CHAPTER 4

Linear models: continuous version

4.1. The exponential function

There are many definitions of the exponential function e^x for real numbers. It is often defined in Calculus as the inverse of the natural logarithm or as the limit of $\left(1+\frac{x}{n}\right)^n$ as $n \to \infty$; often it is just understood (somewhat circularly) as "e to the x^{th} power". These definitions do not generalize very well.

However, we will find it very useful to work with the following definition:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots \tag{4.1}$$

It is shown in Calculus that this series converges absolutely for all real numbers x and that it coincides with the other definitions of e^x.

The basic idea is that this series definition can be used for any objects x for which we can make sense of powers (x^n for positive integers n), vector space operations (scalar multiplication and addition), and limits. We also need a suitable notion of a multiplicative unit, since the series begins with 1, and we would like x^0 to equal this multiplicative unit.

Here is one situation where we can use this idea: Suppose that A is an $m \times m$ matrix. Then we define the *matrix exponential* by the following formula:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \dots \tag{4.2}$$

where I is the $m \times m$ identity matrix.

This definition will make sense as long as the series converges. This is a series of $m \times m$ matrices, so the partial sums $\sum_{n=0}^{N} \frac{A^n}{n!}$ can be calculated; each partial sum is an $m \times m$ matrix. The infinite series is the limit of these partial sums, and we interpret limits of matrices by taking the limits of the entries. The following is the basic justification for the convergence of the series for e^A, but in practice we will have explicit methods for calculating e^A so this is only useful for reassurance:

PROPOSITION 4.1. *For any square matrix A, each entry in e^A is an absolutely convergent series. Hence the series for e^A converges.*

PROOF. Let M be an upper bound for the absolute values of the entries of A. We need an estimate on the size of the entries in A^n.

Suppose B is another $m \times m$ matrix and L is an upper bound for the entries of B. The ik entry in AB is $\sum_j A_{ij}B_{jk}$, and in absolute value this is bounded by $\sum_j |A_{ij}| \cdot |B_{jk}|$. There are m terms in this sum and each term is bounded by $M \cdot L$, so the sum is $\leq mML$. In other words, each entry of AB is bounded, in absolute value, by mML.

Now if we apply this with $B = A$ we see that each entry of $A^2$is bounded, in absolute value, by mM^2. Repeating the argument with $B = A^2$, we see that each entry of A^3 is bounded, in absolute value, by $m \cdot M \cdot mM^2 = m^2M^3$. Prodeeding by induction we find that each entry of A^n is bounded by $m^{n-1}M^n$. This is a crude estimate, and we can make it even cruder (and simpler): Each entry of A^n is bounded, in absolute value, by $(mM)^n$.

Now consider any entry in the series for e^A. It is a sum of entries from various powers A^n, divided by $n!$. Replacing each term in this series by its absolute value and using the estimate above, we get a series of the form $\sum_{n=0}^{\infty} \frac{(mM)^n}{n!}$. But this is just the series for e^{mM}, and we know that this converges absolutely. Hence, by the comparison test, the series for each entry of e^A converges absolutely. □

Many of the manipulations with matrix exponentials can be reduced to the corresponding facts about real exponentials, and similar estimates are often necessary. We will not give any further details for such limit arguments.

As a first example we calculate e^A where $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. This was considered in Exercise 3.4; the n^{th} power is $A^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$. Then

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \ldots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} + \frac{1}{4!}\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} + \ldots \end{aligned}$$

The $1,1$ and $2,2$ entries can be converted into standard exponentials with some algebraic manipulation:

$$\begin{aligned} A_{11} = A_{22} &= 1 + 1 + \frac{1}{2!} \cdot 2 + \frac{1}{3!} \cdot 2^2 + \frac{1}{4!} \cdot 2^3 + \dots \\ &= 1 + \frac{1}{2}\left(2 + \frac{1}{2!} \cdot 2^2 \frac{1}{3!} \cdot 2^3 + \frac{1}{4!} \cdot 2^4 + \dots\right) \\ &= \frac{1}{2} + \frac{1}{2}\left(1 + 2 + \frac{1}{2!} \cdot 2^2 \frac{1}{3!} \cdot 2^3 + \frac{1}{4!} \cdot 2^4 + \dots\right) = \frac{1}{2} + \frac{1}{2} \cdot e^2. \end{aligned}$$

The $1,2$ and $2,1$ entries are the same, except that they do not have the first 1 (which came from the identity matrix in the $1,1$ and $2,2$ entries). Hence we have $A_{21} = A_{12} = \left(\frac{1}{2} + \frac{1}{2}e^2\right) - 1 = -\frac{1}{2} + \frac{1}{2}e^2$. Putting this together, we have

$$e^A = \exp\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^2 & -\frac{1}{2} + \frac{1}{2}e^2 \\ -\frac{1}{2} + \frac{1}{2}e^2 & \frac{1}{2} + \frac{1}{2}e^2 \end{bmatrix}.$$

Of course, most matrix powers are not as simple as this example, so this is not really a representative result. Even for this case, it is not easy to see what the general form will be.

There is an important case in which it is easy to calculate the matrix exponential. Suppose that Λ is a diagonal matrix, with λ_k in the k,k entry. Then Λ^n is still diagonal, with λ_k^n as the k,k entry. If we put this into the series for e^λ we see that the result is still diagonal, with $\sum_{n=0}^{\infty} \lambda_k^n$ as the k,k entry. In other words,

$$e^\Lambda = \exp\left(\begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_m \end{bmatrix}\right) = \begin{bmatrix} e^{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & e^{\lambda_2} & 0 & \dots & 0 \\ 0 & 0 & e^{\lambda_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e^{\lambda_m} \end{bmatrix}$$

This example becomes most useful if we consider a diagonalizable matrix A. In this case $A = P\Lambda P^{-1}$ for a diagonal matrix Λ and an invertible matrix P. Remembering that $A^n = P\Lambda^n P^{-1}$, we find

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n = \sum_{n=0}^{\infty} \frac{1}{n!} P\Lambda^n P^{-1} = P\left(\sum_{n=0}^{\infty} \frac{1}{n!}\Lambda^n\right) P^{-1} = Pe^\Lambda P^{-1}.$$

Here is another calculation of e^A for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$: The eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 2$, with corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then the matrix $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ has the eigenvectors as columns, and we calculate $P^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Hence

$$e^A = Pe^{\Lambda}P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^0 & 0 \\ 0 & e^2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^2 & -\frac{1}{2} + \frac{1}{2}e^2 \\ -\frac{1}{2} + \frac{1}{2}e^2 & \frac{1}{2} + \frac{1}{2}e^2 \end{bmatrix}.$$

In the future we will need to calculate expressions like e^{tA} and e^{tA} where t is a real number and x is a vector, so we record the following formulas:

Proposition 4.2. *Suppose A is diagonalizable, with eigenvalues $\lambda_1, \ldots, \lambda_m$ and corresponding eigenvectors $v_1, \ldots, v_m$. Let P be the matrix with the eigenvectors as columns and let Λ be the diagonal matrix with the eigenvalues on the diagonal. Then*

(a) $e^{tA} = Pe^{t\Lambda}P^{-1} = P \cdot \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \ldots & 0 \\ 0 & 0 & e^{\lambda_3 t} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & e^{\lambda_m t} \end{bmatrix} \cdot P^{-1}.$

(b) If x is expressed as a linear combination of the eigenvectors, $x = c_1v_1 + \cdots + c_mv_m$, then $e^{tA}x = c_1e^{t\lambda_1}v_1 + \cdots + c_me^{t\lambda_m t}v_m$.

