

Math 471-02 – Test 2 – April 29, 2010

There are 9 problems, equally weighted. Do ANY 8. If you turn in all 9 then you will get credit for the best 8.

1. (a) Show that $X_1(x) = x$ and $X_2(x) = x^4$ are orthogonal on $[-1, 1]$.
 (b) Find $\|X_2\|$.
 (c) Find a polynomial of the form $x^2 + Bx + C$ which is orthogonal to both X_1 and X_2 .

Solution: $(X_1, X_2) = \int_{-1}^1 x \cdot x^4 dx = \int_{-1}^1 x^5 dx = 0$ so X_1 and X_2 are orthogonal.

$$(X_2, X_2) = \int_{-1}^1 (x^4)^2 dx = \int_{-1}^1 x^8 dx = \frac{2}{9} \text{ so } \|X_2\| = \sqrt{(X_2, X_2)} = \boxed{\sqrt{2/3}}.$$

Let $P(x) = x^2 + Bx + C$. Then $0 = (P, X_1) = \int_{-1}^1 (x^3 + Bx^2 + Cx) dx = \frac{2}{3}B$ so $B = 0$, so $P(x) = x^2 + B$. Now $0 = (P, X_2) = \int_{-1}^1 (x^6 + Cx^4) dx = \frac{2}{7} + \frac{2}{5}C$ so $C = -7/5$. Hence $P(x) = \boxed{x^2 - \frac{7}{5}}$.

2. The following heat problem has a non-homogeneous boundary condition:

$$u_t = ku_{xx}, \quad u(0, t) = 0, \quad u(1, t) = e^{-t}, \quad u(x, 0) = 0.$$

Convert this to a problem with homogeneous boundary conditions by making the substitution $u(x, t) = V(x, t) + xe^{-t}$.

You must write the PDE and the boundary and initial conditions for V . **Do not solve for V .**

Solution: $u_t = \frac{\partial}{\partial t} (V + xe^{-t}) = V_t - xe^{-t}$, and $u_{xx} = \frac{\partial^2}{\partial x^2} (V + xe^{-t}) = V_{xx} + 0$. Plug these into $u_t = ku_{xx}$ to get $V_t - xe^{-t} = kV_{xx}$, or $\boxed{V_t = kV_{xx} + xe^{-t}}$.

Plugging $t = 0$ into $u = V + xe^{-t}$ leads to $u(x, 0) = V(x, 0) + x$. Since $u(x, 0) = 0$ the initial condition for V becomes $\boxed{V(x, 0) = -x}$.

Plugging $x = 0$ into $u = V + xe^{-t}$ leads to $u(0, t) = V(0, t) + 0$. Since $u(0, t) = 0$ the first boundary condition for V becomes $\boxed{V(0, t) = 0}$.

Plugging $x = 1$ into $u = V + xe^{-t}$ leads to $u(1, t) = V(1, t) + e^{-t}$. Since $u(1, t) = e^{-t}$ the second boundary condition for V becomes $\boxed{V(1, t) = 0}$.

3. The following heat problem has a non-homogeneous PDE:

$$u_t = ku_{xx} + xe^{-t}, \quad u(0, t) = 0, \quad u(1, t) = 0, \quad u(x, 0) = 0.$$

Separation of variables leads to the eigenfunctions $X_n(x) = \sin(nx)$ with corresponding eigenvalues $\lambda_n = n^2$. Set up a variation of parameters problem by assuming that $u(x, t) = \sum_{n=1}^{\infty} B_n(t) \sin(nx)$.

You will need a Fourier expansion of xe^{-t} using the eigenfunctions $X_n(x)$. You can get this by multiplying $x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$ by e^{-t} .

You must write the ODE satisfied by $B_n(t)$. **Do not worry about the initial conditions, and do not solve for $B_n(t)$.**

Solution: $u_t = \frac{\partial}{\partial t} \left(\sum_{n=1}^{\infty} B_n(t) \sin(nx) \right) = \sum_{n=1}^{\infty} \frac{d}{dt} B_n(t) \sin(nx) = \sum_{n=1}^{\infty} B_n'(t) \sin(nx),$

$$u_{xx} = \frac{\partial^2}{\partial x^2} \left(\sum_{n=1}^{\infty} B_n(t) \sin(nx) \right) = \sum_{n=1}^{\infty} B_n(t) \frac{d^2}{dx^2} \sin(nx) = \sum_{n=1}^{\infty} B_n(t) (-n^2 \sin(nx)) =$$

$$\sum_{n=1}^{\infty} (-n^2) B_n(t) \sin(nx), \text{ and } x e^{-t} = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} e^{-t} \sin(nx)$$

