CHAPTER 2

Continuous population models

2.1. The basic model

We now reconsider Example 1.1 with the assumption that the birth and death properties occur at the same rate throughout the year. In this case we still have a net growth parameter r, which is again interpreted as the net change in the population per unit of population, per unit of time. In other words, starting with a population x, the change in the population after one unit of time (a year in the rabbit example) is rx. But now we reinterpret this to mean that the population changes at a constant rate of r throughout the year. This means that, in a small interval Δt of time, the change in the population is given by

$$\Delta x = rx\Delta t.$$

In fact, this formula is not correct for large values of Δt, but it becomes more precise as Δt becomes smaller. This leads us to rewrite this equation as

$$\frac{\Delta x}{\Delta t} = rx$$

and then take a limit:

Example 2.1. $\dfrac{dx}{dt} = rx$, where $x = x(t)$ is a real function of t. The parameter r is the instantaneous net growth rate; it can be any real number.

Just as in the discrete case we want to investigate what happens in the future according to this model. However, there is no transition function F, so there is nothing that we can iterate to determine F^t. Rather, we must determine F^t by solving the *differential equation* $\dfrac{dx}{dt} = rx$. We will often use the notation x' for the derivative of x with respect to t, so this differential equation is more compactly written as $x' = rx$.

There is a notational complication here. In the case of a discrete system we interpreted x as a simple variable, giving our population at a fixed time, and then later values of the population were determined as $F^t(x)$, by iteration of the transition function. However, in the differential equation $x' = rx$, x is a *function* of t, not a simple variable. Our interpretation is that, at an initial time $t = 0$ the function x has a fixed value, say $x(0) = x_0$, and then $x(t)$, for later time, represents the population t units of time later. In other words,

$$F^t(x_0) = x(t),) \text{ where } x' = rx \text{ and } x(0) = x_0.$$

Here we resolve the ambiguity in interpreting x by using x_0 to indicate a fixed value of x.

Thus in order to see the future population $F^t(x_0)$ we need to solve the differential equation $x' = rx$, together with the *initial value condition* $x(0) = x_0$.

Here is a procedure for solving this, and many similar initial value problems:

(1) Rewrite the equation $\dfrac{dx}{dt} = rx$ in differential form:

$$dx = rx\,dt.$$

(2) Rearrange the equation so that all references to x, *including* dx, are on the left hand side and all references to t, *including* dt, are on the right hand side:

$$\frac{dx}{x} = r\,dt.$$

(In this case it does not really matter whether the parameter r is kept with the t's or with the x's.)

(3) Find the indefinite integrals of both side. **Do not forget the constant of integration!**

$$\int \frac{dx}{x} = \int r\,dt$$
$$\ln|x(t)| = rt + C$$

Only one constant of integration is necessary: in effect, the two constants are combined into one and put on the right hand side.

(4) Solve for x:

$$|x(t)| = e^{rt+C} = e^C e^{rt}$$
$$x(t) = \pm e^C e^{rt} = Be^{rt}.$$

We defined the constant B as $\pm e^C$ to simplify the expression.

(5) Plug in $t = 0$, remember that $x(0) = x_0$, and solve for the arbitrary constant B in terms of x_0:

$$x_0 = x(0) = Be^{r\cdot 0} = Be^0 = B,$$

so $B = x_0$.

(6) Plug in this value for B in the formula for $x(t)$ and interpret $x(t)$ as $F^t(x_0)$:

$$x(t) = Be^r t = x_0 e^{rt},$$

so $F^t(x_0) = x_0 e^{rt}$.

(7) Finally, since there were some dubious steps in this procedure (remember, we divided by x, which might be 0), check the answer by verifying that $x(0) = x_0$ and that $\frac{d}{dt}x(t) = \frac{d}{dt}e^{rt}$ is equal to $rx(t) = r \cdot x_0 e^{rt}$. When you do this you will find that the solution is valid even when $x_0 = 0$; but the derivation above must exclude $x = 0$ and, also, $x_0 = B = \pm e^C \neq 0$. So we found an extra solution by verifying the answer.

We now have a formula which describes how an initial value of the population changes as a function of t: $F^t(x_0) = x_0 e^{rt}$. As in the discrete case, we interpret both x_0 and t as variables in this expression. However, now t can be any real number, not just an integer.

It is important to realize that the parameter r has a different meaning here than in Example 1.1. For example, in the numeric example following Example 1.1 we used $r = 0.1$ and $x_0 = 1000$ and obtained $1000 \cdot (1 + .1) = 1100$ as the population after one year. However, in our continuous model, the population after one year is $1000 \cdot e^{r \cdot 1} = 1000 \cdot e^{0.1} = 1105.17$.

This difference might be familiar in terms of interest calculations. If we reinterpret 1000 rabbits as 1000 dollars and the island as a savings account, then the net growth rate of .1 corresponds to an interest rate of 10% per year. In the discrete case we consider that this interest is calculated and added to the money in the account once, at the end of one year. A more common practice is to calculate the interest once per quarter; in this case the interest rate per quarter would be $r \cdot \Delta t = 0.1 \cdot \frac{1}{4} = 0.025$, so interest of 2.5% is calculated and added to the account at the end of each quarter. Since the added interest is used when calculating the next interest payment the bank says that this interest is "compounded quarterly"; the effect is that, at the end of four quarters, the amount in the account is $1000 \cdot (1.025)^4 = 1103.81$ instead of 1100. Of course, some accounts might compound the interest monthly, in wich case the amount would be $1000 \cdot \left(1 + \frac{0.1}{12}\right)^{12} = 1000 * (1.008333...)^{12} = 1104.71$ after 12 months. If the interest is compounded daily then the amount after a year is $1000 \cdot \left(1 + \frac{0.1}{365}\right)^{365} = 1105.16$. Many accounts take this subdivision of the year to the limit and say that the interest is "compounded continuously". In this case the amount after one year would be $1000 \cdot e^{0.1} = 1105.17$.

