MATH 508: COMPLEX ANALYSIS, FALL 2003

1. 2003-09-03 (Dennis Pixton)

There are many ways to construct the complex number system from the reals. They all lead to the following:

(1) $\mathbb{C}$ is a field, containing $\mathbb{R}$ as a subfield.
(2) $\mathbb{C}$ contains an element i which satisfies $i^2 = -1$.
(3) Every element of $\mathbb{C}$ can be written as $x + iy$ where x and y are in $\mathbb{R}$.

It is easy to check that the real numbers x and y in the equation $z = x + iy$ are uniquely determined by z. Hence we can identify $\mathbb{C}$ with $\mathbb{R}^2$ via the correspondence $z \mapsto (x, y)$; this correspondence is a $\mathbb{R}$-vector space isomorphism.

Some definitions, starting with $x = x + iy$:

(1) $\operatorname{Re} z = x$, $\operatorname{Im} z = y$, the *real and imaginary parts of* z.
(2) $\overline{z} = x - iy$, the *conjugate of* z.
(3) $|z| = \sqrt{x^2 + y^2} = \sqrt{z\overline{z}}$, the *modulus of* z.

1.1. **Proposition.** (1) *$z \mapsto \overline{z}$ is a field automorphism of $\mathbb{C}$; that is,*

$$\overline{z + w} = \overline{z} + \overline{w}, \quad \overline{z \cdot w} = \overline{z} \cdot \overline{w}$$

(2) *z is real $\iff z = \operatorname{Re} z \iff \operatorname{Im} z = 0 \iff z = \overline{z}$. Moreover,*

$$\operatorname{Re} z = \frac{1}{2}(z + \overline{z}), \quad \operatorname{Im} z = \frac{1}{2i}(z - \overline{z})$$

(3) *$z \mapsto |z|$ is a norm on $\mathbb{C}$ (considered as a real vector space); in fact,*

$$|z| \geq 0, \quad |z| = 0 \iff z = 0, \quad |zw| = |z| \cdot |w|, \quad |z + w| \leq |z| + |w|.$$

A *polynomial* is a function $f\colon \mathbb{C} \to \mathbb{C}$ so that there are constants $c_k \in \mathbb{C}$, $0 \leq k \leq n$, so that

$$f(z) = c_0 + c_1 z + \cdots + c_n z^n$$

for all $z \in \mathbb{C}$. Any polynomial is continuous on $\mathbb{C}$.

1.2. **Lemma.** *The coefficients $c_0, \ldots, c_n$ are uniquely determined by f.*

Proof. It is enough to show that if $f(z) = c_0 + c_1 z + \cdots + c_n z^n = 0$ for all $z \in \mathbb{C}$ then $c_0 = c_1 = \cdots = c_n = 0$. Plugging in $z = 0$ gives $f(0) = c_0 = 0$, so $f(z) = c_1 z + \cdots + c_n z^n = 0$ for all $z \in \mathbb{C}$. Define $g(z) = c_1 z + \cdots + c_n z^n = 0$. Then $f(z) = z g(z) = 0$ for all $z \in \mathbb{C}$ so $g(z) = 0$ for all $z \in \mathbb{C} \setminus \{0\}$. Then, by continuity of g, $g(z) = 0$ for all $z \in \mathbb{C}$. Now the proof continues by induction on n. ☐

1. **Problem.** Give a proof of Lemma 1.2 that does not use continuity, so it will work for polynomials defined over other fields. You will need to make some assumptions about the field; for example, $x^2 + x = 0$ for all $x \in \mathbb{Z}_2$.

Hence we can define the *degree* of a polynomial as the largest indes of a non-zero coefficient. This does not work for the 0 polynomial; our convention is that the zero polynomial has degree $-\infty$.

1.3. **Definition.** If $D \subset \mathbb{C}$ is unbounded then

(1) $\lim\limits_{z\to\infty} f(z) = L$ iff for any $\epsilon > 0$ there is $M > 0$ so that $|f(z) - L| < \epsilon$ for all $z \in D$ satisfying $|z| > M$.
(2) $\lim\limits_{z\to\infty} f(z) = \infty$ iff for any $B > 0$ there is $M > 0$ so that $|f(z)| > B$ for all $z \in D$ satisfying $|z| > M$.

