Math 471: Selected solutions, Spring, 2010

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p. 17, 2:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\left(2+\frac{2x}{\pi}\right)dx + \frac{1}{\pi}\int_{0}^{\pi} 2\,dx = \frac{1}{\pi}\left[2x+\frac{x^2}{\pi}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[2x\right]_{0}^{\pi} =$$
$$\frac{1}{\pi}\left(0-2(-\pi)-\frac{(-\pi)^2}{\pi}\right)+\frac{1}{\pi}\cdot 2\pi = 2-1+2=3$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\left(2+\frac{2x}{\pi}\right)\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2\cos(nx)\,dx =$$

Use integration by parts for the first integral, with $u = 2+\dfrac{2x}{\pi}$:

$$\frac{1}{\pi}\left[\left(2+\frac{2x}{\pi}\right)\frac{1}{n}\sin(nx)\right]_{-\pi}^{0} - \frac{1}{\pi}\int_{-\pi}^{0}\frac{1}{n}\sin(nx)\cdot\frac{2}{\pi}\,dx + \frac{1}{\pi}\left[2\cdot\frac{1}{n}\sin(nx)\right]_{0}^{\pi} =$$
$$0-\frac{1}{\pi}\left[-\frac{1}{n}\cdot\frac{1}{n}\cos(nx)\cdot\frac{2}{\pi}\right]_{-\pi}^{0}+0 = \frac{2}{n^2\pi^2}\left(1-\cos(n\pi)\right) = 2\cdot\frac{1-(-1)^n}{(n\pi)^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\left(2+\frac{2x}{\pi}\right)\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2\cos(nx)\,dx =$$

Use integration by parts for the first integral, with $u = 2+\dfrac{2x}{\pi}$:

$$\frac{1}{\pi}\left[\left(2+\frac{2x}{\pi}\right)\left(-\frac{1}{n}\cos(nx)\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\int_{-\pi}^{0}\frac{1}{n}\cos(nx)\cdot\frac{2}{\pi}\,dx + \frac{1}{\pi}\left[2\cdot\left(-\frac{1}{n}\cos(nx)\right)\right]_{0}^{\pi} =$$
$$\frac{1}{\pi}\left[2\cdot\left(-\frac{1}{n}\right)-0\right]+\frac{1}{\pi}\left[\frac{1}{n}\cdot\frac{1}{n}\sin(nx)\cdot\frac{2}{\pi}\right]_{-\pi}^{0}+\frac{1}{\pi}\left[2\cdot\left(-\frac{1}{n}\cos(n\pi)\right)-2\cdot\frac{1}{n}\cdot(-1)\right] =$$
$$-\frac{2}{n\pi}+0-\frac{2}{n\pi}\cos(n\pi)+\frac{2}{n\pi} = -\frac{2}{n\pi}(-1)^n = 2\cdot\frac{(-1)^{n+1}}{n\pi}$$

Hence

$$f(x) \sim \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right) = \frac{3}{2}+2\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{(n\pi)^2}\cos(nx)+\frac{(-1)^{n+1}}{n\pi}\sin(nx)\right]$$

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p. 17, 7:

If $0 < x \le \pi$ then $\sin x \ge 0$ so $|\sin x| = \sin x$, and hence

$$\frac{\sin x + |\sin x|}{2} = \frac{\sin x + \sin x}{2} = \sin x = f(x)$$

If $-\pi \le x \le 0$ then $\sin x \le 0$ so $|\sin x| = -\sin x$, and hence

$$\frac{\sin x + |\sin x|}{2} = \frac{\sin x - \sin x}{2} = 0 = f(x)$$

This establishes $f(x) = \dfrac{\sin x + |\sin x|}{2}$ for $-\pi \le x \le \pi$.

According to Prob 10, Sec 5, the Fourier sine series for $\sin x$ on the interval $[0, \pi]$ is $\sin x$. Since the function $\sin x$ is odd, this is also the full Fourier series for $\sin x$ on $[-\pi, \pi]$.