2.2. Continuous dynamical systems

Before modifying this model we isolate the features of the model that constitute a *discrete dynamical system*:

n general, we will work with two variables, x and t. We always think of t as time, but the interpretation of x will depend on the particular application. In general, the values of x are called the *states* of the system. In most cases it will be necessary to specify more than just one number to describe the state of the system, so in most cases we will consider x to be a vector, and we'll refer to it as the *state vector* of the system. The evolution of the system is governed by a differential equation of the form $\frac{dx}{dt} = f(x)$. The general solution of the differential equation expresses x as a function of t together with a constant C of integration. Using the initial value x_0 of x we can write C in terms of x_0, so we can write the formula for $x(t)$ as a function of x_0 and t. This function is written $F^t(x_0)$ and is called the *flow* of the dynamical system. As in the discrete case, F^t transforms the state x_0 at one time to the state $F^t(x_0)$ of the system after t units of time have elapsed. Unlike in the discrete case, time is measured by a real number, not an integer.

In the discrete case the values of $F^t(x_0)$ are calculated by iterating the transition rule F, so there is no problem about defining it as long as the state stays in the domain of the transition rule. In the continuous case we do not calculate $F^t(x_0)$ by a simple iteration procedure, so we shall require some conditions on the differential equation $x' = f(x)$ to ensure that the flow $F^t(x_0)$ can be properly defined.

Just as in the discrete case, we usually need to restrict the set of possible state vectors, either to conform to the physical process that we are modelling or to avoid mathematical difficulties. Also, the differential equation $x' = f(x)$ usually involves parameters, and the qualitative characteristics of the solution may change as the values of the parameters are varied.

Here's the analog of Proposition 1.2.

Proposition 2.2. $F^t \circ F^s = F^{t+s}$.

In fact, this is the same statement as Proposition 1.2, but with a different meaning, since t is now a continuous variable. This version actually requires proof, which we postpone until after discussing the existence and uniqueness theorem.

2.3. Some variations

We now consider the continuous analogs of the two discrete models from section 1.3.

The first variation involves external interaction with the population. We envision a steady migration to (or from) the population. This means that in a single year a number A of rabbits move to (if $A > 0$) or from (if $A < 0$) the island. [This is not too reasonable if we think of the rabbits as beinc confined to an island, but it is not too unreasonable if we consider a group of rabbits that is mostly isolated from the larger population.]

We are thinking of this migration as occurring at a steady rate throughout the year, so if A individuals migrate in one year, then $A\Delta t$ will migrate in a time interval of length Δt. Adding this to our derivation of Example 2.1 gives a change of

$$\Delta x = rx\Delta t + A\Delta t$$

for a small change in time. Dividing by Δt and taking the limit as $\Delta t \to 0$ produces the following model:

EXAMPLE 2.3. $\dfrac{dx}{dt} = rx + A$, so $f(x) = rx + A$. The instantaneous growth rate r and the annual migration rate A can be any real numbers.

We follow the procedure outlined after Example 2.1 to determine the flow $F^t(x_0)$:

(1) Rewrite the equation $\dfrac{dx}{dt} = rx + A$ in differential form:

$$dx = (rx + A)\,dt.$$

(2) Separate the variables:

$$\frac{dx}{rx + A} = dt.$$

(3) Integrate:

$$\int \frac{dx}{rx + A} = \int dt$$
$$\frac{1}{r}\ln|rx + A| = t + C_1.$$

Here C_1 is the constant of integration and the factor of $\frac{1}{r}$ comes from integration using the substitution $u = rx + A$.

(4) Solve for x:

$$\begin{aligned}
\ln|rx + A| &= r(t + C_1) = rt + C_2 && \text{substitute } C_2 = rC_1\\
|rx + A| &= e^{rt+C_2} = e^{rt}e^{C_2} = C_3e^{rt} && \text{exponentiate, substitute } C_3 = e^{C_2}\\
rx + A &= \pm C_3e^{rt} = C_4e^{rt} && \text{substitute } C_4 = \pm C_3\\
x &= -\frac{A}{r} + \frac{C_4}{r}e^{rt} = -\frac{A}{r} + Ce^{rt} && \text{algebra, substitute } C = \frac{C_4}{r}.
\end{aligned}$$

(5) Plug in $t = 0$, $x = x_0$ and solve for C in terms of x_0:

$$x_0 = -\frac{A}{r} + Ce^{r\cdot 0} = -\frac{A}{r} + Ce^0 = -\frac{A}{r} + C$$
$$C = x_0 + \frac{A}{r}.$$

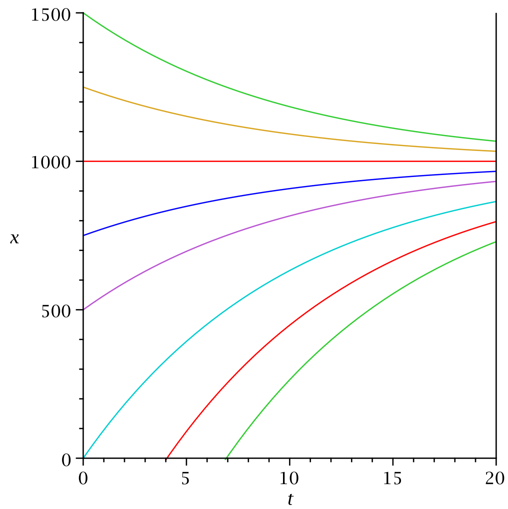

FIGURE 2.1. $x' = rx + A,\ r = -0.1,\ A = 100$

(6) Replace C in the formula for $x(t)$ with its expression in terms of x_0:

$$F^t(x_0) = x(t) = -\frac{A}{r} + Ce^{rt} = -\frac{A}{r} + \left(x_0 + \frac{A}{r}\right) e^{rt}$$

(7) Check your work. Notice that the derivation above assumes that $rx + A \neq 0$, so $x \neq \frac{A}{r}$, so it is a good idea to check that the answer still works if $x_0 \neq \frac{A}{r}$.

