APPENDIX C

Extra problems

C.1. Let $F(x) = x^2 - 2x + 2$. Find the steady states.

C.2. Let $F(x) = \dfrac{x}{2x+1}$, for $x \geq 0$.
(a) Find the steady state. (There is only one; remember $x \geq 0$.)
(b) Calculate $F^t(1)$ for several values of t. You should see a pattern. Can you write a non-recursive formula for $F^t(1)$? What is $\lim_{t\to\infty} F^t(1)$?
(c) Can you generalize part (b) by replacing 1 with any $x > 0$?

C.3. Let $F(x) = \dfrac{x}{x^2+1}$, for any real x.
(a) Find the steady state. (There is only one.)
(b) Show that $F(x) < x$ for all $x \geq 0$.
(c) Why is $\lim_{t\to\infty} F^t(x) = 0$ for any $x \geq 0$?

C.4. Find the flow defined by each of the following defferential equations:
(a) $\dfrac{dx}{dt} = \sec(x)$.
(b) $\dfrac{dx}{dt} = e^{ax}$, where a is a non-zero parameter.
(c) $\dfrac{dx}{dt} = \sqrt{a^2 - x^2}$, where a is a positive parameter and $|x| < a$.
(d) $\dfrac{dx}{dt} = \dfrac{r}{x-a}$ where r and a are non-zero parameters.

C.5. This question concerns the differential equation $\dfrac{dx}{dt} = f(x) = (x-1)^2(x^2-9)$. In the following consider all possible initial values $-\infty < x_0 < \infty$. Do not try to solve the differential equation.
(a) Find the steady state solutions.
(b) Determine the intervals on which $f(x)$ is positive or negative.
(c) Sketch a number of solution curves using this information. Indicate the steady state solutions, and indicate the limiting behavior of the other solutions.

(d) Determine $\lim_{t\to\infty} x(t)$ for all possible initial conditions. Your answer will depend on different ranges of the initial conditions.

C.6. Radioactive isotopes decay according to the basic law $\dfrac{dx}{dt} = -rx$ where x is the amount of the isotope and r is a constant. Here x is measured in grams and t in years. It is known that the most common isotope of radium has a half life of about 1600 years. This means that if $x(t)$ is the solution of the differential equation with initial condition $x(0) = x_0$ then $x(1600) = \frac{1}{2}x_0$.

(a) Calculate r. [Solve the differential equation, then use logarithms. Give a numeric value for your answer.]

(b) Now suppose that, in an initially empty storage facility, 10 grams of this isotope are deposited each year. Modify the differential equation to model this situation. Explain how you arrived at your modifications.

(c) The amount of radium in the storage facility has a finite limit as $t \to \infty$. Find the limit.

(d) How long does it take until the amount of radium reaches 90% of its limit?

C.7. In each of the following, what restrictions are needed on the domain of x so that any initial value problem with x_0 in this domain will have a unique answer? Explain your answer.

(a) $\dfrac{dx}{dt} = (1 - x^2)^{1/3}$.

(b) $\dfrac{dx}{dt} = (1 - x^2)^{4/3}$.

(c) $\dfrac{dx}{dt} = (1 - x^2)^{-1/3}$.

C.8. A is a diagonalizable matrix. Using Maple, I calculated

$$A^{30} = \begin{bmatrix} -11318 & 1617 & 8893 \\ -22582 & 3227 & 17743 \\ -11327 & 1618 & 8900 \end{bmatrix}, \quad A^{31} = \begin{bmatrix} -14127 & 2018 & 11100 \\ -28309 & 4043 & 22242 \\ -14118 & 2017 & 11093 \end{bmatrix} \qquad \text{(rounded)}$$

From this, determine an eigenvalue λ_1 and corresponding eigenvector v_1 for A. You should normalize v_1 so that its third component is 1. You should be able to determine λ_1 to the nearest hundredth. (Do not use anything except a simple calculator, for doing arithmetic.)

C.9. Let $B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$.

(a) What are the eigenvalues and corresponding eigenvectors of B?

(b) Is B diagonalizable?

(c) Calculate a few powers of B. Do you see a pattern? Can you find a non-recursive formula for B^n?

C.10. There are three towns, North Apple, South Apple, Apple Core (abbreviated N, S, C). There is considerable migration between these towns. In each year
from N: 10% move to S, 10% move to C, and the rest stay in N;
from S: 5% move to N, 15% move to C, and the rest stay in S;
from C: 15% move to N, 20% move to S, and the rest stay in C.
This is a very simple model; we are ignoring births and deaths, and migration from or to the outside.
(a) Draw the network for this Markov chain, and write down the corresponding matrix W. Follow the same scheme as in (3.5) and (3.6).
(b) Find the limiting probability distribution p, using the fact that it is an eigenvector of $M = W^T$ corresponding to the eigenvalue 1. Be sure that the entries in p sum to 1.
(c) Initially there are 1000000 people in each of the cities. What do you expect the population in each city to be after many years?

C.11. Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$. Find the solution to $\dfrac{dx}{dt} = Ax$, $x(0) = x_0 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ by the following method:
(a) Find the eigenvalues and eigenvectors.
(b) Find the change of basis matrix P and its inverse.
(c) Find the matrix exponential e^{tA} by calculating $Pe^{t\Lambda}P$ where Λ is the diabonal matrix with the eigenvalues on the diagonal.
(d) Now calculate $x(t) = e^{tA}x_0 = e^{tA} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

C.12. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$. Find the solution to $\dfrac{dx}{dt} = Ax$, $x(0) = x_0 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ by the following method:
(a) Find the eigenvalues, λ_1 and λ_2, and the eigenvectors v_1 and v_2.
(b) x_0 is a linear combination of v_1 and v_2, say $x_0 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = c_1v_1 + c_2v_2$. You know the vectors v_1 and v_2, so this is a system of equations for the unknowns c_1 and c_2 in terms of the "known" constants a_1 and a_2. Solve for c_1 and c_2 by row reduction or another method.
(c) Now $x(t) = c_1e^{\lambda_1}tv_1 + c_2e_{\lambda_2 t}v_2$. You know $v_1, v_2, \lambda_1, \lambda_2$ and you have formulas for c_1 and c_2 in terms of a_1 and a_2. Put it all together.

C.13. For each of the following differential equations, determine whether the equilibrium at the origin is a saddle, a source, a sink, a center, or none of these. In case it is a source or sink, determine whether it is a spiral source or sink.

(a) $\dfrac{dx}{dt} = Ax,\ A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}.$

(b) $\dfrac{dx}{dt} = Ax,\ A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}.$

(c) $\dfrac{dx}{dt} = Ax,\ A = \begin{bmatrix} 2 & -5 \\ 1 & 0 \end{bmatrix}.$

(d) $\dfrac{dx}{dt} = Ax,\ A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}.$

(e) $\dfrac{dx}{dt} = Ax,\ A = \begin{bmatrix} 1 & -5 \\ 2 & -1 \end{bmatrix}.$