According to Example 2, Sec 7, the full Fourier series for $|\sin x|$ on $[-\pi, \pi]$ is $\dfrac{2}{\pi} - \dfrac{4}{\pi} \displaystyle\sum_{n=1}^{\infty} \dfrac{\cos(2nx)}{4n^2 - 1}$.

Now plug these two series into $f(x) = \dfrac{1}{2} \sin x + \dfrac{1}{2} |\sin x|$ to get the series for $f(x)$ in the book.

X2:

(a) Find the indefinite integral of $x = 2 \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$:

$$\frac{x^2}{2} = C + 2 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \left(-\frac{1}{n} \cos(nx) \right) = C + 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

Then the constant of integration must be $\dfrac{a_0}{2} = \dfrac{1}{2} \cdot \dfrac{2}{\pi} \displaystyle\int_0^{\pi} \frac{x^2}{2} dx = \frac{1}{\pi} \frac{\pi^3}{6} = \frac{\pi^2}{6}$

Alternatively, find the definite integral from 0 to x:

$\dfrac{x^2}{2} = \displaystyle\int_0^x s\, ds = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_0^x \sin(ns)\, ds = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[-\frac{\cos(ns)}{n} \right]_0^x =$

$\displaystyle\sum_{n=1}^{\infty} \frac{-(-1)^{n+1}}{n^2} [\cos(nx) - 1] = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) + \frac{\pi^2}{6}$

(using one of the formulas in p. 39, Prob. 6)

(b) Integrate again; I'll jsut show the definite integral method:

$\dfrac{x^3}{6} = \displaystyle\int_0^x \frac{s^2}{2}\, ds = \int_0^x \frac{\pi^2}{6}\, ds + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \int_0^x \cos(ns)\, ds = \frac{\pi^2}{6} x + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(nx)$

This is not a Fourier series, because it has a linear term. To fix this, substitute the original series for x:

$\dfrac{x^3}{6} = \dfrac{\pi^2}{6} \cdot 2 \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(nx) = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{\pi^2}{3n} - \frac{2}{n^3} \right) \sin(nx) =$

$\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n\pi)^2 - 6}{3n^3} \sin(nx)$

(c) Multiply by 6.

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p. 31, 3:

Copy the text from "In treating the" at the beginning of Section 10 to "as n tends to infinity", about two thirds of the way down p. 28. Set a_0 to 0 wherever you see it. Change all $\cos nx$ to $\sin nx$ and change all a_n to b_n.

X4: The function f is continuous, and is linear except at the "break points" -1, 0, 1. At these points its slope changes:

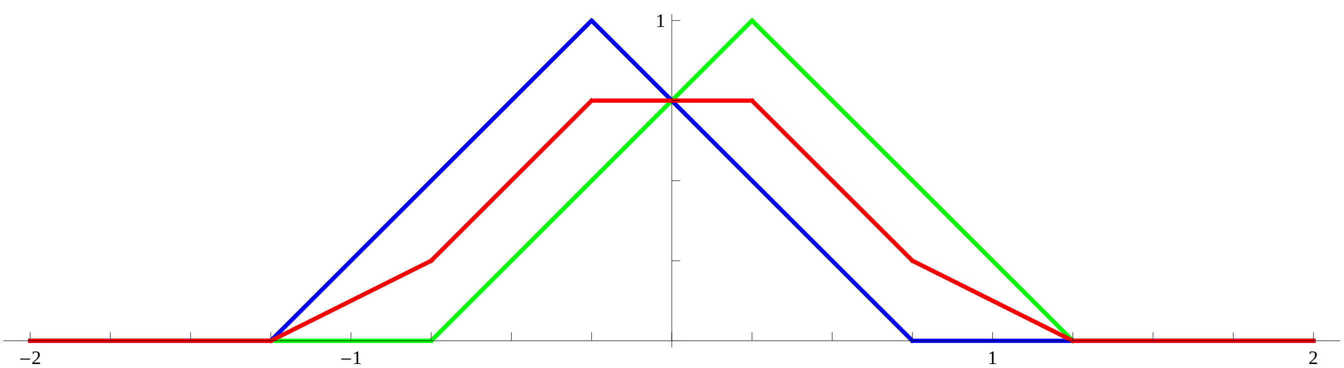
(a) from 0 to 1 at -1;
(b) from 1 to -1 at 0;
(c) from -1 to 0 at 1.