We can visualize the flow $F^t(x_0)$: We graph a number of solutions $x(t)$ of the differential equation, for different values of x_0, on the same plot, with a horizontal T-axis and a vertical X-axis. The initial value, x_0, for each curve is the x value where the curve crosses the X-axis. we can then see what F^{t_1} does for a fixed value of t_1: Locate an x_0 value on the vertical line $t = 0$, and follow the solution curve $x(t)$ through this point until it meets the vertical line $t = t_1$.

Figures 2.1 and 2.2 provide such a visualization for our example, the first with a negative growth rate and the second with a positive growth rate. For example, it is clear from Figure 2.1 that, in this case, F^{20} compresses the interval $0 \leq x \leq 1500$ when $t = 0$ to an interval given approximately by $850 \leq x \leq 1050$ when $t = 20$.

Note that in Figure 2.1 it appears that the migration rate $A = 100$ is sufficient to balance the negative net growth rate of -0.1, and any initial population seems to converge to a stable population of 1000. In Figure 2.2 there is a nagative migration rate $A = -100$ and a positive growth rate $r = 0.1$. It is true that the initial population $x_0 = 1000$ remains constant, but any population less than 1000 eventually dies out, while any population greater than 1000 increases without bound.

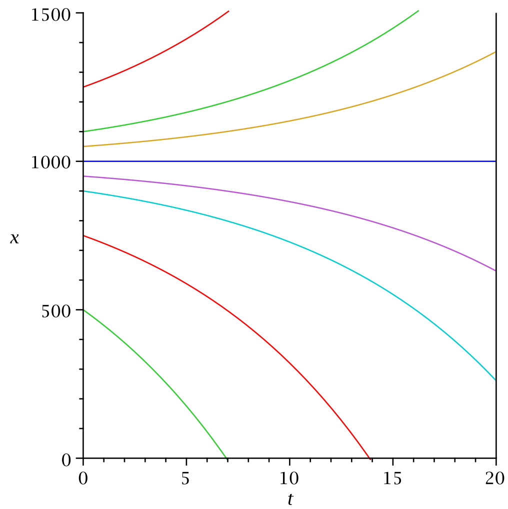

FIGURE 2.2. $x' = rx + A,\ r = 0.1,\ A = -100$

These graphical features can be easily checked by taking the limit as $t \to \infty$ in the formula for the flow. In both examples the quotient $A/r = -1000$, so

$$F^t(x_0) = -\frac{A}{r} + \left(x_0 + \frac{A}{r}\right)e^{rt} = 1000 + (x_0 - 1000)e^{rt}$$

In Figure 2.1 we have $r < 0$ so $e^{rt} \to 0$, and, no matter what the initial condition x_0, we obtain $\lim_{t\to\infty} F^t(x_0) = 1000$. In Figure 2.2 we have $r > 0$ so $e^{rt} \to +\infty$. If $x_0 > 1000$ then $x_0 - 1000$ is positive, so $(x_0 - 1000)e^{rt} \to +\infty$. If $x_0 < 1000$ then $x_0 - 1000$ is negative, so $(x_0 - 1000)e^{rt} \to -\infty$. Since a population cannot be negative we see that, if $0 < x_0 < 1000$, then at some finite time t_* we must have $x(t_*) = 0$. In fact, using logarithms we can solve

$$x(t_*) = 1000 + (x_0 - 1000)e^{0.1 \cdot t_*} = 0$$

to get $t_* = 10 \ln\left(\dfrac{1000}{1000 - x_0}\right)$.

Our second variation on the basic population model is a straightforward reinterpretation of Example 1.4 as a continuous system. The idea there was that the growth rate should decrease with increasing population. The simplest version of such a variable growth rate is $r - \alpha x$, and if we consider this growth rate to be applied at a steady rate throughout the year we obtain

$$\Delta x = (r - \alpha x)x\Delta t$$

for the approximate population change over a small time interval Δt. Dividing by Δt and taking the limit as $\Delta t \to 0$ leads to the following:

EXAMPLE 2.4. $\dfrac{dx}{dt} = x(r - \alpha x)$, so $f(x) = x(x - \alpha x)$. The parameters r and α are both positive; α should be small compared with r.

This differential equation is known as the *logistic equation*.

We now follow our standard procedure to determine the flow $F^t(x_0)$ of the logistic equation.

(1) $dx = x(r - \alpha x)dt$.