2. **Problem.** If f is a polynomial of positive degree then $\lim\limits_{z\to\infty} f(z) = \infty$.

1.4. **Lemma** (Division Algorithm). *If f and d are polynomials with $d \neq 0$ then there are unique polynomials q and r satisfying*

$$f = dq + r, \quad \deg r < \deg d.$$

Proof. This is just long division: There is a unique monomial q_1 so that $f_1 = f - dq_1$ has lower degree than f. Continue by induction. ☐

1.5. **Corollary.** *Suppose f is a polynomial of positive degree and $a \in \mathbb{C}$.*

(1) *There is a unique polynomial g so that*

$$f(z) = (z-a)g(z) + f(a) \text{ for all } z \in \mathbb{C}.$$

(2) *There is a unique polynomial h and a unique positive integer m so that*

$$f(z) = (z-a)^m h(z) + f(a) \text{ for all } z \in \mathbb{C}, \text{ and } h(a) \neq 0.$$

(3) *$f(a) = 0$ iff there is a polynomial g so that $f(z) = (z-a)g(z)$ for all $z \in \mathbb{C}$.*

1.6. **Theorem** (Fundamental Theorem of Algebra, or FTA). *If f is a polynomial of positive degree then there is $z_1 \in \mathbb{C}$ so that $f(z_1) = 0$.*

1.7. **Corollary.** *If f is a polynomial of degree $n > 0$ then there are a complex number c, distinct complex numbers $z_1, \ldots, z_k$, and positive integers $m_1 \ldots m_k$ so that*

$$f(z) = c(z-z_1)^{m_1}(z-z_2)^{m_2}\ldots(z-z_k)^{m_k}$$

for all $z \in \mathbb{C}$.

2. 2003-09-04 (DENNIS PIXTON)

Before proving the FTA we preview some facts about the exponential function:

For $z \in \mathbb{C}$ define

$$\exp z = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Formally, for $y \in \mathbb{R}$,

$$\exp(iy) = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \cos y + i \sin y.$$

This is Euler's formula.

The main properties of the exponential formula now follow (at least formally) from the definition:

1. $\exp 0 = 1$, $\exp(z+w) = \exp(z)\exp(w)$.
2. $\exp(-z)\exp(z) = 1$, so $\exp(z) \neq 0$ and $\exp(z)^{-1} = \exp(-z)$.
3. By induction, $\exp(z)^n = \exp(nz)$ for $n \in \mathbb{Z}$.
4. $\overline{\exp z} = \exp \overline{z}$.
5. For $x, y \in \mathbb{R}$, $\exp(x+iy) = e^x(\cos y + i \sin y)$.
6. $|\exp z| = e^{Rez}$, $|\exp(iy)| = 1$ if $y \in \mathbb{R}$.
7. $(\cos y + i \sin y)^n = e^{iny} = \cos ny + i \sin ny$ for $y \in \mathbb{R}$ and $n \in \mathbb{Z}$.
8. $(\cos y + i \sin y)^{-1} = \overline{(\cos y + i \sin y)} = \cos y - i \sin y$ for $y \in \mathbb{R}$.

First proof of the FTA. Assume f is a polynomial of positive degree which is never zero; we shall derive a contradiction.

First we claim that $|f|$ has a minimum value. That is, there is $a \in \mathbb{C}$ so that $|f(a)| \leq |f(z)|$ for all $z \in \mathbb{C}$. To see this, remember that $f(z) \to \infty$ as $z \to \infty$. From the definition there is $M > 0$ so that $|f(z)| > |f(0)|$ for all $z \in \mathbb{C}$ satisfying $|z| > M$. Now $D = \{\, z \in \mathbb{C} \colon |z| \leq M \,\}$ is compact and $|f|$ is continuous so $|f|$ has a minimum value on D. That is, there is $a \in D$ so that $|f(z)| \leq |f(a)|$ for all $z \in D$. In particular, $|f(z)| \leq |f(0)|$ But also, if $z \notin D$ then $|z| > M$ so $|f(z)| \geq |f(0)| \geq |f(a)|$. So $|f(a)|$ is the minimum value of $|f|$ on $\mathbb{C}$.

