Math 330 notes and problems, Fall, 2008

X1. Joe decided that he could construct a number system with exactly 4 elements, namely $0, 1, 2, 3$. (This means that $0, 1, 2, 3$ are all different, so $3 \neq 1$, etc.) His system is supposed to satisfy all the axioms of Chapter 1, but with $4 = 0$. In order to describe his system he produced operation tables for addition, multiplication, and negation. Here is the table for addition:

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Fill in the tables for negation and multiplication:

x	$-x$
0	
1	
2	
3	

$\cdot$	0	1	2	3
0				
1				
2				
3				

Joe's number system violates one of the axioms from Ch. 1. Which one?

X2. Jane had a strange idea about doing arithmetic with 4 elements: perhaps the four elements **aren't** $0, 1, 2, 3$. She invented a system with exactly 4 elements, $0, 1, \Box, \Diamond$. Her system **does** satisfy all the axioms of Ch. 1. Complete the following operation tables:

$+$	0	1	$\Box$	$\Diamond$
0				
1		0		
$\Box$				
$\Diamond$				

x	$-x$
0	
1	
$\Box$	
$\Diamond$	

$\cdot$	0	1	$\Box$	$\Diamond$
0				
1				
$\Box$			$\Diamond$	
$\Diamond$				

An integer n is said to be *even* if it is divisible by 2. This means that there is an integer q satisfying $n = 2q$. An equivalent statement is that the equation $2x = n$ has a solution x in the integers.

An integer is said to be *odd* if it is not even.

X3. If m and n are even then $m + n$ and mn are even.

X4. Let B be the set of all natural numbers m which satisfy the condition that $m^2 + m$ is even. That is,

$$B = \left\{ m \in \mathbb{N} \colon m^2 + m \text{ is even} \right\}$$

Prove:

(i) $1 \in B$.

(ii) If $k \in B$ then $k + 1 \in B$.

Now, using Proposition 3.5, prove that $m^2 + m$ is even if m is any natural number.

X5. Let C be the set of all natural numbers m that satisfy the following condition: Either m is even or $m - 1$ is even. That is,

$$C = \{ m \in \mathbb{N} \colon m \text{ is even or } m - 1 \text{ is even} \}$$

Prove:

(i) $1 \in C$.

(ii) If $k \in C$ then $k + 1 \in C$.

Now, using Proposition 3.5, prove that any natural number is even or is equal to an even integer plus 1.

Here's a "mathematical" variation on the "all horses in a herd have the same color" argument. Say that two numbers *have the same parity* if they are both even or both odd. Say that *all numbers in a set A have the same parity* if each pair of numbers chosen from the set A have the same parity.

False Theorem 1. *Suppose $n \in \mathbb{N}$. If S is set of integers with n elements then all elements of S have the same parity.*

Proof. This is a proof by induction on n. The statement that we are proving is

$P(n)$: If S is a set of integers with n elements then all elements of S have the same parity.

First, the initial case: $P(1)$ is true since if a set has only one element then any two elements in the set are identical, so they have the same parity.

Now here is the induction step. We assume that $P(k)$ is true for some $k \in \mathbb{N}$. Note that $P(k)$ applies to *any* set with k elements. We need to show that $P(k+1)$ is true, so we start with *any* set of natural numbers, say T, with $k+1$ elements.

Let x be the largest element of T. If we remove x from T then the result is a set S_0 having k elements. The inductive hypothesis applies to S_0 and we can conclude that all elements of S_0 have the same parity. For definiteness, suppose that all the elements of S_0 are even. (The alternative is that all the elements of S_0 are odd, and it is easy to change the rest of the proof to handle this case.)

Now return the element x to S_0 and then remove the smallest element. This results in a set S_1 with k elements: One of them is x and the others are all even. The inductive hypothesis applies to S_1 and we can conclude that all the elements of S_1 have the same parity. Since all but x are even, we conclude that all the elements of S_1, including x, are even.

Now all the elements of S_0 are even and x is even, and this covers all the elements of T. Hence all the elements of T have the same parity.

This completes the proof that $P(k)$ implies $P(k+1)$.

Hence, by the Principle of Mathematical induction, $P(n)$ is true for all natural numbers n. ☐

X6. The proof above is, of course, INCORRECT, since the Theorem is FALSE. Find the error.

X7. Here are a few more facts about even and odd integers.

(a) **X5** shows that any natural number m has the form $2k$ or $2j+1$ where k and j are integers. Show that this is also true if m is any integer.

(b) Show that the alternatives in **X5** are exclusive. That is, an integer cannot be both of the form $2k$ and of the form $2j+1$. [As a consequence, a number is odd if and only if it has the form $2j+1$.]

(c) **X3** shows that mn is even if m and n are even, and a similar proof shows that mn is even if either m or n is even. Suppose mn is even and show that either m or n is even.

X8. Suppose x is an integer and let D_x be the set of all integers which are divisible by x. Suppose x divides y, and show that $D_y \subseteq D_x$. Is the converse true?

X9. Using the terminology of **X8**, show that $D_2 \cap D_3 = D_6$. Does this suggest a generalization about $D_x \cap D_y$? Is your generalization true?