(2) $\dfrac{dx}{x(r - \alpha x)} = dt$

(3) Using partial fractions (look it up in your calculus book):

$$\begin{aligned}
\int \frac{dx}{x(r-\alpha x)} &= \int \left(\frac{\frac{1}{r}}{x} + \frac{\frac{\alpha}{r}}{r - \alpha x} \right) dx \\
&= \frac{1}{r} \ln|x| - \frac{\alpha}{r} \frac{1}{(-\alpha)} \ln|r - \alpha x| \\
&= \frac{1}{r} \left(\ln|x| - \ln|r - \alpha x| \right) \\
&= \int dt = t + C_1
\end{aligned}$$

(4) Solve for x, using "difference of logs = log of quotient":

$$\begin{aligned}
\frac{1}{r} \left(\ln|x| - \ln|r - \alpha x| \right) &= t + C_1 \\
\ln|x| - \ln|r - \alpha x| &= rt + rC_1 = rt + C_2 \\
\ln \frac{|x|}{|r - \alpha x|} &= rt + C_2 \\
\frac{|x|}{|r - \alpha x|} &= e^{rt + C_2} = C_3 e^{rt} \\
(*) \qquad \frac{x}{r - \alpha x} &= \pm C_3 e^{rt} = C e^{rt} \\
x &= (r - \alpha x) \cdot C e^{rt} = rCe^{rt} - \alpha x C e^{rt} \\
x + \alpha x C e^{rt} &= rCe^{rt} \\
x(1 + \alpha C e^{rt}) &= rCe^{rt} \\
x(t) &= \frac{rCe^{rt}}{1 + \alpha C e^{rt}}
\end{aligned}$$

(5) To determine C in terms of x_0 just plug $t = 0$ into equation (*): $C = \dfrac{x_0}{r - \alpha x_0}$.

(6) Plug this into the formula for $x(t)$ and simplify:

$$x(t) = \frac{r\frac{x_0}{r-\alpha x_0}e^{rt}}{1 + \alpha\frac{x_0}{r-\alpha x_0}e^{rt}} = \frac{rx_0e^{rt}}{r - \alpha x_0 + \alpha x_0 e^{rt}}$$

This is $F^t(x_0)$. We can get an alternate form by dividing top and bottom by e^{rt}, so

$$F^t(x_0) = \frac{rx_0e^{rt}}{r - \alpha x_0 + \alpha x_0 e^{rt}} \quad \text{or} \quad \frac{rx_0}{(r - \alpha x_0)\, e^{-rt} + \alpha x_0}. \tag{2.1}$$

(7) Check your work. It is not easy to plug $x(t)$ into the differential equation and verify that it works; you might want to use a symbolic math package like Maple or Mathematica to do this. On the other hand, during the derivation we divided by both x and $r - \alpha x$, so it is a good idea to check that the constant solutions $x = 0$ and $x = \dfrac{r}{\alpha}$ both work.

Examining the solution shows that there are two steady state solutions, $x = 0$ and $x = \dfrac{r}{\alpha}$. Figure 2.3 is a sample plot of solution curves.

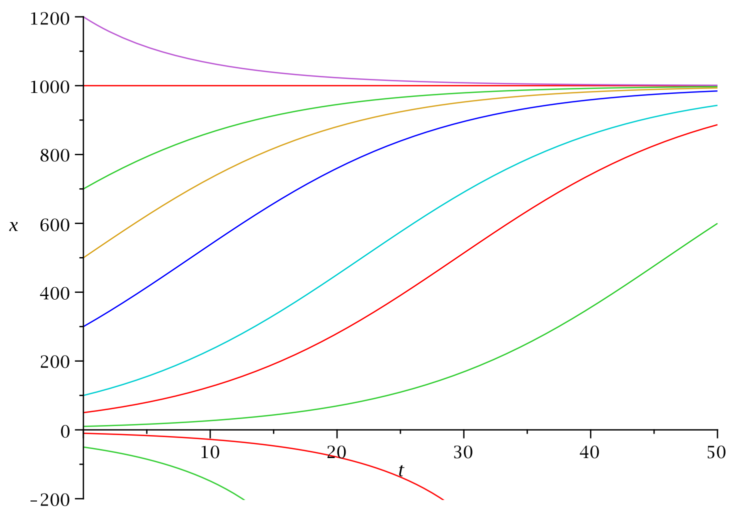

FIGURE 2.3. $x' = x(r - \alpha x),\ r = 0.1,\ \alpha = 0.0001$

From the plot it seems that all positive initial populations eventually converge to the steady state solution $x = \dfrac{r}{\alpha}$. This is correct, and can be verified by taking the limit of the flow (2.1) as $t \to \infty$: Since $r > 0$ we have $e^{-rt} \to 0$ as $t \to \infty$ so, using the second form in (2.1), we have

$$\lim_{t\to\infty} F^t(x_0) = \lim_{t\to\infty} \frac{rx_0}{(r - \alpha x_0)\, e^{-rt} + \alpha x_0} = \frac{rx_0}{0 + \alpha x_0} = \frac{r}{\alpha}$$

This doesn't work if $x_0 = 0$ because of division by zero, and we wouldn't expect this since $x = 0$ is a constant solution.

In the logistic population model the limiting state, $\dfrac{r}{\alpha}$, is called the *carrying capacity* of the system. This is a positive stable population, at which the net growth rate is zero, and any other initial population will decrease (if $x_0 > r/\alpha$) or increase (if $0 < x_0 < r/\alpha$) towards the carrying capacity. This is one explanation for how a population stabilizes over time without external intervention.

There is one strange feature of this limit analysis. This concerns negative initial states x_0. We don't need to worry about these if we are concerned only with this system as a population model, but similar mathematical issues occur in other applications.

Here's the issue: The limit calculations above seem to work just as well if $x_0 < 0$. However, it seems from the plot that the solution $x(t)$ for negative x_0 always decreases, so it stays negative. In fact, it is impossible for such a solution to converge to the carrying capacity, which is possible, for then it would have to cross over the T-axis, and this would violate the basic premise that the system is deterministic, so two different solution curves can't cross.

Here's the resolution: If $x_0 < 0$ then the solution $x(t)$ is *not* defined for all positive values of t. If x_0 is negative then the denominator of the fraction in (2.1) becomes zero at some finite value of t, say when $t = t_*$. This means that the solution *only exists for* $0 \le t < t_*$, so it makes no sense to take the limit as $t \to \infty$. In effect the graph of such a function has a vertical asymptote; $x(t) \to -\infty$ on the left of the asymptote and $x(t) \to +\infty$ on the right of the asymptote. The curve to the right of the asymptote converges to the carrying capacity, but this part of the curve is not part of the solution of the differential equation with initial value x_0, since a solution must be defined and satisfy the differential equation on an entire interval starting at $t = 0$.