Now apply Corollary 1.5.2, so $f(z) = f(a) + (z-a)^m g(z)$ where m is a positive integer, g is a polynomial, and $g(a) \neq 0$. Write $f(a) = b$. This is not zero so we can factor it out, so, with $h = b^{-1}g$,

$$f(z) = b(1 + (z-a)^m h(z)) = b\left[1 + (z-a)^m h(a) + (z-a)^m (h(z) - h(a))\right]. \tag{2.1}$$

Using polar coordinates we write $h(a) = r(\cos\theta + i\sin\theta) = re^{i\theta}$; we now consider values of z of the form $z = a + te^{iy}$ for suitable real $t > 0$ and y. We first choose $y = (\pi - \theta)/m$. Then

$$\begin{aligned} 1 + (z-a)^m h(a) &= 1 + (te^{iy})^m h(a) = 1 + t^m e^{imy} re^{i\theta} = 1 + rt^m e^{i(\pi-\theta)+i\theta} = 1 + rt^m e^{i\pi} \\ &= 1 - rt^m. \end{aligned}$$

We shall require that $t < \delta_0 = r^{-\frac{1}{m}}$, so $1 - rt^m > 1 - rr^{-1} = 0$, so $1 - rt^m$ is positive. Hence

$$|1 + (z-a)^m h(a)| = |1 - rt^m| = 1 - rt^m. \tag{2.2}$$

The last thing we need to use is the continuity of h at a: Let $\epsilon = \frac{r}{2}$ and choose $\delta > 0$ so that $|h(z) - h(a)| < \epsilon$ if $|z - a| < \delta$. Hence, with $z = a + te^{iy}$ as above,

$$|(z-a)^m(h(z)-h(a))| = |te^{iy}|^m|h(z)-h(a)| \leq t^m\epsilon = \tfrac{1}{2}t^m r \tag{2.3}$$

Now, if $0 < t < \min(\delta_0, \delta)$,

$$\begin{aligned}
|f(z)| &= |b\,(1 + (z-a)^m h(a) + (z-a)^m(h(z)-h(a)))| && \text{by (2.1)}\\
&\leq |b|\,(|1 + (z-a)^m h(a)| + |(z-a)^m| \cdot |h(z)-h(a)|) && \text{triangle inequality}\\
&\leq |b|\left(1 - rt^m + \tfrac{1}{2}t^m r\right) && \text{by (2.2) and (2.3)}\\
&= |b|\left(1 - \tfrac{1}{2}rt^m\right)\\
&< |b| = |f(a)|.
\end{aligned}$$

This contradicts the minimality of $|f(a)|$. □

It is always a good idea to try to improve a theorem by weakening the assumptions so that the proof still works. This leads to the following.

2.1. Theorem. *Suppose $f\colon \mathbb{C} \to \mathbb{C}$ satisfies the following:*

1. $\lim_{z\to\infty} f(z) = \infty$.
2. *For each $a \in \mathbb{C}$ there is a positive integer m and a continuous function $g\colon \mathbb{C} \to \mathbb{C}$ so that $f(z) = f(a) + (z-a)^m g(z)$ for all $z \in \mathbb{C}$ and $g(a) \neq 0$.*

Then there is $z_1 \in \mathbb{C}$ so that $f(z_1) = 0$.

The proof above also proves this theorem. However, this generality is illusory: we shall see later that the hypotheses of the theorem actually imply that f is a polynomial.