Plug these into $u_t = k u_{xx} + x e^{-t}$:

$$\sum_{n=1}^{\infty} B_n'(t) \sin(nx) = k \sum_{n=1}^{\infty} (-n^2) B_n(t) \sin(nx) + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} e^{-t} \sin(nx) =$$

$$\sum_{n=1}^{\infty} \left(-n^2 k B_n(t) + \frac{2(-1)^{n+1}}{n} e^{-t} \right) \sin(nx)$$

Now equate coefficients of $\sin(nx)$ to get $B_n'(t) = -n^2 k B_n(t) + \frac{2(-1)^{n+1}}{n} e^{-t}$, or

$$B_n'(t) + n^2 k B_n(t) = 2 \frac{(-1)^{n+1}}{n} e^{-t}$$

4. Determine the eigenvalue problem that arises from

$$u_t = x(xu)_{xx} + e^{-t}, \quad u(0, t) = 0, \quad u(1, t) = 0, \quad u(x, 0) = x.$$

That is, perform separation of variables on the corresponding homogeneous PDE.

You must write the ODE satisfied by $X(x)$, and its boundary conditions. **Do not solve for the eigenfunctions.**

Solution: The homogeneous PDE is $u_t = x(xu)_{xx}$. Separate variables by setting $u(x, t) = X(x)T(t)$:

$$\frac{\partial}{\partial t} (X(x)T(t)) = x \frac{\partial^2}{\partial x^2} (xX(x)T(t))$$

$$X(x) \frac{d}{dt} T(t) = x \frac{d^2}{dx^2} (xX(x)) T(t)$$

$$X(x)T'(t) = x(xX(x))'' T(t)$$

$$\frac{T'(t)}{T(t)} = \frac{x(xX(x))''}{X(x)}$$

$$\frac{T'(t)}{T(t)} = -\lambda \text{ and } \frac{x(xX(x))''}{X(x)} = -\lambda$$

$$T' + \lambda T = 0 \text{ and } x(xX)'' + \lambda X = 0$$

We only want the equation for X . The boundary conditions for u are $u(0, t) = X(0)T(t) = 0$ and $u(1, t) = X(1)T(t)$, so $X(0) = 0$, $X(1) = 0$. So the eigenvalue problem is

$$x(xX)'' + \lambda X = 0, \quad X(0) = 0, \quad X(1) = 0$$

Note: This Sturm-Liouville problem has a second order singularity at the origin, and it is not possible to find eigenfunctions satisfying the boundary condition at 0. However, the problem does not ask for the solutions of the eigenvalue problem; it just asks you to set it up.

5. The only solutions of the Sturm-Liouville problem

$$(x^{-1}X'(x))' + \lambda xX(x) = 0, \quad X(0) = 0, \quad X(1) = 0$$

are of the form $B \sin(\alpha x^2)$, where B is a constant.

- (a) Determine the eigenfunctions and corresponding eigenvalues. **Do not normalize the eigenfunctions.**
- (b) Eigenfunctions for different eigenvalues are orthogonal, if you use the correct weight function. What is the weight function for these eigenvalues?

Solution: Let $X = \sin(\alpha x^2)$.

First, use the boundary conditions to determine α : $X(0) = 0$ is automatic. $X(1) = \sin \alpha = 0$ is true (with a positive α) if and only if $\alpha = n\pi$, with n a positive integer.

Next check the differential equation: $X' = 2\alpha x \cos(\alpha^2 x)$ so $x^{-1}X'(x) = 2\alpha \cos(\alpha^2 x)$. Now differentiate this: $(x^{-1}X'(x))' = -4\alpha^2 x \sin(\alpha x^2)$. Plug this into the differential equation: $(x^{-1}X'(x))' + \lambda xX(x) = 0$ becomes $-4\alpha^2 x \sin(\alpha x^2) + \lambda x \sin(\alpha x^2) = 0$. Cancel the $x \sin(\alpha^2 x)$ factor (since it is not identically zero) to get $\lambda = 4\alpha^2$.

Hence the eigenfunctions are $X_n = \sin(n\pi x^2)$, with corresponding eigenvalues $\lambda_n = 4(n\pi)^2$, where $n = 1, 2, 3, \dots$

The differential equation $(x^{-1}X'(x))' + \lambda xX(x) = 0$ is written in standard S-L form. The generic S-L equation is $[r(x)X'(x)]' + [q(x) + \lambda p(x)]X(x) = 0$, so $r(x) = x^{-1}$, $q(x) = 0$, $p(x) = x$. The weight function is $p(x) = x$.