This value t_* can be determined by setting the denominator to 0 and solving for t_0, using logarithms. The result is $t_* = \dfrac{1}{r} \ln \left(\dfrac{\alpha x_0 - r}{\alpha x_0} \right)$. (This is defined since x_0 is negative, so the numerator and denominator of the fraction are both negative, so the fraction is positive.) Note that the "lifetime" of the solution depends on the value of x_0; values closer to 0 will be defined for a longer interval of t values, but eventually every solution starting at a negative value blows up.

2.4. Steady states and limit states

Just as in the discrete case we define a steady state to be a solution $x(t)$ that is constant as a function of t. We saw several example of steady states in the population examples.

Steady states (also called *equilibrium states*) are usually the first solutions that we should consider when studying a dynamical system. We can find them even without solving for the flow! To do this, just notice that a solution of the differential equation $fderivxt = f(x)$ is a steady state solution if it is constant as a function of time, and that just means that $\frac{dx}{dt} = 0$. Comparing these two conditions, we see that a steady state solution $x(t) = x_0$ of the differential equation is determined by the solution $x = x_0$ of the equation $f(x) = 0$. This equation just refers to x as a variable, ont as a function of t, so solving $f(x) = 0$ is usually just an exercise in algebra.

For example, the steady states of the logistic equation $x' = x(r-\alpha x)$ are obtained by solving the algebraic equation $x(r - \alpha x) = 0$ for x, and we immediately obtain $x = 0$ or $x = \frac{r}{\alpha}$.

We can also adapt the definition of limit state from the discrete case. A state x_1 is a *limit state* of the dynamical system if there is an initial condition x_0, not equatl to x_1, so that the solution $F^t(x_0)$ converges to x_1 as $t \to \infty$. As in the discrete case we have a relation between limit and steady states:

Proposition 2.5. *Any limit state is a steady state.*

However, we will require more details about the solutions of differential equations before we can prove this.

In one dimensional dynamical systems we can actually determine the limit states without solving the differential equation. As an example, consider again the logistic equation $x' = f(x)$ with $f(x) = x(r - \alpha x)$. We already determined the steady state solutions by solving the equation $f(x) = 0$. Now we need to analyze the sign of $f(x)$ for various ranges of x values. We can do this graphically by plotting $y = f(x)$. This curve will cross the X-axis at the points corresponding to the steady states, and in between these states it will be either positive or negative (presuming that f is a continuous function.)

Figure 2.4 shows $y = f(x)$ with the same parameters as in Figure 2.3, where the carrying capacity is 1000. It is clear form the figure that $f(x) > 0$ if $0 < x < 1000$ and $f(x) < 0$ if $x < 0$ or $x > 1000$.

From this we can produce enough of the information in Figure 2.3 to determine limiting behavior. For example, if $0 < x(t) < 1000$ then $x' = f(x) > 0$ so $x(t)$ is increasing. so any solution curve that starts with $0 < x_0 < 1000$ will continue to

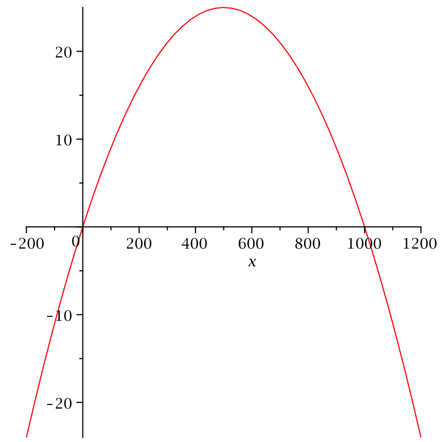

FIGURE 2.4. $y = f(x) = x(r - \alpha x),\ r = 0.1,\ \alpha = 0.0001$

increase as long as it remains less than the steady state solution at $x = 1000$. However, because the dynamical system is deterministic, it is impossible for the solution $x(t)$ to ever cross (or even touch) the steady state solution $x = 1000$. A fundamental fact about the real number system is that any bounded increasing function has a limit, so our solution $x(t)$ must converge to a limit state $x = x_1$, and $x_1 \leq 1000$ because $x(t) < 1000$ for all t. But Proposition 2.5 implies that $x = x_1$ must be a steady state solution. The only two steady state solutions are $x = 0$ and $x = 1000$ and $x_1 \geq x_0 > 0$, so we must have $x_1 = 1000$.

This analysis shows that any solution curve that starts with $0 < x_0 < 1000$ must increase and converge to 1000. A similar analysis shows that any solution curve starting at $x_0 > 1000$ must decrease (because $f(x) < 0$ for $x > 1000$) and must converge to the steady state solution $x = 1000$. We can also conclude that any solution that starts with $x_0 < 0$ must decrease. There is no negative steady state, so such a solution cannot have a finite limit as $t \to \infty$. However, we do not have enough information from this analysis to distinguish between two possibilities: either $x(t)$ is defined for all $t \geq 0$, in which case it will have limit $-\infty$; or it is only defined for a finite range of t values ($0 \leq t < t_*$), in which case it will have a vertical asymptote at $t = t_*$ (this requires some of the theory from the next section).

A more general analysis leads to the same conclusions about the general logistic equation with $f(x) = x(r - \alpha x)$. Solutions with positive initial conditions must converge to the carrying capacity r/α, while solutions with negative initial conditions decrease and do not converge to steady states. Thus the carrying capacity is a limit state, while the other steady state, $x = 0$, is not a limit state.