Further analysis of the proof yields the following:

2.2. Theorem. *Suppose f is a polynomial of positive degree, $K \subset \mathbb{C}$ is compact, and $f(z) \neq 0$ for all $z \in K$. Then the minimum value of $|f|$ on K occurs at a boundary point of K but not at an interior point.*

Proof. If the minimum value of f occured at a point of the interior of K then we could, just as in the proof of the FTA, find z arbitrarily near z with $|f(z)| < |f(a)|$, and all z sufficiently near to a are in the interior of K. □

This theorem is true in greater generality and is usually called the Minimum Modulus Theorem.

Here's another proof of the FTA, due to Paul Loya. It requires some calculus, including Leibniz's theorem on differentiation under the integral sign, but should be understandable by Calculus students.

Second proof of the FTA. Suppose, again, that $f(z) = 0$ has no solutions, and we'll get a contradiction.

For $t \geq 0$ define

$$F(t) = \int_0^{2\pi} \frac{d\theta}{f(te^{i\theta})}.$$

Since f is continuous and never zero the integrand is a continuous function of θ, so is integrable. Technically, it is necessary to split the integrand into real and imaginary parts and integrate them separately. These are just integrals of messy rational functions of $\sin\theta$ and $\cos\theta$.

We have $F(0) = 2\pi/f(0)$. On the other hand, $\lim_{z\to\infty} f(z) = \infty$, so, given any B, there is M so that $|f(te^{i\theta})| > B$ for all $t = |te^{i\theta}| > M$. Since $|F(t)| \le \int_0^{2\pi} \frac{d\theta}{|f(te^{i\theta})|}$ we have $|F(t)| \le 2\pi/B$ for $t > M$. Hence $\lim_{t\to\infty} F(t) = 0$. $\square$

3. 2003-09-05 (DENNIS PIXTON)

I should have mentioned this yesterday. If f is a polynomial of degree $n > 0$ then $f(z)$ has at least one solution (by the FTA) and it has at most n solutions (since each solution corresponds to a linear factor of f. If $b \in \mathbb{C}$ then $1 \leq \# f^{-1}\{b\} \leq n$, since the elements of $f^{-1}\{b\}$ are just the solutions of $f(z) - b = 0$.

3. **Problem.** Let B be the set of points $b \in \mathbb{C}$ for which $\# f^{-1}\{b\} < n$. Show that B is finite. Can you say anything interesting about the numbers $\# f^{-1}\{b\}$ for b in this set? (I'm not sure whether this has an interesting answer.)

We now need to discuss differentiation.

3.1. **Definition.** Suppose $V \subset \mathbb{C}$ is open and $f\colon V \to \mathbb{C}$. We say f is *differentiable at* a iff

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. In this case the limit is called the *derivative of* f *at* a, written $f'(a)$. If $U \subset V$ we say f is *differentiable on* U iff f is differentiable at each point of U, and we say that f is *differentiable* iff f is differentiable on V.

Note that part of the definition of a differentiable function is that its domain is open; it is generally not interesting to talk about differentiation of complex functions on non-open sets. A differentiable function is also called *holomorphic* or *analytic*. These terms are often defined somewhat differently, but eventually all the definitions are proved to be equivalent.

Here are a few equivalent formulations; the proofs are immediate from the definitions.

3.2. **Lemma.** *Suppose $f\colon V \to \mathbb{C}$ with V open, and suppose $a \in V$. The following are equivalent:*

1. *f is differentiable at a with derivative L.*
2. *For any $\epsilon > 0$ there is $\delta > 0$ so that for all $z \in V$, if $0 < |z - a| < \delta$ then*
$$\left| \frac{f(z) - f(a)}{z - a} - L \right| < \epsilon.$$
3. *For any $\epsilon > 0$ there is $\delta > 0$ so that for all $z \in V$, if $|z - a| < \delta$ then*
$$|f(z) - f(a) - L(z - a)| \leq \epsilon |z - a|.$$
4. *The function g defined by*
$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & \text{if } z \in V,\ z \neq a \\ L & \text{if } z = a \end{cases}$$
is continuous at a.
5. *There is a function g defined on V and continuous at a so that, for all $z \in V$, $f(z) = f(a) + (z - a)g(z)$ and $g(z) = L$.*

The elementary computational tools for derivatives are proved just as they are in a decent Calculus textbook. This includes linearity, the product and quotient rules, the power rule (for integer exponents), and the chain rule. Thus polynomials are differentiable, as are rational functions (that is, quotients of polynomials).