Proof. Since $tA = P(t\Lambda)P^{-1}$, part (a) is just a restatement of our calculations above.

For part (b), let c be the vector with entries c_j. Then Pc is the linear combination of the columns of P with coefficients given by c; but this linear combination is $x = c_1v_1 + \cdots + c_mv_m$. That is, $x = Pc$. Hence $e^{tA}x = P \cdot e^{t\Lambda}P^{-1}Pc = P \cdot e^{t\Lambda}c$. Now this is the linear combination of the columns of P with coefficients given by $e^{t\Lambda}c$, and this is the column vector with entries $e^{t\lambda_j}c_j$. Therefore $e^{tA}x = c_1e^{t\lambda_1}v_1 + \cdots + c_me^{t\lambda_m t}v_m$. ☐

There are many properties of the real exponential function, and many of them are still true for the matrix exponential. Here is a list:

Proposition 4.3. *Suppose A and B are $m \times m$ matrices.*

(a) $e^O = I$, where O is the zero matrix.
(b) $e^{A+B} = e^A \cdot e^B$ if A and B commute; that is, $AB = BA$.
(c) e^A is invertible, with inverse e^{-A}.
(d) $e^{(s+t)A} = e^{sA} \cdot e^{tA}$, if s and t are scalars.

(e) $\dfrac{d}{dt}\left(e^{tA}\right) = A \cdot e^{tA}$.
(f) A *and* e^{tA} *commute.*

Warning: $e^{A+B} = e^A \cdot e^B$ is **false** if A and B do not commute.

Proof. Part (a) is obvious. Part (c) follows from parts (a) and (b), since A and $-A$ commute, so $e^A \cdot e^{-A} = e^{-A} \cdot e^A = e^{A+(-A)} = e^O = I$. Part (d) follows from part (b) since sA and tA commute.

To prove part (b) we multiply the series for e^A and e^B and compare the result to the series for e^{A+B}. Grouping terms of the same degree, and being careful to preserve the order of multiplication,

$$\begin{aligned}
e^A \cdot e^B &= \left(I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \dots\right) \cdot \left(I + B + \frac{1}{2}B^2 + \frac{1}{6}B^3 + \dots\right) \\
&= I + (A+B) + \frac{1}{2}\left(A^2 + 2AB + B^2\right) \\
&\quad + \frac{1}{6}\left(A^3 + 3A^2B + 3AB^2 + B^3\right) + \dots \\
e^{A+B} &= I + (A+B) + \frac{1}{2}(A+B)^2 + \frac{1}{6}(A+B)^3 + \dots \\
&= I + (A+B) + \frac{1}{2}(A^2 + AB + BA + B^2) \\
&\quad + \frac{1}{6}\left(A^3 + A^2B + ABA + BA^2 + AB^2 + BAB + B^2A + B^3\right) + \dots
\end{aligned}$$

It should now be clear why we require that A and B commute: In order to have $e^Ae^B = e^{A+B}$ we would need the quadratic terms toThe exponential function agree; but if

$$(A+B)^2 = A^2 + AB + BA + B^2 \quad \text{equals} \quad A^2 + 2AB + B^2$$

then we must have $AB + BA = 2AB$, so $BA = AB$. Also, it should be clear that we have no reason to expect the terms of higher degree to be equal unless A and B commute.

On the other hand, if A and B commute then the parts of the expansions that we have calculated are the same, and this is true for the rest of the terms, using the binomial formula to expand $(A+B)^n$. [A rigorous proof requires a bit more work, since we need to prove that it is possible to multiply series of matrices as we did above.]

To prove part (e) we differentiate the series for e^{tA} term-by-term; this is justified since it is a power series (in each entry) with infinite radius of convergence:

$$\begin{aligned}\frac{d}{dt}\left(e^{tA}\right) &= \frac{d}{dt}\left(\sum_{n=0}^{\infty}\frac{(tA)^n}{n!}\right) = \frac{d}{dt}\left(\sum_{n=0}^{\infty}\frac{t^nA^n}{n!}\right)\\ &= \sum_{n=1}^{\infty}\frac{nt^{n-1}A^n}{n!} = \sum_{n=1}^{\infty}\frac{t^{n-1}A^n}{(n-1)!}\\ &= A\cdot\left(\sum_{n=1}^{\infty}\frac{t^{n-1}A^{n-1}}{(n-1)!}\right) = A\cdot\left(\sum_{k=0}^{\infty}\frac{t^kA^k}{k!}\right) = Ae^{tA}.\end{aligned}$$

To prove part (f) we notice that A commutes with any power of A, since $A \cdot A^n$ and $A^n \cdot A$ are both equal to A^{n+1}. Hence

$$A\cdot\left(\sum_{n=0}^{\infty}\frac{(tA)^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{t^n\,A\cdot A^n}{n!} = \sum_{n=0}^{\infty}\frac{t^n\,A^n\cdot A}{n!} = \left(\sum_{n=0}^{\infty}\frac{(tA)^n}{n!}\right)\cdot A. \qquad \square$$

In the discussion so far we have assumed that our matrices are real matrices. However, if we want to use eigenvalues and eigenvectors to calculate matrix exponentials we have to be prepared to handle e^{λ} if λ is a non-real eigenvalue. There are no real problems with this – the calculations proceed just as above – except that we need a definition for e^z where z is a complex number. Of course, all we need to do is plug a complex number into the series for the exponential function. The analog of Proposition 4.3 is true for exponentials of complex numbers, and, since complex multiplication is commutative, $e^{z+w} = e^z e^w$ is true for all complex numbers. There are some special rules for exponentials of complex numbers:

PROPOSITION 4.4. *In addition to the analogs of the properties in Proposition 4.3 we have the following. Here z is a complex number and x and y are real.*

(a) $\overline{e^z} = e^{\overline{z}}$.
(b) $e^{x+iy} = e^x e^{iy}$.
(c) Euler's formula: $e^{iy} = \cos(y) + i\,\sin(y)$.
(d) $\left|e^{x+iy}\right| = e^x$.

PROOF. Part (a) follows by conjugating the series for e^z. Part (b) is a consequence of $e^{z+w} = e^z e^w$. Part(d) follows from parts (b) and (c), together with $|\cos(y) + i\,\sin(y)| = \sqrt{\cos^2(y) + \sin^2(y)} = 1$.