X10. Using the terminology of **X8**, show that $D_{12} \cup D_{30} \subseteq D_6$ but $D_{12} \cup D_{30} \neq D_6$. Is it ever true that $D_x \cup D_y = D_z$?

X11. Suppose $x \in \mathbb{N}$ and p is a prime. Show that there is a unique factorization $x = p^k x_1$ where $k \in \mathbb{Z}_{\geq 0}$ and p does not divide x_1. There are two parts to this:

(a) Uniqueness: If $x = p^k x_1$ and $x = p^j x_2$ so that x_1 and x_2 are not divisible by p then $k = j$ and $x_1 = x_2$. To prove this you may assume $j \leq k$ (since a symmetric argument will work for $k \leq j$). Then cancel p^j from both sides in $p^k x_1 = p^j x_2$. Now what?

(b) Existence: Define x_1 to be the *smallest* natural number such that $x = p^k x_1$ holds for some k. That is, you are using the Well Ordering Principle; check that it is applicable, and then check that p does not divide x_1.

X12. Write $x = p^k x_1$ and $y = p^j y_1$ using the factorization of **X11**. Show that y divides x if and only if $0 \leq j \leq k$ and y_1 divides x_1.

X13. Our goal is to define the function N, so that $N(x)$ is the number of positive divisors of the natural number x. For example, $N(36) = 9$ because the divisors of 36 are $1, 2, 3, 4, 6, 9, 12, 18, 36$. Explain informally, using **X12**, why the following is a correct recursive definition for N:

$$N(p^k) = k+1, \quad N(p^k x_1) = N(p^k)N(x_1)$$

where p is a prime and $x = p^k x_1$ is the factorization of **X11**. [This uses the form of recursive definition based on "strong induction".]

X14. Prove $N(xy) \leq N(x)N(y)$, using the recursive definition from **X13** and either strong induction or the Well Ordering Principle. In either case the reduction has the following form: Suppose p is a prime that divides xy. Then $xy = p^{j+k} x_1 y_1$ and $x_1 y_1 < xy$, and so $N(x_1 y_1) \leq N(x_1)N(y_1)$ by the inductive hypothesis. Fill in the details.

X15. Suppose that r is a rational number. The natural number b is defined as the smallest positive integer m satisfying $r = \dfrac{n}{m}$ for some integer n, and then a is defined as $a = rb$, so $r = \dfrac{a}{b}$. We showed in class that a and b have no common factor except ± 1.
Also, suppose c and d are integers, with $d > 0$, so that c and d have no common factor except ± 1, and $r = \dfrac{c}{d}$.

Prove, using strong induction on b, that $b = d$ (and, hence, $a = c$).

The base case, $b = 1$, was done in class. Your job is to assume that $b > 1$ and that the result is true for all natural numbers $b_1 < b$ and all corresponding rational numbers r_1, and then show that the result is true for b. As a suggestion, select a prime p so that p divides b, and let $b_1 = \dfrac{b}{p}$.

X16. Let $A = \left\{ \dfrac{x}{x+1} : x \in \mathbb{R}_{\geq 0} \right\}$. Find $\sup A$ and prove that your answer is correct. This does not require the Completeness Axiom.

X17. Let $B = \left\{ \dfrac{x}{x+1} : x \in \mathbb{Z}_{\geq 0} \right\}$. Find $\sup B$ and prove that your answer is correct. This requires the Completeness Axiom (or one of its consequences).

X18. This is 13.15, with hints.
(a) Prove, by induction, that $(1+b)^n \geq 1 + bn$ if $n \in \mathbb{Z}_{\geq 0}$ and b is a real number and $b \geq -1$.
(b) Suppose $0 < \alpha < 1$. Define $b = \alpha^{-1} - 1$ and show that $\alpha^n \leq \dfrac{1}{1+bn}$.
(c) Show that α^n converges to 0 if $0 < \alpha < 1$.

The followong two sequences were discussed in class. The sequence P_n converges, very slowly, to $\pi/2$, and the sequence Q_n diverges.

X19. Define the sequence P_n recursively as follows:

$$P_1 = \frac{4}{3}, \qquad P_{k+1} = \frac{(2k+2)^2}{(2k+2)^2 - 1} \cdot P_k$$

That is, $P_n = \frac{4}{3} \cdot \frac{16}{15} \cdot \frac{36}{35} \cdot \cdots \cdot \frac{4n^2}{4n^2 - 1}$. For example, $2P_{100} = 3.133787\ldots$

(a) Show that $P_n \leq \frac{4n-1}{n}$ for all $n \geq 1$.

(b) Show that P_n is a bounded increasing sequence and conclude (by citing a theorem from Ch 15) that it converges.

X20. Define the sequence Q_n recursively by

$$Q_1 = \frac{2}{1}, \qquad Q_{k+1} = \frac{2k}{2k-1} \cdot Q_k$$

That is, $Q_n = \frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \cdots \cdot \frac{2n}{2n-1}$. For example, $Q_{100} = 17.7467079\ldots$

(a) Show that $Q_n^2 \geq 2n + 1$ for all $n \geq 1$.

(b) Show that the sequence Q_n is an unbounded sequence and conclude (by citing a theorem from Ch 13) that it does not converge.