2.5. Existence and uniqueness

It is hard to establish general facts about continuous dynamical systems without some very basic results. In fact, the very definition of the flow $F^t(x_0)$ relies on the idea that a differential equation $x' = f(x)$ always has a unique solution $x(t)$ satisfying $x(0) = x_0$, for at least some time interval $[0, T)$ with positive T. That is, it must be possible to follow the solution for a non-empty time interval, for otherwise $x(t)$ is useless for predicting the future; and there can't be two different solutions with the same initial state, for then there is no way to decide which solution to follow.

If the differential equation can be solved explicitly, as we did for examples 2.1, 2.3 and 2.4, then we don't need to worry about existence of a solution. But we do need a general existence criterion to guarantee solutions in case we can't find an explicit solution.

It is harder to establish uniqueness of a solution, since we would need to rule out all other possible solutions. Here is an example where we can find explicit solutions of the differential equation satisfying any initial conditions; but the solutions are not unique:

Example 2.6. The differential equation is $\frac{dx}{dt} = f(x)$ with $f(x) = 3x^{2/3}$ has solutions satisfying any initial condition, defined for all $t \in \mathbb{R}$. However the solutions are not unique. In fact, for any pair (x_1, t_1) there are infinitely many solutions $x(t)$ satisfying $x(t_1) = x_1$.

To check these properties, we first follow the standard procedure for solvint $x' = f(x)$:

(1) $dx = 3t^{2/3}\, dt$.

(2) $\frac{dx}{x^{2/3}} = 3dt$.

(3) $\int \frac{dx}{x^{2/3}} = \int x^{-2/3}\, dx = 3x^{1/3} = \int 3dt = 3t + C_1$.

(4) $x^{1/3} = t + C_1/3 = t + C$ so $x(t) = (t + C)^3$.

(5) For $t = 0$, $x(0) = x_0 = (0 + C)^3 = C^3$, so $C = x_0^{1/3}$.

(6) $x(t) = \left(t + x_0^{1/3}\right)^3$.

However, this doesn't allow us to define $F^t(x_0)$ since there are other solutions, so we can't use the rule that $F^t(x_0) =$ *the* solution starting with $x(0) = x_0$, evaluated at time t. In fact, if $x_0 = 0$ then the formula gives $x(t) = t^3$, but there is another solution: $\tilde{x}(t) = 0$ for all t also satisfies the differential equation (just plug it in to the differential equation and check that it works) and satisfies $\tilde{x}(0) = 0$.

So we have two solutions satisfying $x(0) = 0$. To see infinitely many solutions, consider Figure 2.5. The cubics $x = (t+C)^3$ cover all points in the plain (for different

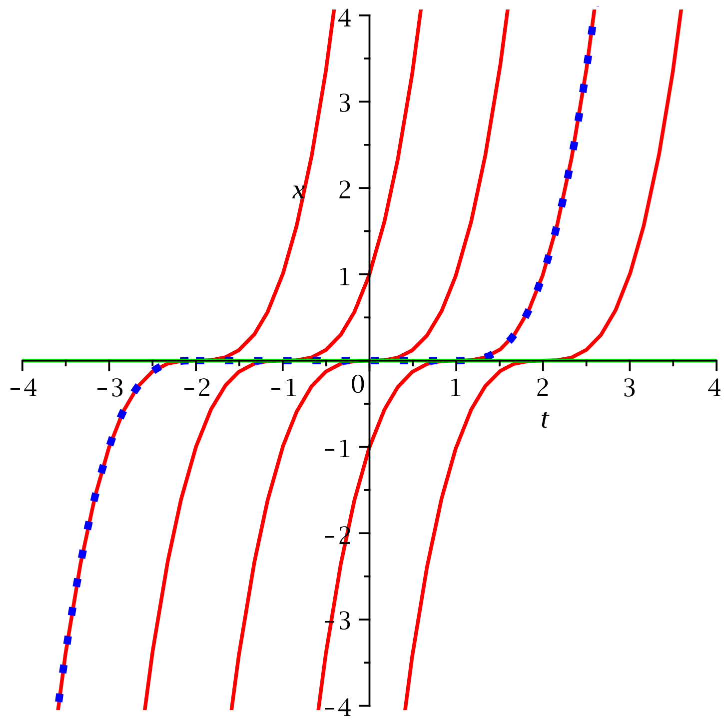

FIGURE 2.5. Solutions of $x' = 3x^{2/3}$

values of C) and they are all tangent to the T-axis. We can find more solutions of $x' = x^{2/3}$ with $x(0) = 0$ as follows: Follow a cubic from $t = -\infty$ until it touches the T-axis at a negative t value; then follow the T asis to a point with a positive t value; then leave the T-axis along the cubic which touches the T-axis there and continue on the cubic as $t \to +\infty$. One such solution is indicated on Figure 2.5.

In general, you can find infinitely many solutions through any point (x_1, t_1) by pasting together three partial solution curves: A "half-cubic" through (t_1, x_1), a segment of the T-axis, and another "half-cubic".

You can see from this example that existence of solutions is not enough to guarantee uniqueness. This is reflected in the general theorems regarding existence and uniqueness: The most general existence theorems require much weaker assumptions about the differential equation than the corresponding uniqueness theorems. However, for our purposes, existence is not good for much withot uniqueness, so our basic theorem imposes conditions which imply both existence and uniqueness.