Part (c) requires the series expansions of the sine and cosine, plus the fact that the sequence i^n follows the periodic pattern $1, i, -1, -i, 1, i, -1, -i, \ldots$:

$$\begin{aligned} e^{iy} &= 1 + (iy) + \frac{1}{2!}(iy)^2 + \frac{1}{3!}(iy)^3 + \frac{1}{4!}(iy)^4 + \frac{1}{5!}(iy)^5 + \frac{1}{6!}(iy)^6 + \ldots \\ &= 1 + iy + -\frac{1}{2!}y^2 - i\frac{1}{3!}y^3 + \frac{1}{4!}y^4 + i\frac{1}{5!}y^5 - \frac{1}{6!}y^6 + \ldots \\ &= \left(1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 - \frac{1}{6!}y^6 + \ldots\right) + i\left(y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 + \ldots\right) \\ &= \cos(y) + i\sin(y). \end{aligned}$$

□

Here's an example of calculating a matrix exponential when the eigenvalues are complex. Let $A = t\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}t & -t\\ t & t\end{bmatrix}$, where t is a real number. This has eigenvalues $\lambda = t+it$ and $\overline{\lambda} = t-it$, with corresponding eigenvectors $v = \begin{bmatrix}i\\ 1\end{bmatrix}$ and $\overline{v} = \begin{bmatrix}-i\\ 1\end{bmatrix}$. The change of basis matrix is $P = \begin{bmatrix}i & -i\\ 1 & 1\end{bmatrix}$, with inverse $P^{-1} = \begin{bmatrix}-\frac{1}{2}i & \frac{1}{2}\\ \frac{1}{2}i & \frac{1}{2}\end{bmatrix}$. Then we have

$$\begin{aligned} \exp\left(\begin{bmatrix}t & -t\\ t & t\end{bmatrix}\right) &= P\cdot\exp\left(\begin{bmatrix}t+it & 0\\ 0 & t-it\end{bmatrix}\right)\cdot P^{-1} = P\cdot\begin{bmatrix}e^{t+it} & 0\\ 0 & e^{t-it}\end{bmatrix}\cdot P^{-1} \\ &= \begin{bmatrix}i & -i\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}e^t(\cos(t) + i\sin(t)) & 0\\ 0 & e^t(\cos(t) - i\sin(t))\end{bmatrix}\cdot\begin{bmatrix}-\frac{1}{2}i & \frac{1}{2}\\ \frac{1}{2}i & \frac{1}{2}\end{bmatrix} \\ &= \begin{bmatrix}e^t\cos(t) & -e^t\sin(t)\\ e^t\sin(t) & e^t\cos(t)\end{bmatrix} \end{aligned}$$

Exercise 4.2 provides a direct derivation of the general formula for the exponential of the matrix $C = \begin{bmatrix}a & -b\\ b & a\end{bmatrix}$:

$$e^C = \exp\left(\begin{bmatrix}a & -b\\ b & a\end{bmatrix}\right) = e^a\begin{bmatrix}\cos(b) & -\sin(b)\\ \sin(b) & \cos(b)\end{bmatrix} = \begin{bmatrix}e^a\cos(b) & -e^a\sin(b)\\ e^a\sin(b) & e^a\cos(b)\end{bmatrix}, \tag{4.3}$$

which agrees with the calculations above. This formula is convenient for dealing with complex eigenvalues via real canonical form, as in section 3.5.

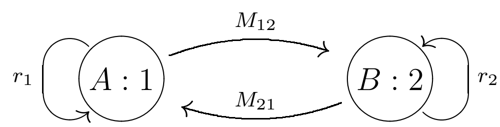
4.2. Some models

We first look at a population model for two (or more) separated populations in which migration occurs, as well as natural population changes doe to births and deaths.

Consider two cities, A and B. For each city there is an intrinsic rate of growth, given by the difference between birth and death rates; these rates may be different. Also, there is a migration rate for individuals moving from A to B, and also a migration rate in the other direction; these rates may be different.

If we number the cities as 1 and 2 then we can use the parameters r_1 and r_2 for the intrinsic growth rates, and we can use M_{12} and M_{21} for the migration rates from 1 to 2 and from 2 to 1. We can visualize this information as the following weighted digraph.

(4.4)



We need to interpret this diagram correctly in order to construct the corresponding dynamical system. We use two state variables, x_1 aand x_2, for the populations of the two cities. If we consider the changes in x_1 and x_2 during a small time interval Δt then we find, approximately,

$$\begin{aligned}\Delta x_1 &= (r_1 x_1 - M_{12}x_1 + M_{21}x_2)\Delta t = (M_{11}x_1 + M_{21}x_2)\Delta t\\ \Delta x_2 &= (r_2 x_2 - M_{21}x_2 + M_{12}x_1)\Delta t = (M_{12}x_1 + M_{22}x_2)\Delta t,\end{aligned}$$

where $M_{11} = r_1 - M_{12}$ and $M_{22} = r_2 - M_{21}$.

Caution: The multiplier of $x_j \Delta t$ is not just r_j, but r_j minus the rate of migration out of city j.

We divide Δx_1 and Δx_2 by Δt to obtain, in matrix terms,

$$\frac{\Delta x}{\Delta t} = \frac{1}{\Delta t}\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix} = \begin{bmatrix}M_{11}x_1 + M_{21}x_2\\ M_{12}x_1 + M_{22}x_2\end{bmatrix} = \begin{bmatrix}M_{11} & M_{21}\\ M_{12} & M_{22}\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

As in the one dimensional case in Chapter 2, this is not correct for large values of Δt, but it becomes more accurate as Δt gets smaller, so we take the limit as $\Delta t \to 0$. So we have our next example:

EXAMPLE 4.5. The state of the system is an m-dimensional vector x representing populations in m cities. The j^{th} city has intrinsic growth rate r_j and the migration rate from the j^{th} city to the k^{th} city is M_{jk}, for $j \neq k$. We assume the migration rates M_{jk} are non-negative. We define M_{jj} to be r_j minus the sum of all migration *out* of the j^{th} city, so

$$M_{jj} = r_j - \sum_{k \neq j} M_{jk}.$$

Finally, we let M be the matrix with entries M_{jk}. Then the dynamical system is defined by the vector differential equation $\frac{dx}{dt} = M^T x$.

As we saw in Example 2.1, the one-dimensional differential equation $\frac{dx}{dt} = ax$, with initial condition $x(0) = x_0$, has the solution $x(t) = e^{at}x_0$. We can't use the method of solution that we used in Example 2.1, since that involved division and we are using vectors, but we can verify that essentially the same solution works:

PROPOSITION 4.6. *If A is a square matrix then the vector differential equation $\frac{dx}{dt} = Ax$, with initial condition $x(0) = x_0$, has the unique solution $x(t) = e^{tA}x_0$.*

PROOF. Define $x(t) = e^{tA}x_0$. Using Proposition 4.3e, we have

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left(e^{tA}\right) \cdot x_0 = A \cdot e^{tA}x_0 = A \cdot x(t),$$

so $x(t)$ satisfies the differential equation. Also, $x(0) = e^O \cdot x_0 = I \cdot x_0 = x_0$, so $x(t)$ satisfies the initial condition.