The solution, according to D'Alembert, is $u(x,t) = \frac{1}{2}f(x+2t) + \frac{1}{2}f(x-2t)$ (since $a = 2$).

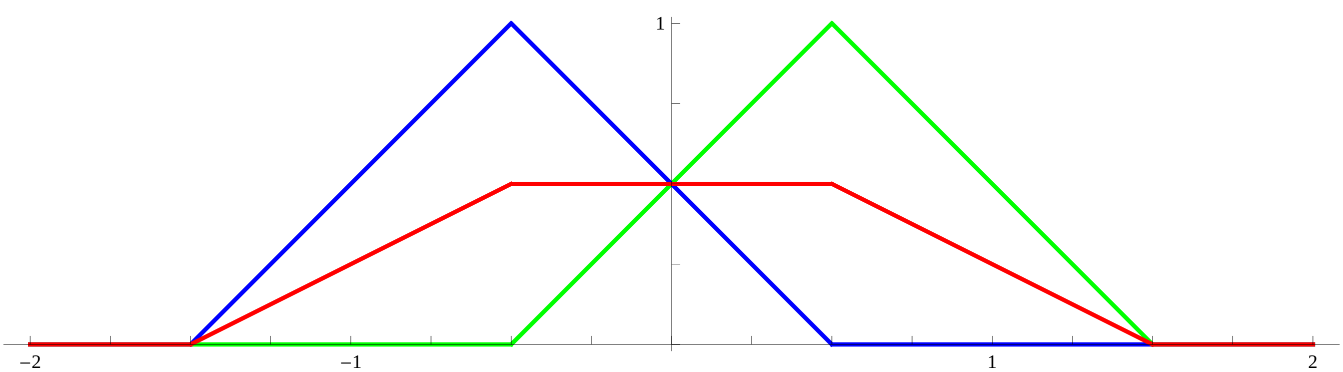
For $t = \frac{1}{8}$ this is $u(x, \frac{1}{8}) = \frac{1}{2}f(x + \frac{1}{4}) + \frac{1}{2}f(x - \frac{1}{4})$. The graph of $u = \frac{1}{2}f(x + \frac{1}{4})$ is the graph of $u = \frac{1}{2}f(x)$, shifted to the left by $\frac{1}{4}$ units. Hence this graph is linear except at the shifted breakpoints $-1-\frac{1}{4} = -\frac{5}{4}$, $0-\frac{1}{4} = -\frac{1}{4}$, and $1-\frac{1}{4} = \frac{3}{4}$. Similarly, the graph of $u = \frac{1}{2}f(x-\frac{1}{4})$ is the graph of $u = \frac{1}{2}f(x)$ shifted to the right by $\frac{1}{4}$ units, so it is linear except at the break points $-\frac{3}{4}$, $\frac{1}{4}$, and $\frac{5}{4}$. Since the sum of two linear functions is again linear, the solution $\frac{1}{2}f(x+\frac{1}{4})+\frac{1}{2}f(x-\frac{1}{4})$ is linear on the intervals bounded by the combined break points $-\frac{5}{4}$, $-\frac{3}{4}$, $-\frac{1}{4}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{4}$.

So, in order to graph $u(x, \frac{1}{8})$ it is enough to plot the points on the graph corresponding to the break points, and then connect them by straight lines.