Also, just as in your Calculus textbook,

3.3. **Lemma.** *If f is differentiable at a then f is continuous at a.*

However, there are many very simple functions that are not differentiable. Here is useful criterion:

3.4. **Theorem** (Cauchy-Riemann equations). *If f is differentiable at a then*

$$f'(a) = \frac{\partial f}{\partial x}(a) = \frac{1}{i}\frac{\partial f}{\partial y}(a).$$

Proof. If we restrict h to the real number Δx in the definition of the differentive then we get

$$\frac{\partial f}{\partial x}(a) = \lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{\Delta x} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = f'(a).$$

Similarly, restricting h to $i\Delta y$ for real Δy gives

$$\frac{1}{i}\frac{\partial f}{\partial x}(a) = \lim_{\Delta y\to 0}\frac{f(a+i\Delta y)-f(a)}{i\Delta y} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = f'(a).$$

□

The converse of this theorem is true with extra assumptions on f. For example, if

$$\frac{\partial f}{\partial x}(a) = \frac{1}{i}\frac{\partial f}{\partial y}(a)$$

and the partial derivatives of f exist and are continuous in an open set containing a then f is differentiable at a. There are many other such theorems, differing in the extra hypotheses required. However, if

$$f(z) = \begin{cases} \dfrac{z^2-\overline{z}^2}{|z|^2} & z\neq 0 \\ 0 & z = 0 \end{cases}$$

then

$$\frac{\partial f}{\partial x}(0) = \frac{1}{i}\frac{\partial f}{\partial y}(0)$$

but f is not differentiable at 0. In fact, it is not even continuous at 0 since $f((1+i)t) = 2i$ for $t \neq 0$.

4. **Problem.** Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{R}$-linear, so it can be written in terms of matrix operations as

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix} \text{ with } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for real numbers a, b, c, d. Find necessary and sufficient conditions on A for f to be differentiable. [Necessity should come from the CR equations; for sufficiency is is simplest to show that the conditions on A mean that f can be written in the form $f(z) = \alpha z$ for some complex constant α.]

One important fact about differentiation in the reals is not available in the complex domain:

5. **Problem.** The "exact MVT" would say the following: For a differentiable function $f\colon V \to \mathbb{C}$, if $a \neq b$ and the segment $[a, b]$ from a to b lies in V then

$$\frac{f(b) - f(a)}{b - a} = f'(c) \text{ for some } c \in [a, b].$$

Show that this is true if f is any quadratic polynomial, and give an example to show that it is not always true for cubic polynomials.

We can still prove a version of the MVT involving an inequality rather than equality, but we probably won't need it. For example, we have the following:

3.5. **Proposition.** *If $f\colon V \to \mathbb{C}$ is analytic and $f'(z) = 0$ for all $z \in V$ then f is constant, provided one of the following holds:*

1. *V is convex, meaning $[a, b] \subset V$ whenever a and b are in V.*
2. *V is star-shaped, meaning that there is some $c \in V$ so that, for each $a \in V$, $[c, a] \subset V$.*
3. *V is polygonally connected, meaning that for each a and b in V there is a finite sequence $z_0 = a, z_1, \ldots, z_n = b$ so that $[z_k, z_{k+1}] \subset V$.*
4. *V is path connected, meaning that for each a and b in V there is a continuous function $\gamma\colon [0, 1] \to V$ so that $\gamma(0) = a$ and $\gamma(1) = b$.*
5. *V is connected, meaning that V is not the union of two disjoint non-empty open sets.*

As an application of this result, note that by the CR equations, if $f\colon V \to \mathbb{R}$ is analytic then $f'(z) = 0$ for all $z \in V$. Hence the only analytic real-valued functions on a connected open set are constant. Thus $\operatorname{Re} z$ and $\operatorname{Im} z$ are not analytic. Since $z + \overline{z} = 2 \operatorname{Re} z$, $\overline{z}$ is not analytic. Although $|z|^2$ is differentiable at 0 it is not analytic in any non-empty open set.