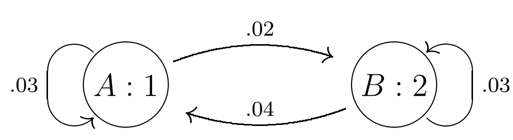
For uniqueness, suppose that $\tilde{x}(t)$ is any other function that satisfies the differential equation and initial conditions, and let $z(t) = e^{-tA} \cdot \tilde{x}(t)$. Since $\tilde{x}(t)$ satisfies the differential equation we have $\frac{d}{dt}\tilde{x}(t) = A\tilde{x}(t)$. Using this plus the product rule and Proposition 4.3e, we find

$$\frac{d}{dt}z(t) = \frac{d}{dt}\left(e^{-tA}\right) \cdot \tilde{x}(t) + e^{-tA} \cdot \frac{d}{dt}\tilde{x}(t) = -A \cdot e^{-tA}\tilde{x}(t) + e^{-tA} \cdot A\tilde{x}(t).$$

Since A and e^{-tA} commute, by Proposition 4.3f, we can simplify this last expression to $-Ae^{-tA}\tilde{x}(t) + Ae^{-tA}\tilde{x}(t) = 0$. Since $\frac{dz}{dt} = 0$, the function $z(t) = e^{-tA} \cdot \tilde{x}(t)$ is a constant. Plugging in $t = 0$ and $\tilde{x}(0) = x_0$, we have $z(0) = e^O \cdot x_0 = x_0$, so $z(t) = e^{-tA} \cdot \tilde{x}(t) = x_0$ for all t. Multiplying by e^{tA} gives $\tilde{x}(t) = e^{tA}x_0$. $\square$

Vector differential equations of the form $\frac{dx}{dt} = Ax$, where A is a (constant) square matrix, are called *constant coefficient linear differential equations*, and these form the main class of differential equations that can be explicitly solved in terms of elementary functions.

Here is a specific case of Example 4.5. We start with the digraph



Following the recipe of Example 4.5, we have $M_{12} = .02$ and $r_1 = .03$, so $M_{11} = .03 - .02 = .01$. Also, $M_{21} = .04$ and $r_2 = .03$ so $M_{22} = .03 - .04 = -.01$. Therefore $M = \begin{bmatrix} .01 & .02 \\ .04 & -.01 \end{bmatrix}$. If we let $A = M^T$ then the differential equation is

$$\frac{dx}{dt} = Ax, \quad A = \begin{bmatrix} .01 & .04 \\ .02 & -.01 \end{bmatrix}. \tag{4.5}$$

To solve this we need to calculate e^{tA}, so we start by diagonalizing A. We find that the eigenvalues are $\lambda_1 = .03$ and $\lambda_2 = -.03$, with corresponding eigenvectors $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. As usual, let P be the matrix with the eigenvectors as columns and let Λ be the diagonal matrix with the eigenvalues on the diagonal. According to Proposition 4.2a, $e^{tA} = Pe^{t\Lambda}P^{-1}$. From this we can calculate

$$e^{tA} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{.03t} & 0 \\ 0 & e^{-.03t} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}e^{.03t} + \frac{1}{3}e^{-.03t} & \frac{2}{3}e^{.03t} - \frac{2}{3}e^{-.03t} \\ \frac{1}{3}e^{.03t} - \frac{1}{3}e^{-.03t} & \frac{1}{3}e^{.03t} + \frac{2}{3}e^{-.03t} \end{bmatrix}.$$

If we write the initial populations as $x_1(0) = p_1$, $x_2(0) = p_2$ then computing $e^{tA}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ gives the flow $x = F^t(p)$:

$$\begin{aligned} x_1 &= \left(\frac{2}{3}e^{.03t} + \frac{1}{3}e^{-.03t}\right)p_1 + \left(\frac{2}{3}e^{.03t} - \frac{2}{3}e^{-.03t}\right)p_2 \\ x_2 &= \left(\frac{1}{3}e^{.03t} - \frac{1}{3}e^{-.03t}\right)p_1 + \left(\frac{1}{3}e^{.03t} + \frac{2}{3}e^{-.03t}\right)p_2. \end{aligned}$$

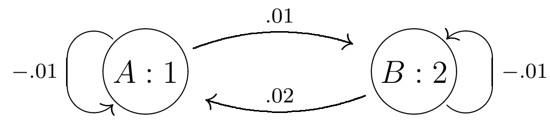
For large t the terms involving $e^{-.03t}$ are negligible, so the solution is approximately

$$x_1 \approx \frac{2}{3}(p_1 + p_2)e^{.03t}, \; x_2 \approx \frac{1}{3}(p_1 + p_2)e^{.03t}.$$

So, for large t, both city populations are growing exponentially, with twice as many in city 1 as in city 2.

A simpler way of seeing this is to use Proposition 4.2b. If the initial vector p is written in terms of the eigenvectors as $p = c_1v_1 + c_2v_2$ then $x = e^{tA}p = c_1e^{.03t} + c_2e^{-.03t}$. From this it is clear that, for large t, $x \approx c_1e^{.03t}v_1$, so $x(t)$ is growing exponentially, at a rate of 3%, in the direction given by $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Here is another "two cities" example:



The corresponding differential equation is

$$\frac{dx}{dt} = Bx, \quad B = \begin{bmatrix} -.02 & .02 \\ .01 & -.03 \end{bmatrix}. \tag{4.6}$$

The matrix B has eigenvalues $\lambda_1 = -.01$ and $\lambda_2 = -.04$, with corresponding eigenvectors $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$. As above we write the initial condition as $x(0) = p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ and then express p as a linear combination of the eigenvectors, so $p = c_1v_1 + c_2v_2$. We can determine c_1 and c_2 from $Pc = p$ by solving for c using row reduction, or by calculating P^{-1}, so that $c = P^{-1}p$. (Since p is arbitrary these two approaches are equivalent, but with a specific choice of p the first is usually simpler.) We obtain $c_1 = \frac{1}{3}p_1 + \frac{1}{3}p_2$, $c_2 = -\frac{1}{3}p_1 + \frac{2}{3}p_2$. Then the solution is $x = e^{tB}p = c_1e^{-.01t}v_1 + c_2e^{-.04}v_2$. From this it is clear that the populations in the two cities are dying out. For large t the two exponentials $e^{-.01t}$ and $e^{-.04t}$ approach zero, but the second goes to zero faster. Hence, for large t, $x \approx c_1e^{-.01t}v_1 = c_1e^{-.01t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, so the population is very small and the first city has approximately twice as many individuals as the second.

Some simple physics problems lead to constant coefficient linear differential equations. The basis for many physical applications is Newton's Second Law, which says that the position of a simple object of mass m satisfies

$$m\frac{d^2x}{dt^2} = F \tag{4.7}$$

where F is the force acting on the object. The force is generally a function of the position x and the velocity $v = \dfrac{dx}{dt}$, and sometimes depends on t. (We will continue to postpone consideration of systems that depend explicitly on t to a later chapter.) In order to specify a single solution $x(t)$ we must specify initial values for both $x(0)$ and for $\dfrac{d}{dt}x(0)$.

We will always convert higher order differential systems into first order systems by introducing extra state variables to represent derivatives. In case of Newton's law we can use both the position and the velocity v to represent the state of the system.

In this case we can replace the second order system (4.7), where x is the position vector, by the first order system

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ \frac{1}{m}F \end{bmatrix}. \tag{4.8}$$

An equivalent formulation is to represent the system in terms of the position and the *momentum* mv.

One of the very first problems encountered in elementary physics is the following example.