Theorem 2.7 (Basic Existence and Uniqueness). *Suppose the function f is defined on the interval (a, b) and suppose that its derivative $\frac{df}{dx}$ exists and is bounded on the interval. Then, for any t_1 and any $x_1 \in (a, b)$, the differential equation $\frac{dx}{dt} = f(x)$ has a solution $x(t)$, defined for t in some open interval containing t_1, satisfying $x(t_1) = x_1$. The domain of $x(t)$ extends indefinitely in either direction, or until $x(t)$ "leaves" the interval (a, b). The solution is unique.*

There is a simpler version of this theorem, which is the one one that is almost always used in paractice:

Corollary 2.8 (Existence and Uniqueness). *The conclusions of Theorem 2.7 hold under the assumption that $\frac{df}{dx}$ exists and is continuous on the interval (a, b).*

This is not an immediate corollary of Theorem 2.7, since the assumption does not imply that $\frac{df}{dx}$ is bounded. The argument is based on the fact that (a, b) can be written as an increasing union of closed bounded intervals, and on each closed bounded interval the continuous function f' is bounded (this is a theorem from Calculus). Theorem 2.7 can be applied to each of these closed and bounded intervals, giving parts of the solutions, and the full solutions, for x in (a, b), can be pieced together from these partial solutions.

Here are some examples:

(1) The logistic equation $x' = x(r - \alpha x)$ with $x(0) = x_0$ has unique solutions for all choices of x_0, since $f(x) = x(r - \alpha x)$ has the continuous derivative $r - 2\alpha x$. The solution is not necessarily defined for all t; see the discussion of Example 2.4.

(2) The equation $x' = x + \sin(x)$ has unique solutions for any initial condition, since $x + \sin(x)$ has the continuous derivative $1 + \cos(x)$. However, we can't find a formula for the solution, since that would require evaluating $\int \frac{dx}{x + \sin(x)}$, and this cannot be expressed in terms of elementary functions.

(3) We know that the equation $x' = 3x^{2/3}$ of Example 2.6 has uniqueness problems. The derivative of $f(x) = 3x^{2/3}$ is $f'(x) = 2x^{-1/3} = \frac{2}{x^{1/3}}$, and this is not defined for $x = 0$. Moreover, $f'(x)$ blows up as $x \to 0$, so it is not bounded near 0. However, Corollary 2.8 guarantees unique solutions *if we restrict x to an interval which does not contain* 0. Specifically, there are unique solutions if x is restricted to the interval $(0, \infty)$ (these are the top "half-cubics" in Figure 2.5) or if x is restricted to the interval $(-\infty, 0)$ (the bottom "half-cubics"). In other words, the Theorem only guarantees unique solutions as long as they avoid touching the T axis.

We will leave discussion of the existence of solutions to a later chapter, but we will give a proof of uniqueness, since this proof relies on some inequalities that will be useful later.

A key ingredient in the uniqueness proof is the following inequality:

LEMMA 2.9 (Gronwall's Inequality). *Suppose that M is a constant and that g is a differentiable function defined on an interval containing the value t_1. If $|g'(t)| \leq M|g(t)|$ holds on this interval then*

$$|g(t)| \leq |g(t_1)| e^{M|t-t_1|}$$

PROOF. First we prove a version without the absolute inequalities (this version is also known as Gronwall's inequality). Suppose that h is a differentiable function defined on an interval containing the value t_1, and $h'(t) \leq Mh(t)$ holds on this interval. Let $H(t) = e^{-Mt}h(t)$ and calculate the derivative,

$$(*) \qquad H'(x) = -Me^{-Mt}h(t) + e^{-Mt}h'(t).$$

Now since e^{-Mt} is positive we can multiply it times the inequality $h'(t) \leq Mh(t)$ to get $e^{-Mt}h'(t) \leq e^{-Mt}Mh(t)$. Plugging this into (*) gives $H'(t) \leq -Me^{-Mt}h(t) + e^{-Mt}Mh(t) = 0$. Therefore H is non-increasing, so, if $t > t_1$, $H(t) \leq H(t_1)$. This means $e^{-Mt}h(t) \leq e^{-Mt_1}h(t_1)$. If we multiply this inequality by the positive number e^{Mt} and remember $e^{Mt}e^{-Mt_1} = e^{Mt-Mt_1} = e^{M(t-t_1)}$, we get

$$(**) \qquad h(t) \leq h(t_1)e^{M(t-t_1)}.$$

Now return to the original setup. Since differentiation does not work well with absolute values we use a trick. Define $h = g^2$. Then

$$\begin{aligned} h'(t) = 2g(t)g'(t) &\leq 2|g(t)||g'(t)| \leq 2|g(t)| \cdot M|g(t)| \\ &= 2M|g(t)|^2 = 2M(g(t))^2 = 2Mh(t). \end{aligned}$$

Hence h satisfies the condition for inequality (**), after we replace M with $2M$. Hence, $h(t) \leq h(t_1)e^{2M(t-t_1)}$ for $t \geq t_1$. This is an inequality between non-negative numbers, so we can take square roots of both sides. The square root of $h(t) = (g(t))^2$ is $|g(t)|$, so after taking square roots we have $|g(t)| \leq |g(t_1)| e^{M(t-t_1)}$ if $t \geq t_1$.

Finally, we need to consider the possibility that $t < t_1$. This is really just a matter of "letting time run backward". This will have the effect of changing $g'(t)$ to its negative, but we don't care since we are taking the absolute value. It also has the effect of changing $t - t_1$ in the exponent to $t_1 - t$, and this is why the lemma is stated with $|t - t_1|$ in the exponent. □

The other main ingredient is the following weak version of the Mean Value Theorem (MVT):

LEMMA 2.10. *(Mean Value Inequality) If f is differentiable on the closed interval between x and $\tilde{x}$ and satisfies $|f'(z) \leq M|$ for all z between x and $\tilde{x}$ then $|f(\tilde{x}) - f(x)| \leq |\tilde{x} - x|$.*