6. Suppose X is a smooth function satisfying $X(0) = 0$ and $X(1) = 0$. Use integration by parts to show that $(X, X'') = -\|X'\|^2$. (Use the standard inner product for functions defined on $[0, 1]$.)

Solution: $(X, X'') = \int_0^1 X(x)X''(x) dx$. Use integration by parts:

$$\begin{aligned} u &= X(x) & dv &= X''(x) dx \\ du &= X'(x) dx & v &= \int X''(x) dx = X'(x) \end{aligned}$$

$$\begin{aligned} \text{So } (X, X'') &= \int_0^1 X(x)X''(x) dx = \int_0^1 u dv = uv \Big|_0^1 - \int_0^1 v du = X(x)X'(x) \Big|_0^1 - \int_0^1 X'(x) \cdot X'(x) dx = \\ &= X(1)X'(1) - X(0)X'(0) - (X', X') = 0 \cdot X'(1) - 0 \cdot X'(0) - \|X'\|^2 = -\|X'\|^2 \end{aligned}$$

7. Show that 0 is **not** an eigenvalue of

$$X''(x) + (\lambda - x)X(x) = 0, \quad X(0) = 0, \quad X(1) = 0.$$

To do this, suppose that 0 is an eigenvalue, with corresponding eigenfunction $X(x)$, so $X''(x) + (0 - x)X(x) = 0$, or $X''(x) - xX(x) = 0$. Multiply this equation by $X(x)$ and integrate from 0 to 1. Apply problem #6 and rearrange the terms to get

$$-\int_0^1 [X'(x)]^2 dx = \int_0^1 x [X(x)]^2 dx.$$

Why is this impossible?

Solution: Multiply $X''(x) - xX(x) = 0$ by $X(x)$ and integrate: $\int_0^1 X(x)X''(x) dx - \int_0^1 x[X(x)]^2 dx = \int_0^1 0 dx = 0$. The first integral is (X, X'') , which equals $-\|X'\|^2$ by problem #6. So the equation becomes $-\|X'\|^2 - \int_0^1 x[X(x)]^2 dx = 0$, so

$$-\|X'\|^2 = -\int_0^1 [X'(x)]^2 dx = \int_0^1 x[X(x)]^2 dx$$

Now $\int_0^1 [X'(x)]^2 dx \geq 0$, since the integrand is non-negative, so the left hand side is ≤ 0 . On the other hand, $\int_0^1 x[X(x)]^2 dx > 0$ since the integrand is non-negative and continuous but not identically 0. It is impossible for a number which is ≤ 0 to also be positive.

8. Define functions f and f_n with domain $[0, 1]$ by $f(x) = x$ and $f_n(x) = x + n^{1/4}x^n$. Show that f_n converges in the mean to f .

Solution: $\|f_n - f\|^2 = \int_0^1 (f_n(x) - f(x))^2 dx = \int_0^1 (n^{1/4}x^n)^2 dx = \int_0^1 n^{1/2}x^{2n} dx = n^{1/2} \frac{x^{2n+1}}{2n+1} \Big|_0^1 = \frac{\sqrt{n}}{2n+1} = \frac{1}{2\sqrt{n} + 1/\sqrt{n}}$. Hence $\lim_{n \rightarrow \infty} \|f_n - f\|^2 = 0$. Now take square roots and use continuity of the square root function to get $\lim_{n \rightarrow \infty} \|f_n - f\| = 0$. According to the definition, this means that f_n converges to f in the mean.

9. Find the generalized Fourier expansion of the function 1 on $[0, 1]$, using the orthonormal sequence $X_n(x) = \sqrt{2} \cos(\alpha_n x)$, $\alpha_n = \frac{2n-1}{2}\pi$, $n = 1, 2, 3, \dots$

Solution: $c_n = (1, X_n) = \int_0^1 1 \cdot \sqrt{2} \cos(\alpha_n x) dx = \sqrt{2} \cdot \frac{1}{\alpha_n} \sin(\alpha_n x) \Big|_0^1 = \frac{\sqrt{2}}{\alpha_n} \sin(\alpha_n) = \frac{2\sqrt{2}}{(2n-1)\pi} \sin\left(\frac{2n-1}{2}\pi\right) = \frac{2\sqrt{2}}{\pi} \frac{(-1)^{n+1}}{(2n-1)}$. So the series is

$$1 = \sum_{n=1}^{\infty} c_n X_n(x) = \sum_{n=1}^{\infty} \frac{2\sqrt{2}}{\pi} \frac{(-1)^{n+1}}{(2n-1)} \sqrt{2} \cos\left(\frac{2n-1}{2}\pi x\right) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \cos\left(\frac{2n-1}{2}\pi x\right)$$

(Table of formulas goes here)