EXAMPLE 4.7. This is a model of an object moving in one dimension, under the influence of a spring. The position x is the displacement of the object from its rest position (where the spring is neither stretched nor compressed) and the force exerted by the spring is proportional to the displacement and acts to move the object bact towards its rest position. We assume unit mass, so $m = 1$, and we write, traditionally, k^2 for the spring constant, so the equations are

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ -k^2 x \end{bmatrix} = S \cdot \begin{bmatrix} x \\ v \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix}.$$

To solve this we first find the eigenvalues and eigenvectors: We find $\lambda_1 = ki$, $\lambda_2 = -ki$ and $v_1 = \begin{bmatrix} -i \\ k \end{bmatrix}$, $v_2 = \begin{bmatrix} i \\ k \end{bmatrix}$. As usual we form the diagonal matrix of eigenvalues Λ and the change of basis matrix P and calculate $e^t S$. In the calculation we use Euler's formula to evaluate $e^{t\lambda_k}$:

$$\begin{aligned} e^{tS} &= P \cdot e^{t\Lambda} \cdot P^{-1} = \begin{bmatrix} -i & i \\ k & k \end{bmatrix} \cdot \begin{bmatrix} e^{ikt} & 0 \\ 0 & e^{-ikt} \end{bmatrix} \cdot \begin{bmatrix} -i & i \\ k & k \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -i & i \\ k & k \end{bmatrix} \cdot \begin{bmatrix} \cos(kt) + i\sin(kt) & 0 \\ 0 & \cos(kt) - i\sin kt \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2}i & \frac{1}{2k} \\ -\frac{1}{2}i & \frac{1}{2k} \end{bmatrix} \\ &= \begin{bmatrix} \cos(kt) & \frac{1}{k}\sin(kt) \\ -k\sin(kt) & \cos(kt) \end{bmatrix} \end{aligned}$$

Hence the flow is given by

$$\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \cos(kt) & \frac{1}{k}\sin(kt) \\ -k\sin(kt) & \cos(kt) \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} x_0\cos(kt) + \frac{1}{k}v_0\sin(kt) \\ -kx_0\sin(kt) + v_0\cos(kt) \end{bmatrix}.$$

This kind of solution in general is called a *simple harmonic oscillator.*

Notice that every solution in Exampleex:spring is periodic, with period $\frac{2\pi}{k}$. Hence any solution will repeat forever. This is an idealized model; in the real world there

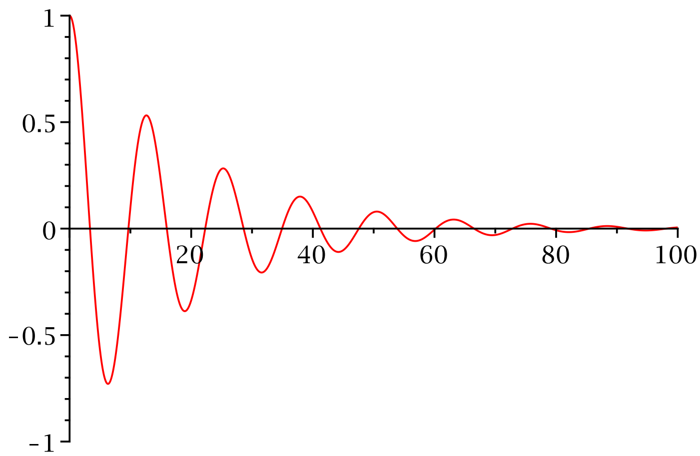


FIGURE 4.1. $x(t) = c_1 e^{-.05t} \cos(.497t)$

are always *dissipative* forces at work. These forces include such things as friction, air resistance, metal fatigue, etc., and always work to retard the motion of a physical system, converting its energy into heat. Here is a modified spring model that gives one approach for taking such forces into account.

EXAMPLE 4.8. In some cases a first approximation to frictional forces is a force which opposes the motion and is proportional to the velocity. We modify Example 4.7 by adding term $-rv$ to the force, where r is a positive constant. Hence our differential equation becomes

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ -k^2 x - rv \end{bmatrix} = T \cdot \begin{bmatrix} x \\ v \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 1 \\ -k^2 & -r \end{bmatrix}.$$

It turns out that the behavior of this system is different for different values of the parameters, so we give a numeric example: Let $k = .5$ and $r = .1$. We then calculate the eigenvalues as $\lambda = -0.05 + 0.497i$ and $\overline{\lambda} = -0.05 - 0.497i$; the corresponding eigenvectors are $z = \begin{bmatrix} -0.05 - 0.497i \\ .25 \end{bmatrix}$ and $\overline{z} = \begin{bmatrix} -0.05 + 0.497i \\ .25 \end{bmatrix}$. Rather than finish solving for $x(t)$ (which will be messy) we consider the form of the solution. Eventually $x(t)$ appears as a linear combinations of the complex exponentials $e^{\lambda t}$ and $e^{\overline{\lambda} t}$. Using Euler's formula, we can replace these complex exponentials with $e^{-.05t}(\cos(.497t) \pm \sin(.497t))$, and these are in turn linear combinations of $e^{-.05t}\cos(.497t)$ and $e^{-.05t}\sin(.497t)$. When we are all finished, there will only be real coefficients, since $x(t)$ is real. Hence

$$x(t) = c_1 e^{-.05t}\cos(.497t) + c_2 e^{-.05t}\cos(.497t) = e^{-.05t}\left(c_1 \cos(.497t) + c_2 \sin(.497t)\right)$$

for some constants c_1 and c_2. The velocity $v(t)$ will also have this general form; but we can calculate it more quickly from $x(t)$ by differentiating, since $v = \frac{dx}{dt}$.

We can now see how $x(t)$ behaves as time increases: It is the product of a periodic function, $c_1 \cos(.497t) + c_2 \sin(.497t)$, with period $\frac{2\pi}{0.497}$, times an exponential $e^{-.05t}$ which converges to 0 as $t \to +\infty$. This system is an example of a *damped harmonic oscillator*: In each period $x(t)$ comes back almost to its starting position, but with a smaller value. For large t the object is esentially stationary at the rest position of the spring. See Figure 4.1 for a sample.

4.3. Phase portraits

A graph like Figure 4.1 can help to visualize a dynamical system, but it shows only a small part of the picture. Usually we can get considerable insight into the overall behavior of a dynamical system by looking at another type of graph.

Suppose we have a vector differential equation $fderivxt = f(x)$. This defines a flow F^t as in the one-dimensional case: $F^t(x_0)$ is equal to $x(t)$ where $x(t)$ is the solution of the differential equation which satisfies the initial value condition $x(0) = x_0$. The existence of such a flow requires an m-dimensional version of the Existence and Uniqueness Theorem. We will postpone discussion of this until the next chapter; it is valid for all systems that we will study. The analogs of Proposition 2.2 and 2.5 are also true.

If x_1 is a fixed state vector then we define the *orbit* or *trajectory* passing through x_1 to be the set of all points that are connected to x_1 by the flow. That is, x_2 is on the orbit through x_1 if $F^{t_1}(x_1) = x_2$ for some time value t_1. We note some general properties of orbits:

PROPOSITION 4.9. *For any state vectors x_1 and x_2:*
(a) x_1 is on the orbit through x_1.
(b) If two orbits intersect then they are the same set.