PROOF. The MVT states that $\dfrac{f(\tilde{x}) - f(x)}{\tilde{x} - x} = f'(z)$ for some z between x and $\tilde{x}$, assuming $x \neq \tilde{x}$. Multiplying this out gives $f(\tilde{x}) - f(x) = M(\tilde{x} - x)$, and this remains true if $x = \tilde{x}$. Now just take absolute values of both sides and use $|f'(z)| \leq M$. □

It is now easy to give the main idea in the uniqueness part of Theorem 2.7. Suppose that $x(t)$ and $\tilde{x}(t)$ are two solutions of the differential equation $\dfrac{dx}{dt} = f(x)$, and $x(t_1) = x_1$ and $\tilde{x}(t_1) = x_1$. Define $g(t) = \tilde{x}(t) - x(t)$. Since $x(t)$ and $\tilde{x}(t)$ both satisfy the differential equation we have $x'(t) = f(x(t))$ and $\tilde{x}'(t) = f(\tilde{x}(t))$, so $|g'(t)| = |\tilde{x}'(t) - x'(t)| = |f(\tilde{x}(t)) - f(x(t))|$. Since we have a bound on $f'(x)$ we can apply the Mean Value inequality. This gives $|g'(t)| = |f(\tilde{x}(t)) - f(x(t))| \leq M\,|\tilde{x}(t) - x(t)| = M\,|g(t)|$. Now we have the conditions for Gronwall's inequality, so $|g(t)| \leq e^{M|t-t_1|}\,|g(t_1)|$. But $g(t_1) = \tilde{x}(t_1) - x(t_1) = x_1 - x_1 = 0$, so Gronwall's inequality becomes $|g(t)| \leq 0$, so $g(t) = 0$. Since $g(t) = \tilde{x}(t) - x(t)$ we conclude $\tilde{x}(t) = x(t)$, and this is exactly what the uniqueness property means.

Exercises

2.1. Find the flow defined by each of the following differential equations:

(a) $\dfrac{dx}{dt} = rx^2$, where r is a non-zero parameter.

(b) $\dfrac{dx}{dt} = -c^2x^2 + A^2$, where A and c are positive parameters.

(c) $\dfrac{dx}{dt} = c^2x^2 + A^2$, where A and c are positive parameters.

(d) $\dfrac{dx}{dt} = \dfrac{r}{x}$, where r is a non-zero parameter.

(e) $\dfrac{dx}{dt} = \dfrac{r}{x} + A$, where A and r are non-zero parameters.

2.2. This problem refers to Exercise 2.1a, so the differential equation is $x' = rx^2$. Assume $r < 0$; you might want to make a substitution like $r = -s$, so that $s > 0$.

(a) Find all steady state solutions.

(b) Verify that the flow $F^t(x_0)$, as found in Exercise 2.1a, decreases as t increases (assuming x_0 is positive).

(c) If x_0 is positive, explain why $x(t)$ never reaches 0.

(d) How long does it take until x has half its initial value? Your answer will involve both x_0 and r.

2.3. This question concerns the differential equation $\frac{dx}{dt} = f(x) = x^2(x^2 - 8x + 12)$. In the following consider all possible initial values $-\infty < x_0 < \infty$. Do not try to solve the differential equation.

(a) Find the steady state solutions.

(b) Determine the intervals on which $f(x)$ is positive or negative.

(c) Sketch a number of solution curves using this information. Indicate the steady state solutions, and indicate the limiting behavior of the other solutions.

(d) Determine $\lim_{t\to\infty} x(t)$ for all possible initial conditions. Your answer will depend on different ranges of the initial conditions.

2.4. Solving a differential equation is based on integration, so it is not surprising that changing variables is a useful technique.

(a) Find a change of variables of the form $y = x + c$, where c is a constant, that transforms the equation of Example 2.3 into the equation of Example 2.1.

(b) Try changes of variable of the form $z = cx$ where c is a non-zero constant on the equation $\frac{dx}{dt} = -2x + 4x^2$. You should get an equation of the form $\frac{dz}{dt} = Az + Bz^2$.

(i) Can you a choice of c that will make $B = 1$?

(ii) Can you find a choice of c that will make $B = 0$?

(iii) Can you find a choice of c that will make $A = -4$?

2.5. The logistic equation is often transformed to use the percentage of the carrying capacity y rather than the actual population x as the state variable. (Of course you can't do this until you have first determined that there *is* a carrying capacity.) In other words, $x = Cy$ where $C = \frac{r}{\alpha}$ is the carrying capacity. Use this relation between x and y to transform the logistic equation into an equation for y.

2.6. Modify the logistic equation to also incorporate "migration". That is, assume that there is a steady transfer of A individuals per year int or out of the population. (See Example 2.3.) Start with the specific parameters in Figure 2.3 and add migration at the rate of 50 per year (either in or out of the population – your choice). Find the steady state populations (even if negative) and discuss the limiting behavior. Do not solve the differential equation; however, you will want a calculator to solve for the steady states.

2.7. Let $f(x) = \frac{2}{\pi}x - \sin(x)$ and consider the differential equation $x' = f(x)$.

(a) Show that $|f'(x)| \leq 2$ for all x.

(b) Use the Mean Value inequality, applied to the interval between 0 and x, to show that $|f(x)| \leq 2|x|$ for all x.

(c) Use Gronwall's Inequality to show that any solution $x(t)$ of the differential equation satisfies $|x(t)| \leq |x(0)| e^{2|t|}$ for all t.

(d) Can a solution of the differential equation blow up at a finite value of t?

(e) The differential equation has 3 steady states. Find them, either by inspection or by graphing $y = \frac{2}{\pi}x$ and $y = \sin(x)$ and looking for points of intersection.

(f) For some initial conditions x_0 the solution is bounded for all t, and for others it is unbounded as $t \to +\infty$. Determine the values of x_0 in these two cases.