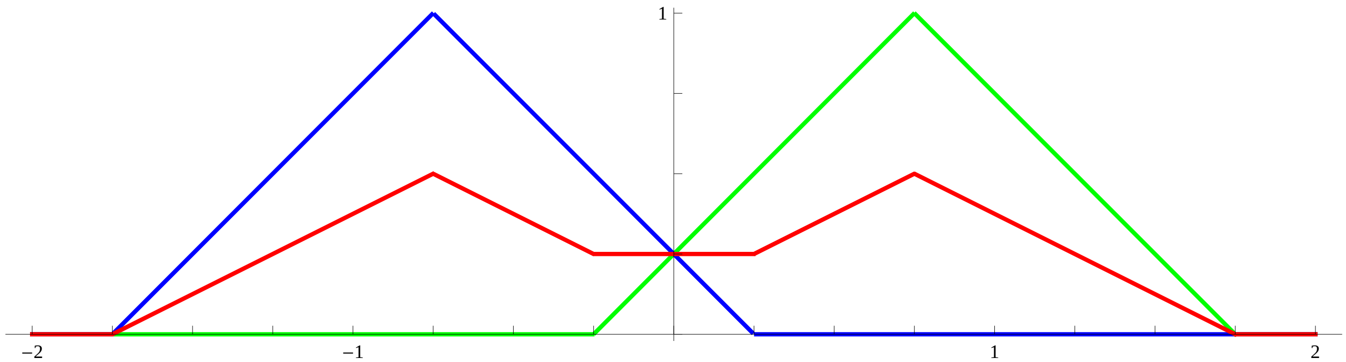
Here are the plots. $f(x + 2t)$ and $f(x - 2t)$ are in blue and green, and the solution $u(x, t)$ is in red.



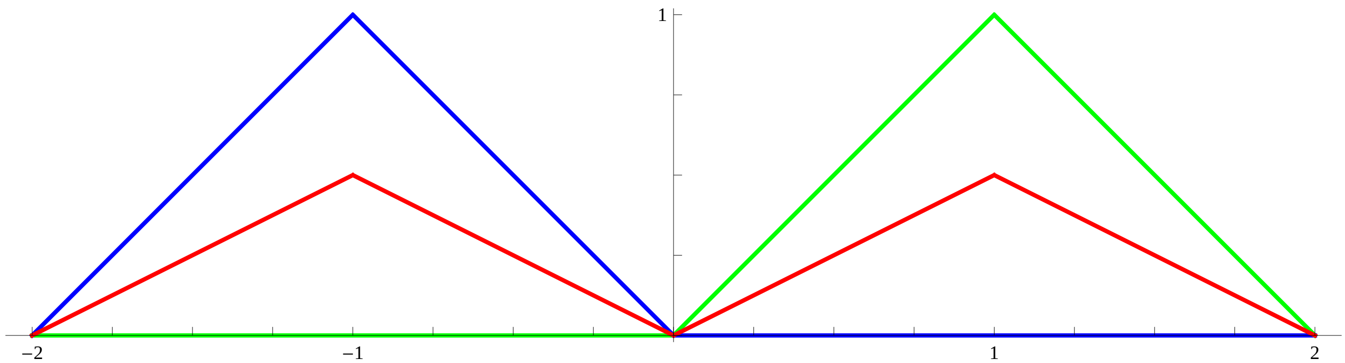
$t = \frac{1}{8}$



$t = \frac{1}{4}$



$t = \frac{3}{8}$



$t = \frac{1}{2}$

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p. 72, 3:

The heat equation is $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$. Since this is a steady state problem u depends only on x, so $\dfrac{\partial u}{\partial t} = 0$ and $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{d^2 u}{dx^2}$. So the PDE reduces to $\dfrac{d^2 u}{dx^2} = 0$. Integrate this with respect to x to get $\dfrac{du}{dx} = B$, and then again: $u = A + Bx$. Here A and B are constants to be determined by the boundary conditions.

Newton's Law of Cooling says that $\Phi_{\text{in}} = H(T_0 - u)$ at a boundary point, where Φ_{in} is the flux of heat *into* the slab, H is a constant, T_0 is the external temperature and u is the internal temperature. Fourier's Law, at the same point, is $\Phi_{\text{out}} = -K\dfrac{du}{dn}$, where K is a constant and $\mathbf{n}$ is the unit vector normal to the boundary and pointing *out* of the slab. Putting these together with $\Phi_{\text{out}} = -\Phi_{\text{in}}$ gives $\dfrac{du}{dn} = h(T_0 - u)$ where $h = H/K$.