PROOF. x_1 is on the orbit through x_1, since $F^0(x_1) = x_1$.

For the second part, suppose that the orbits through x_1 and y_1 have some state in common – say z. Then if y_2 is any point in the orbit through y_1 then we can reach y_2 from x_1 as follows: Since z is on the orbit through x_1 then $F^t(x_1) = z$ for some t. Since z is on the orbit through y_1 then $F^s(y_1) = z$ for some s. Since y_2 is on the orbit through y_1 then $F^u(y_1) = y_2$ for some u. Now use Proposition 2.2, twice:

$$F^{u-s+t}(x_1) = F^{u-s}(F^t(x_1)) = F^{u-s}(z) = F^{u-s}(F^s(y_1)) = F^{u-s+s}(y_1) = F^u(y_1) = y_2.$$

This means that y_2 is on the orbit through x_1. Since y_2 was an arbitrary point of the orbit through y_1 this means that the orbit through x_1 contains the orbit through

y_1. If we repeat this argument, starting with y_1 , we see that the orbit through y_1 contains the orbit through x_1. So the two orbits are the same set. □

Note: You may have seen this kind of argument before. It uses the relation $x_1 \sim x_2$ defined by the condition that $F^t(x_1) = x_2$ for some t. It follows from Proposition 2.2 that this relation is an *equivalence relation*, and the orbits are the *equivalence classes.*

In practical terms, all that Proposition 4.9 says is that the curves which are described parametrically in state space by the solutions of the differential equation are the orbits. A description of all the orbits is called the *phase portrait* of the dynamical system.

We start with a simple example. Suppose $\dfrac{dx}{dt} = Hx$ where $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then the solutions are given by $x = e^{tH}x_0$, and, since H is diagonal, we have $e^{tH} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$. Hence, if $x_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, the solutions are given by

$$x_1 = c_1 e^t, \quad x_2 = c_2 e^{-t}.$$

The orbit through $(c_1, c_2) = (1, 1)$ is given parametrically as $x_1 = e^t$, $x_2 = e^{-t}$. We can put this into a more recognizable form by eliminating t between the two equations. If we multiply the equations together then the exponentials will cancel, leaving us with $x_1x_2 = 1$, which we recognize as a hyperbola, asymptotic to the x_1 and x_2 axes. Notice, though, that a hyperbola has two branches (in this case, in the first and fourth quadrants), but the orbit is just the branch of the hyperbola that passes through $(1, 1)$.

We might ask for the orbit through the point $(2, .5)$, but, since this is on the hyperbola through $(1, 1)$, we do not expect a new orbit. To see this algebraically, note that the orbit with $(c_1, c_2) = (2, .5)$ is given parametrically as $x_1 = 2e^t$, $x_2 = .5e^{-t}$, and if we eliminate t by multiplying the equations, we obtain $x_1x_2 = 2 \cdot .5 = 1$, so we still have the same hyperbola.

On the other hand, the orbit through $(.5, .5)$ is given parametrically as $x_1 = .5e^t$, $x_2 = .5e^{-t}$, and if we eliminate t between these equations we are left with $x_1x_2 = .25$, which is a different hyperbola. Again, the orbit is just the branch in the first quadrant.

These orbits are shown in Figure 4.2.

In fact, three other orbits are shown in Figure 4.2.

First, if the initial point is $(c_1, c_2) = (0, 0)$ then the parametric equations become $x_1 = 0 \cdot e^t = 0$, $x_2 = 0 \cdot e^{-t}$, so $x_1(t) = 0$, $x_2(t) = 0$ for all t. That is, this orbit consists **only** of the point $(0, 0)$. This is a special kind of orbit, called a *stationary point* (or

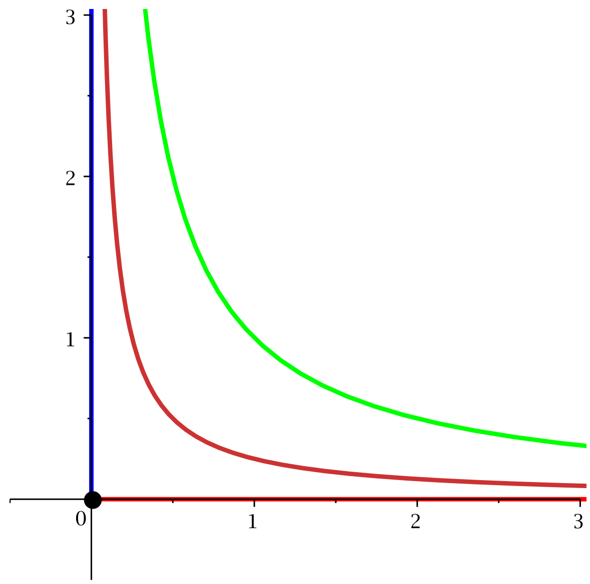


FIGURE 4.2. $x' = Hx$, $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

steady state, as in Chapter 2). When referring to the differential equation $\dfrac{dx}{dt} = f(x)$ such a point is usually called an *equilibrium point*. As in Chapter 2, any such points can be found by solving the equation $f(x) = 0$. In all our examples in this chapter, $f(x) = Mx$ for some matrix M, so the origin will always be an equilibrium point. There may be other equilibrium points. In the next chapter we will look at systems where $f(x)$ is more complicated than just multiplication be a matrix.

Another orbit shown in figure 4.2 is the positive x_1 axis. This corresponds to the initial condition $(c_1, c_2) = (1, 0)$, which leads to the parametric equations $x_1 = e^t$, $x_2 = 0 \cdot e^{-t} = 0$. This represents the positive x_1 axis since x_2 is 0 for all t, while $x_1 = e^t$ ranges over all positive real numbers as t ranges over $(-\infty, \infty)$. Similarly, the positive x_2 axis is an orbit.

The same kind of argument establishes the negative x_1 and x_2 axes as orbits.

Note: These orbits along the axes *do not contain* the origin, since the origin is another orbit and different orbits are disjoint.

When we look at orbits we lose information about the time dependence of the solutions. We can't do much about this, but we can at least indicate in which direction the variable point $(x_1(t), x_2(t))$ moves along the orbit. This can be done by differentiating the parametric equations for the orbit; the result, written as a vector, points in the direction of motion of $(x_1(t), x_2(t))$. For example, the hyperbola through $(1,1)$ has parametric equations $x_1 = e^t$, $x_2 = e^{-t}$, and if we differentiate this we obtain $\frac{dx_1}{dt} = e^t$, $\frac{dx_2}{dt} = -e^{-t}$. Then, for example, the point $(1,1)$ corresponds to $t = 0$ and we have, this point, the vector $(e^0, -e^{-0}) = (1,-1)$. Hence the point $(x_1(t), x_2(t))$ is moving along the hyperbola in this direction when $t = 0$. We conclude that the point $(x_1(t), x_2(t))$ moves from top to bottom (and left to right) along the hyperbola. This is true when $t = 0$, as we just calculated, and, more generally, for all t.

