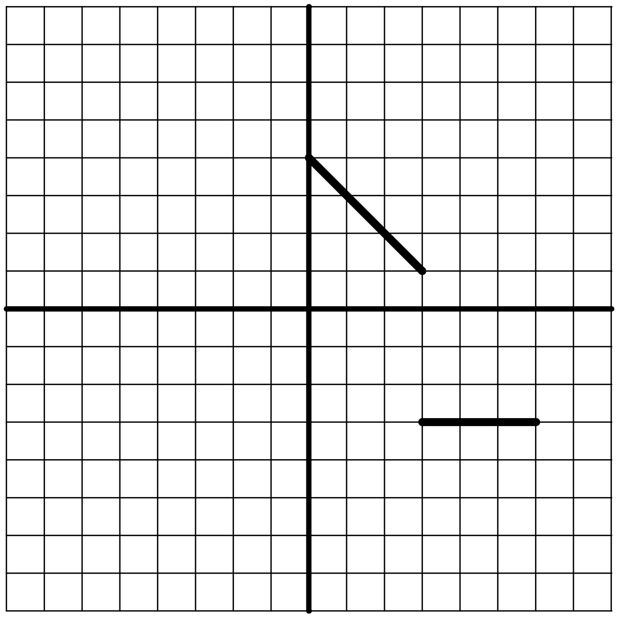
Math 471-02 – Test 1A – March 4, 2010

1. (10) The function $f(x)$ is defined for $0 \le x \le 6$ as below.



$$f(x) = \begin{cases} 4 - x & \text{for } 0 \le x \le 3 \\ -3 & \text{for } 3 < x \le 6 \end{cases}$$

Let $S(x)$ be the sine series for $f(x)$ and let $C(x)$ be the cosine series for $f(x)$. **Do not calculate any terms of either series!** Fill in the following table:

$S(0)$	$C(0)$	$C(3)$	$C(-2)$	$S(-2)$	$S(6)$	$S(8)$

Solution:

(a) $S(0) = 0$ since a sine series *always* converges to 0 at the endpoints (the sines are all 0).

(b) The cosine series is even, so it corresponds to the *even periodic* extension of f. Call this extension $f_e(x)$. Then $f_e(0+) = 4$ and $f_e(0-) = 4$ (draw a picture) so the cosine series converges to $\frac{1}{2}[f_e(0+) + f_e(0-)] = 4$.

(c) $C(3) = \frac{1}{2}\left[f(3-) + f(3+)\right] = \frac{1}{2}\left[1 + (-3)\right] = -1$

(d) Since f is continuous at 2, its even extension f_e is continuous at -2, so $C(-2) = f_e(-2) = f(2) = 2$

(e) The sine series is even, so it corresponds to the *odd periodic* extension of f. Call this extension $f_o(x)$. f is continuous at 2, so f_o is continuous at -2, so $S(-2) = f_o(-2) = -f(2) = -2$

(f) $S(6) = 0$. See the explanation for $S(0)$.

(g) Use the odd periodic extension: $f_o(8) = f_o(8 - 12) = f_0(-4)$ since 12 is the period. Then $f_o(-4) = -f_o(4) = -(-3) = 3$. f is continuous at 4, so f_o is continuous at -4, and hence f_o is continuous at 8. So $S(8) = f_o(8) = 3$.

On the "B" test the answers are 0, 1, 1, 3, −3, 0, 2

2. (15) Let $f(x)$ be defined for $0 < x < \pi$ by

$$f(x) = \begin{cases} 2 & \text{for } 0 < x \le \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$$

Find the Fourier cosine series for $f(x)$. **You do not need to simplify the coefficients.**

Solution: $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\,dx = \dfrac{2}{\pi}\int_0^{\pi/2} 2\,dx + \dfrac{2}{\pi}\int_{\pi/2}^{\pi} 0\,dx = 2 + 0 = 2$

$a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\cos nx\,dx = \dfrac{2}{\pi}\int_0^{\pi/2} 2\cos nx\,dx + \dfrac{2}{\pi}\int_{\pi/2}^{\pi} 0\,dx = \dfrac{2}{\pi}\left[\dfrac{2}{n}\sin nx\right]_0^{\pi/2} + 0 = \dfrac{4}{n\pi}\sin\left(\dfrac{n\pi}{2}\right)$

So the series is $\dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx = 1 + \dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\sin\left(\dfrac{n\pi}{2}\right)\cos nx$

If you simplify this you get $1 + \dfrac{4}{\pi}\displaystyle\sum_{m=1}^{\infty}\dfrac{(-1)^{m+1}}{2m-1}\cos[(2m-1)x]$, but you do not need to simplify.