4. 2003-09-08 (Garry Bowlin)

Proof.

1. V is convex. Take a,b $\in V$, parameterize [a,b] as $\gamma : [a,b] \to V(\gamma$(t)=a(1-t)+bt). Then we have $(f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t) = 0$, where $f \circ \gamma : [0,1] \to \mathbb{C}$ and both $Re \circ f \circ \gamma$ and $Im \circ f \circ \gamma$ have zero derivative on $[0,1]$. By the Mean Value Theorem on $\mathbb{R}$, $F \circ \gamma$ is constant so $f(a) = f(\gamma(0)) = f(\gamma(1)) = f(b)$
2. V is star-shaped. Same argument as for (1).
3. V is polygonally connected. Same argument as for (1), but with induction.
4. V is path connected. We will show later that this is equivalent to V being polygonally connected.
5. V is connected. We will show later that this is equivalent to V being path connected.

☐

4.1. **Lemma** (Lebesque Covering Lemma). *If K is a compact set, $K \subset X$, X is a metric space, then for all open covers O of K there is an α¿0 so that for all x and $y \in X$ if $x \in K$ and $d(x,y) < \alpha$ then there is $U \in O$ so that $x, y \in U$.*

Proof.

1. Let B be a cover of K by balls $B(x, \delta_x)$ so that $B(x, \delta_x) \subset U$ for some $U \in O$.
2. Let B' be a covering by such balls of radius $\delta_x/2$.
3. B' has a finite subcover $\delta = min(\delta_x/2)$. Then we have $B(z, \delta_z/2) \subset B(z, \delta_z) \subset U \in O$. So $d(z,y) < 2\delta \leq \delta_z$, and $x, y \in B(z, \delta_z)$.

☐

4.2. **Proposition.** *If $\gamma : [0,1] \to V$ is continuous and $\epsilon > 0$ $\exists \sigma : [0,1] \to V$ which is piecewise linear so that $\mid \sigma(t) - \gamma(t) \mid < \epsilon$ for all t, $\sigma(0) = \gamma(0)$, and $\sigma(1) = \gamma(1)$.*

Proof. Let $O = \{B(\gamma(t), \epsilon_t) \subset V\} \mid$ I=(0,1] and $\epsilon_t \leq \epsilon/2\}$ Let α be the Lebesque Number for $\{\gamma^{-1}B \mid B \in O\}$ then partition the interval [0,1] into subintervals of length ¡ α. Define σ on $[t_k, t_{k+1}]$ to be the usual parameterization of the segment from $\gamma(t_k)$ to $\gamma(t_{k+1})$. ☐

4.3. **Corollary.** *If V is open in $\mathbb{C}$ and V is path connected if any only if V is polygonally connected.*

4.4. **Theorem.** *If V is open in $\mathbb{C}$ then V is connected if and only if V is path connected.*

Proof. Choos a let A be the set of all points you can get to from a and B be the points you cannot get to from a. A and B are open and so A or B is empty. ☐

5. 2003-09-10 (Ken Ferry)

Suppose $\mathcal{O}$ is a collection of open sets. An $\mathcal{O}$-chain from a to b is a finite sequence $\langle U_0, \ldots U_n \rangle$ of elements of $\mathcal{O}$ so that $a \in U_0$, $b \in U_n$, and $U_j \cap U_{j+1} \neq \emptyset$ for $0 \leq j < n$.

6. Problem. Suppose V is an open set in $\mathbb{C}$. Then

1. If V is connected and $\mathcal{O}$ is any collection of open disks whose union is V then any two points of V can be connected by an $\mathcal{O}$-chain.
2. If there is a collection $\mathcal{O}$ of open disks whose union is V so that any two points of V can be connected by a $\mathcal{O}$-chain then V is connected.