There is an easier way to see the direction of an orbit: We just calculated the vector with coordinates $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$, working with a solution curve as it passes through the point (x_1, x_2). But this is just, in vector terms, $\frac{dx}{dt}$. However, the differential equation is in the form $\frac{dx}{dt} = f(x)$, so we just need to calculate $f(x)$ to find the derivative vector of the solution curve through the point (x_1, x_2). For example, we could calculate the derivative at the point $(1,1)$ by this method, since $f\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = H \cdot \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$.

Using this method, it is easy to see that (x_1, x_2) moves along the second hyperbola in Figure 4.2 in the same direction, from top to bottom (and left to right). The motion along the positive x_1 axis is from left to right, and the motion along the positive x_2 axis is from top to bottom. The origin is an equilibrium point, so it doesn't move at all.

We can also see some limiting behavior. Points on the hyperbolic orbits in the first quadrant go to $+\infty$, asymptotic to the x_1 axis, as $t \to +\infty$, and to $+\infty$, asymptotic to the x_2 axis, as $t \to -\infty$. Points on the positive x_2 asis converge to 0 as $t \to +\infty$, and go to $+\infty$ as $t \to -\infty$. The same kind of thing is true along the positive x_1 axis: points converge to the origin as $t \to -\infty$ and to $+\infty$ as $t \to +\infty$. This is consistent with Proposition 2.5, which says that if a trajectory approaches a limit point in the domain of f then that limit point is an equilibrium point.

Before looking at some more examples we summarize a few facts about phase portraits, in terms of the general equation $\frac{dx}{dt} = f(x)$:

(1) Equilibrium points are orbits; they correspond to solutions of the algebraic equation $f(x) = 0$.
(2) The vector $f(x_1)$ points in the direction of motion along the orbit passing through the ppoint x_1.

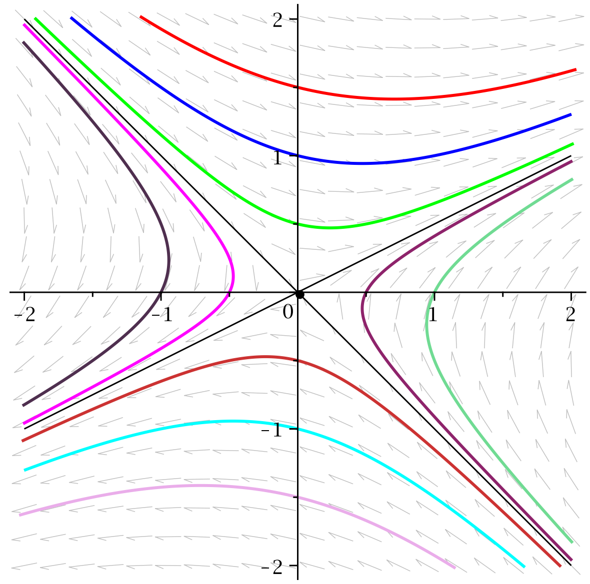


FIGURE 4.3. $x' = Ax$, $A = \begin{bmatrix} .01 & .04 \\ .02 & -.01 \end{bmatrix}$

(3) If a point on an orbit has a limit point x_0 in the domain of f then x_0 is an equilibrium point (assuming the necessary conditions for the Existence and Uniqueness Theorem at x_0).
(4) Orbits are disjoint.

The next example is the "two cities" example (4.5). Several orbits, and the directions of the vector field Ax, are shown in Figure 4.3. This is similar to the picture in Figure 4.2, but the asymptotes of the hyperbolas are at an angle. In fact, the asymptotes are the lines through the eigenvectors v_1 and v_2. The origin is an equilibrium point, since $f(x) = Ax$ where A is a matrix, so $f(0) = 0$. The half lines through the asymptotes are orbits. This is true true for any differential equation of the form $\dfrac{dx}{dt} = Mx$, since, if v is an eigenvector of M corresponding to the eigenvalue λ, then $e^{\lambda t}v$ is a solution of the differential equation. Note that this only gives *positive* multiples of the eigenvector, so the orbit is a half-line. The other half of the line through v is also an orbit, since $-v$ is also an eigenvector, so the set of *positive*multiples of $-v$ is an orbit.

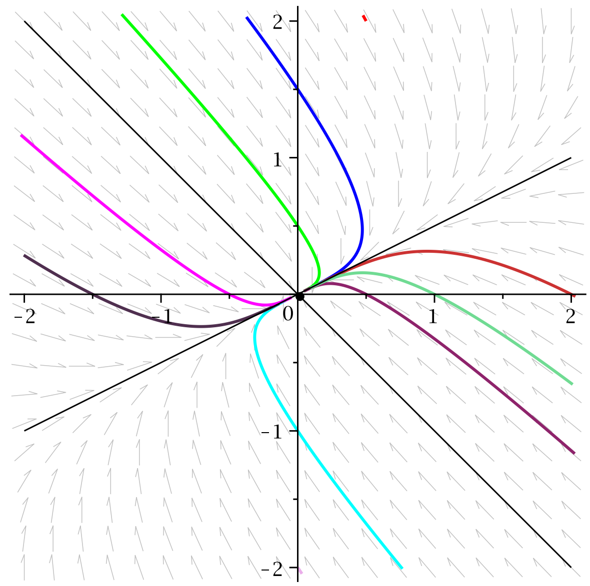


FIGURE 4.4. $x' = Bx$, $B = \begin{bmatrix} -.02 & .02 \\ .01 & -.03 \end{bmatrix}$

From Figure 4.3 we see that all population vectors in the first quadrant move to be asymptotoc to the first eigenvector $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as $t \to +\infty$, as we discovered in our earlier analysis of (4.5).

Equilibrium points that "look like" Figure 4.2 are called *saddle points*. We will explain exactly what this means in the next chapter, but for now we can characterize this picture by the condition that the eigenvalues are real, with one positive and one negative.

The next example is our second "two cities" example, (4.6). Figure 4.4 shows the equilibrium at the origin, and the straight line orbits along the half lines through the eigenvectors. In this example, all orbits converge to the origin as $t \to +\infty$, as we discovered in our earlier analysis of (4.6). As in that analysis, we see that all population vectors in the first quadrant eventually turn and approach the origin asymptotically to the positive line through the eigenvector $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. In fact, all orbits do this except the origin and the two straight line orbits corresponding to the other eigenvector, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

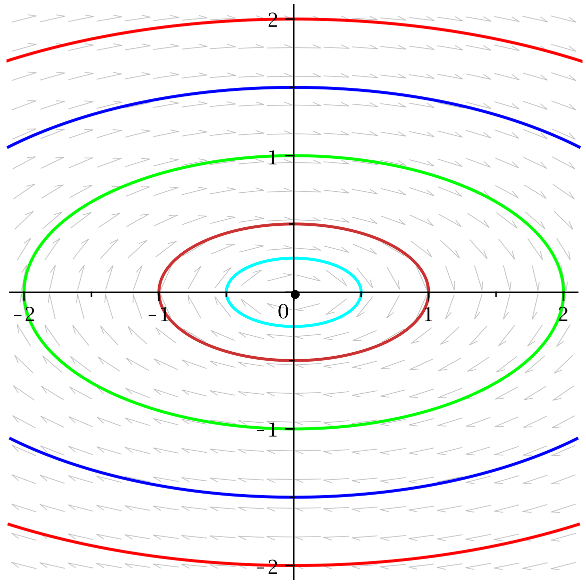


FIGURE 4.5. $x' = Sx$, $S = \begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix}$, $k = .5$

In general, an equilibrium is called a *sink* or *attractor* if all near-by points converge to the equilibrium as $t \to +\infty$. If we reverse the arrows in Figure 4.4 then we will have an example of a *source* or *repellor*, in which all near-by points converge to the equilibrium as $t \to -\infty$.