On the "B" test the answer is $2 + \dfrac{8}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n}\sin\left(\dfrac{n\pi}{2}\right)\cos nx$

3. (10) The Fourier cosine series for $f(x) = x^2$ is $\dfrac{\pi^2}{3} + 4\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cos nx$. What does the series converge to when $x = \pi$? Plug in $x = \pi$ and derive the formula $\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n^2} = \dfrac{\pi^2}{6}$.

Solution: If $f_e(x)$ is the even periodic extension of $f(x) = x^2$, $0 \le x \le \pi$ then f_e is continuous at π (draw a picture) so the series with $x = \pi$ converges to $f_e(\pi) = f(\pi) = \pi^2$.

Now plug in $x = \pi$: $\pi^2 = \dfrac{\pi^2}{3} + 4\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cos n\pi = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cdot(-1)^n = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\dfrac{(-1)^{2n}}{n^2} = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\dfrac{1}{n^2}$. So

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$
$$4\sum_{n=1}^{\infty}\frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3} = \frac{2}{3}\pi^2$$
$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{4}\cdot\frac{2}{3}\cdot\pi^2 = \frac{\pi^2}{6}$$

The problem on the "B" test is the same.

4. (5) Suppose $S(x) = \displaystyle\sum_{n=1}^{20} n^2 \sin nx$. What is $\displaystyle\int_0^{\pi} S(x)\sin(6x)\,dx$?

Solution: $\displaystyle\int_0^{\pi} S(x)\sin(6x)\,dx$ consists of 20 integrals, of the form $\displaystyle\int_0^{\pi} n^2\sin(nx)\sin(6x)\,dx$, where $n = 1, 2, 3, \ldots, 20$. In 19 of these integrals $n \ne 6$, and $n^2\displaystyle\int_0^{\pi}\sin(nx)\sin(6x)\,dx = 0$ if $n \ne 6$. This leaves one term with $n = 6$, so $\displaystyle\int_0^{\pi} S(x)\sin(6x)\,dx = 6^2\int_0^{\pi}\sin^2(6x)\,dx = 36\cdot\dfrac{\pi}{2} = 18\pi$

On the "B" test the answer is 5π

5. (15) Find the steady state solution to the heat equation $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$, for $0 \le x \le 10$, with the boundary conditions $u(0,t) = 100$ and $u_x(10,t) = u(10,t)$.

Solution: Since this is a steady state solution, u is a function only of x, so $u_t = 0$ and $u_{xx} = u''$. So the differential equation becomes $ku'' = 0$, or, dividing by k, $\dfrac{d^2u}{dx^2} = 0$. Integrating this twice leads to the general solution $u(x) = A + Bx$.

The boundary conditions for $u(x)$ become $u(0) = 100$ and $u'(10) = u(10)$. Using the first, $100 = u(0) = A + B \cdot 0$, so $A = 100$. For the second, first calculate $u'(x) = B$. So $u'(10) = B$, and also $u(10) = A + B \cdot 10 = 100 + 10B$. Plugging these into $u'(10) = u(10)$ gives $B = 100 + 10B$. Solve this to get $B = -100/9$. So the answer is $u(x) = 100 - \dfrac{100}{9} \cdot x$

On the "B" test the answer is $u(x) = 100 - \dfrac{100}{11} \cdot x$

6. (15)

(a) Separate variables in $\dfrac{\partial u}{\partial t} - xt\dfrac{\partial^2 u}{\partial x^2} = 0$, for $0 \le x \le 10$, by setting $u(x,t) = X(x)T(t)$. **Do not try to solve the resulting ordinary differential equations.**
Solution: The PDE becomes $X(x)T(t)' - xtX''(x)T(t)$. Rearrange this with all "x" terms on the left and all "t" terms on the right: $x\dfrac{X''(x)}{X(x)} = \dfrac{1}{t}\dfrac{T'(t)}{T(t)}$. The only way this can happen is if both sides are equal to the same constant. For traditional reasons call this constant $-\lambda$, so both $x\dfrac{X''(x)}{X(x)} = -\lambda$ and $\dfrac{1}{t}\dfrac{T'(t)}{T(t)} = -\lambda$. Multiply these out and simplify to get

$$xX''(x) + \lambda X(x) = 0, \qquad T'(t) + \lambda t T(t) = 0$$

On the "B" test the answer is $X''(x) + \lambda x X(x) = 0$, $tT'(t) + \lambda T(t) = 0$

