x^2$, so the coordinate vector of $x+2x^2$ in the standard basis is $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.
- $J_X(\mathbf{e}_3) = 1-x^2$, the third element of the set X. Then $L(1-x^2) = x \cdot (-2x) = -2x^2$, and $-2x^2 = 0 \cdot 1 + 0 \cdot x + (-2) \cdot x^2$, so the coordinate vector of $-2x^2$ in the standard basis is $\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$.

Putting these columns together gives the answer: The matrix representation of $K_S \circ L \circ J_X$ $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & -2 \end{bmatrix}$.