At the boundary $x = c$ the vector $\mathbf{n}$ is pointing to the right, so $\dfrac{d}{dn} = \dfrac{d}{dx}$, and T_0 is given as T, so the boundary condition is $\dfrac{du}{dx}(c) = u'(c) = h(T - u(c))$. At the boundary $x = 0$ the vector $\mathbf{n}$ is pointing to the left, so $\dfrac{d}{dn} = -\dfrac{d}{dx}$, and T_0 is given as 0, so the boundary condition is $-\dfrac{du}{dx}(0) = -u'(0) = h(0 - u(0)) = -hu(0)$, or $u'(0) = hu(0)$.

Now use the BCs to determine the constants A and B, starting with

$$(*) \qquad u(x) = A + Bx \qquad u'(x) = B$$

When $x = 0$ this becomes $u(0) = A$ and $u'(0) = B$, so $u'(0) = hu(0)$ becomes $B = hA$. If $x = c$ then $(*)$ becomes $u(c) = A + Bc$ and $u'(c) = B$. Plug these into $u'(c) = h(T - u(c))$ to get $B = h(T - A - Bc) = hT - hA - hcB$. So there are two linear equations to solve for A and B:

$$\begin{array}{lll} hA & -B & = 0 \\ hA & +(1+hc)B & = hT \end{array}$$

Subtract the first equation from the second to get $(2 + hc)B = hT$, so $B = \dfrac{hT}{2 + hc}$. From the first equation, $A = \dfrac{B}{h} = \dfrac{1}{h} \cdot \dfrac{hT}{2 + hc} = \dfrac{T}{2 + hc}$. So

$$u(x) = A + Bx = \frac{T}{2 + hc} + \frac{hTx}{2 + hc} = \frac{T}{2 + hc}(1 + hx)$$

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p. 72, 4:

The heat equation in spherical coordinates is

$$\frac{\partial u}{\partial t} = k\boldsymbol{\nabla}^2 u = k\left[\frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2 \sin^2\theta} u_{\varphi\varphi} + \frac{1}{r^2 \sin\theta}(\sin\theta\, u_\theta)_\theta\right]$$

This is a steady state problem, so $u_t = 0$, and the directions said you should assume that u depends only on r, so the teerms that involve determinants with respect to φ and ψ are zero. So the equation becomes

$$((*)) \qquad 0 = \frac{d}{dr}\left(r^2 \frac{du}{dr}\right) = 2r\frac{du}{dr} + r^2 \frac{d^2u}{dr^2}$$

On the other hand,

$$\frac{d^2}{dr^2}(ru) = \frac{d}{dr}\left(u + r\frac{du}{dr}\right) = \frac{du}{dr} + \frac{du}{dr} + r\frac{d^2u}{dr^2} = 2\frac{du}{dr} + r\frac{d^2u}{dr^2}$$

If you multiply this by r you get the expression in $(*)$, so $\dfrac{d^2}{dr^2}(ru) = 0$ is equivalent to $(*)$.

Now solve $\dfrac{d^2}{dr^2}(ru) = 0$ by integrating twice:

$$\frac{d^2}{dr^2}(ru(r)) = 0 \implies \frac{d}{dr}(ru(r)) = B \implies ru(r) = A + Br$$

where A and B are constants. Dividing by r gives $u(r) = \dfrac{A}{r} + B$. The boundary conditions are $u(a) = \dfrac{A}{a} + B = 0$ and $u(b) = \dfrac{A}{b} + B = u_0$. Subtract these to get $A\left(\dfrac{1}{a} - \dfrac{1}{b}\right) = -u_0$ and solve for A to get $A = -\dfrac{abu_0}{b-a}$. Then $B = -\dfrac{A}{a} = \dfrac{bu_0}{b-a}$. Put these together to get

$$u(r) = \frac{A}{r} + B = -\frac{1}{r} \cdot \frac{abu_0}{b-a} + \frac{bu_0}{b-a} = \frac{bu_0}{b-a}\left(1 - \frac{a}{r}\right)$$