(b) Translate the boundary conditions $u_x(0,t) = 0$ and $u(10,t) = 0$ to boundary conditions for X.
Solution: $u_x(0,t) = 0$ becomes $X'(0)T(t) = 0$. Then $T(t) \ne 0$ (except for the trivial solution), so $X'(0) = 0$. Similarly, $u(10,t) = 0$ becomes $X(10)T(t) = 0$, so $X(10) = 0$. The translated boundary conditions are

$$X'(0) = 0 \qquad X(10) = 0$$

On the "B" test the answer is $X(0) = 0$, $X'(10) = 0$

7. (20) Find the eigenvalues and corresponding eigenfunctions for the problem

$$X'' + \lambda X = 0, \qquad X(0) = 0,\ X(10) = 0$$

There are no negative eigenvalues for this problem, so **you only need to consider the cases $\lambda = 0$ and $\lambda = \alpha^2 > 0$.** You must show your work.

Solution: First consider $\lambda = 0$. Then $X'' = 0$, which has the general solution $X = A + Bx$. Plug in $x = 0$ to get $X(0) = A + B \cdot 0 = 0$, so $A = 0$ and $X = Bx$. Now plug in $x = 10$ to get $X(10) = 10B = 0$, so $B = 0$. Hence the only solution when $\lambda = 0$ is the trivial solution $X = 0$. Since an eigenfunction cannot be the zero function, 0 is not an eigenvalue.

Now consider $\lambda = \alpha^2 > 0$, with $\alpha > 0$. The general solution of $X'' + \alpha^2 X = 0$ is $X = A\cos\alpha x + B\sin\alpha x$. Plug in $x = 0$ to get $X(0) = A + B \cdot 0 = 0$, so $A = 0$ and $X = B\sin\alpha x$. Now plug in $x = 10$ to get $B\sin 10\alpha = 0$. If $B = 0$ then X is the trivial solution, and so we don't have an eigenfunction.

If $B \neq 0$ then $\sin 10\alpha = 0$. A sine is 0 if and only if its input is an integer multiple of π, so $10\alpha = n\pi$ where n is a positive integer (since $10\alpha > 0$). That is, $\alpha = n\pi/10$, and $X = B\sin \alpha x$ is a non-trivial solution if $B \neq 0$. Choose $B = 1$; then the eigenvalues and corresponding eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{10}\right)^2 \qquad X_n(x) = \sin\left(\frac{n\pi x}{10}\right)$$

The problem on the "B" test is the same.

8. (10) D'Alembert's solution for

$$u_{tt} = a^2 u_{xx}, \qquad u(x,0) = f(x),\ u_t(x,0) = 0$$

is $u(x,t) = \frac{1}{2}f(x+at) + \frac{1}{2}f(x-at)$

(a) Verify that D'Alembert's solution satisfies the **initial conditions**.

Solution: $u(x,0) = \dfrac{1}{2}f(x+0) + \dfrac{1}{2}f(x-0) = \dfrac{1}{2}f(x) + \dfrac{1}{2}f(x) = f(x)$

$u_t(x,t) = \dfrac{\partial}{\partial t}\left(\dfrac{1}{2}f(x+at) + \dfrac{1}{2}f(x-at)\right) = \dfrac{1}{2}f'(x+at)\dfrac{\partial}{\partial t}(x+at) + \dfrac{1}{2}f'(x-at)\dfrac{\partial}{\partial t}(x-at) =$
$\dfrac{1}{2}f'(x+at)\cdot a + \dfrac{1}{2}f'(x-at)\cdot(-a) = \dfrac{a}{2}f'(x+at) - \dfrac{a}{2}f'(x-at)$

So $u_t(x,0) = \dfrac{a}{2}f'(x+0) - \dfrac{a}{2}f'(x-0) = \dfrac{a}{2}f'(x) - \dfrac{a}{2}f'(x) = 0$

This part of the problem on the "B" test is the same.

(b) Suppose $f(x) = \sin(x)$. Use trig identities to write D'Alembert's solution in the form $u(x,t) = \sin(x)T(t)$ where $T(t)$ is a function of t.

Solution: Use $\sin A \cos B = \dfrac{1}{2}\sin(A-B) + \dfrac{1}{2}\sin(A+B)$ with $A = x$ and $B = at$:

$$u(x,t) = \frac{1}{2}\sin(x+at) + \frac{1}{2}\sin(x-at) = \sin(x)\cos(at)$$

This is the desired form, with $T(t) = \cos(at)$.

On the "B" test the answer is $u(x,t) = \cos(x)\cos(at)$, so $T(t) = \cos(at)$.

(Table of formulas goes here)