The spring model, Example 4.7, has very different solutions from the population models. The eigenvectors are purely imaginary, and the solutions are periodic. The phase portrait is shown in Figure 4.5, with the parameter $k = .5$. The origin is still an equilibrium, but there are no straight-line solutions, since the eigenvectors are not real. If $(x(t), v(t))$ is a solution then both $x(t)$ and $v(t)$ are periodic, with period $2\pi/k = 4\pi$. This means that the solution point $(x(t), v(t))$ will return to exactly its starting position after 4π time units; and then it will simply retrace its path. In other words, the orbit is a closed curve. This is the only way a solution curve can ever intersect itself; if it did anything else than retrace its steps then it would violate the uniqueness theorem. e previously calculated that the solution with initial condition $x(0) = x_0$, $v(0) = 0$ is $x(t) = x_0 \cos(kt) = x_0 \cos\left(\frac{1}{2}t\right)$, $v(t) = -kx_0 \sin(kt) = -\frac{1}{2}x_0 \sin\left(\frac{1}{2}t\right)$. Then

$$\frac{x^2}{x_0^2} + \frac{v^2}{x_0^2/4} = \cos^2\left(\tfrac{1}{2}t\right) + \sin^2\left(\tfrac{1}{2}t\right) = 1.$$

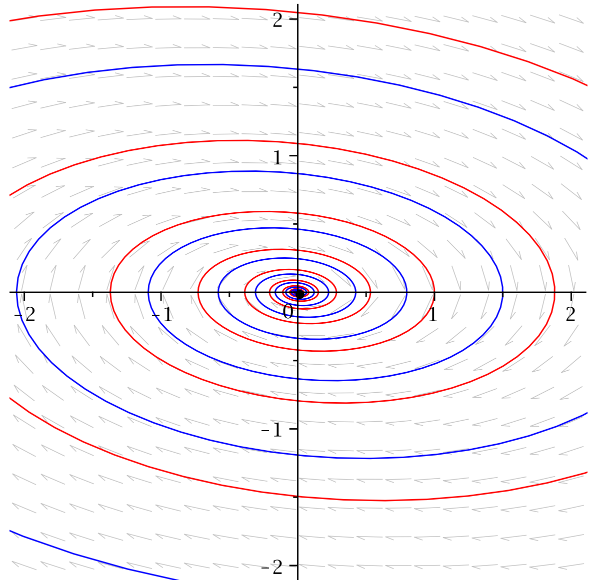


FIGURE 4.6. $x' = Tx$, $T = \begin{bmatrix} 0 & 1 \\ -k^2 & -r \end{bmatrix}$, $k = .5$, $r = .1$

In other words, the orbit through $(x_0, 0)$ is an ellipse, with semi-major axes $|x_0|$ and $\frac{1}{2}|x_0|$.

An equilibrium is sometimes called a *center* if all nearby orbits are closed orbits.

Our last example is the damped oscillator of Example 4.8, with the parameters $k = .5$, $r = .1$. Again the origin is an equilibrium. In our previous analysis we saw that all solutions converge to the origin as $t \to +\infty$, so this is another example of a sink. However, the eigenvalues are not real, and this means that the solutions are given by periodic functions times an exponential factor which converges to 0. This forces the solutions to spiral around the origin as they converge to it, as shown in Figure 4.6.

In this example the origin is called a *spiral sink*. In general, for constant coefficient linear systems, the origin is a sink if all the eigenvalues have negative real part (this includes real eigenvalues that are negative). In two dimensions it is called a spiral sink if, in addition, the eigenvalues are not real. In higher dimensions this spiralling behavior is more complicated, and harder to visualize.

Exercises

4.1. Calculate e^A:

(a) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Use the series.

(b) $A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$. Use the fact that $A = aI + A_1$ where A_1 is the matrix in part (a).

(c) $A = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$. Use eigenvalues and eigenvectors.

4.2. Derive (4.3) by following this outline:

(a) Let $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$. Show that $e^C = e^A \cdot e^B$.

(b) Show that $e^A = e^a I$.

(c) Let $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The powers J^n follow a periodic pattern. What is it?

(d) Since $B = bJ$ we have $B^n = b^n J^n$. Plug this into the series for e^B and use the same idea as in the proof of Euler's formula to show that $e^B = \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}$.

(e) Finish.

4.3. This exercise demonstrates some of the borderline cases of phase portraits for linear systems. For each of the following, prepare a sketch of the phase portrait for $\frac{dx}{dt} = Ax$ and indicate how it compares to the examples in Figures 4.3 – 4.6.

(a) Repeated eigenvalues: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. This doesn't require Maple: analyze the straight-line solutions.

(b) Zero eigenvalue: $A = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}$. First find the equilibrium points.

(c) Repeated eigenvalues: $A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}$. Use Maple.

(d) Repeated zero eigenvectors: $A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$.

4.4. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $T = \operatorname{tr} A = a + d$ (the trace of A) and $D = \det A = ad - bc$ (the determinant of A).

(a) Check that the characteristic polynomial of A is $\lambda^2 - T\lambda + D$.

(b) Use the quadratic formula to solve for the eigenvalues in terms of T and D.

(c) If λ_1 and λ_2 are the eigenvalues then the characteristic equation factors as $(\lambda - \lambda_1)(\lambda - \lambda_2)$. Multiply this out and compare with the formula in (a) to conclude that T is the sum of the eigenvalues and D is their product. (This fact holds for $m \times m$ matrices, for any m.)

4.5. Using the results of Exercise 4.4, determine the conditions on T and D that lead to

(a) Periodic orbits (eigenvalues purely imaginary, not 0).

(b) A saddle (eigenvalues are real, one positive, one negative).

(c) A sink (both eigenvalues have real part less than 0).

(d) A spiral sink (a sink with non-real eigenvalues).

4.6. "Two city" population models like Example 4.5 have the form $\frac{dx}{dt} = Ax$, where A is 2×2 and the terms off the diagonal (corresponding to migration) are non-negative.

(a) Explain why such a system must have real eigenvalues. You might want to use the results of Exercises 4.4 and/or 4.5.

(b) This is harder: Explain why e^{tA} is a positive matrix for all $t > 0$.

4.7. In Example 4.8 the choice of $k = .5$ and $r = 0$ leads to a center, while $k = .5$ and $r = .1$ leads to a spiral sink. In the following, assume $k = .5$.

(a) Find a formula for the eigenvalues of the system in terms of r.

(b) Show that there is a sink at the origin for all $r > 0$.

(c) There is a value r_c so that there is a spiral sink for $0 < r < r_c$, and a non-spiral sink for $r \geq r_c$. Find r_c.

(d) The solution, in case of a spiral sink, involves trig functions times exponential functions. What is the period of the trig functions, as a function of r?