A power series is a formal sum of the form

$$\sum_{n=0}^{\infty} c_n z^n$$

or more generally,

$$\sum_{n=0}^{\infty} c_n (z-a)^n.$$

When given a power series, the natural question is "For what values of z does this power series converge?"

Let's look at a few examples.

(1) The geometric series,

$$\sum_{n=0}^{\infty} z^n \text{ converges to } \frac{1}{1-z} \text{ for} |z-1| < 1.$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} z^n &= \lim_{N\to\infty} \sum_{n=0}^{N} z^n \\ &= \lim_{N\to\infty} 1 + z + z^2 + \ldots + z^N \\ &= \lim_{N\to\infty} \frac{1-z^{N+1}}{1-z}\end{aligned}$$

The limit will converge to $\frac{1}{1-z}$ whenever $z^N \to 0$, which occurs when $|z| < 1$. Observe that the series diverges (i.e. does not converge) for all $|z| \geq 1$; the general term z^n has modulus $|z^n| = |z|^n \geq 1 \not\to 0$ as n goes to infinity, so the series cannot possibly converge. □

(2)

$$\sum_{n=0}^{\infty} (z-1)^n \text{ converges to } \frac{1}{2-z} \text{ for } |z-1| < 1.$$

(3)

$$\sum_{n=0}^{\infty}(1-z)^n.$$

First, is this really a power series? It doesn't have the form we gave. However,

$$\sum_{n=0}^{\infty}(1-z)^n = \sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

and

$$\sum_{n=0}^{\infty}(-1)^n(z-1)^n \text{ converges to } \frac{1}{z} \text{ for } |z-1|<1.$$

In each of the example cases we found a nice form for a set of z on which the series converged. This is characteristic of power series, though having it quite as nice as we found in the first example is not.

5.1. **Theorem.** *Given a power series* $\sum_{n=0}^{\infty} c_n z^n$*, there exists* $R \in [0,\infty]$ *such that*

(1) *the series converges absolutely for all* $|z| < R$
(2) *the series diverges for all* $|z| > R$.

R *is called the radius of convergence of the power series.*

Moreover, $|c_n z^n|$ *is a bounded sequence if* $|z| < R$ *and is not bounded if* $|z| > R$.

More moreover, $c_n z^n \to 0$ *for* $|z| < R$ *and the limit as* n *goes to* ∞ *of* $c_n z^n$ *does not exist.*

How do we calculate R? In practice, other techniques may be more appropriate, but there does exist an explicit formula:

$$R = \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}.$$

R separates $\mathbb{C}$ into two zones with very distinct behaviors. What about the boundary, where $|z| = R$? The theorem says nothing.

In fact, boundary behavior can be quite complicated. Suppose that $R = 1$. Then $|z| = R \implies z = e^{i\theta}$ and $\sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n e^{i\theta}$. This is a completely general Fourier series, and Fourier series convergence can be tricky. Look at $\sum \frac{1}{n} z^n$, which is a power series with radius of convergence 1. At $z = 1$ this is the harmonic series, which does not converge, but at $z = -1$ we get an alternating harmonic series, which does converge. Our first example, $\sum_{n=0}^{\infty} z^n$ diverges for all $|z| = R$, and the series $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$ converges for all such z.

To prove the theorem, we could verify the formula given above, but that might not be very enlightening. We give a proof which illustrates a standard technique when for proving statements involving power series.

Proof. (proof of theorem 5.1)

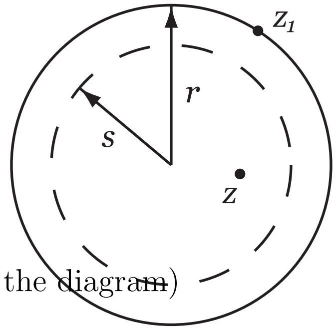
5.2. **Lemma.** *Suppose the sequence* $\{|c_n z_1|^n\}$ *is bounded. Then*

$$|z| < |z_1| \implies \sum c_n z^n \textit{ converges absolutely.}$$

Supposing $\{|c_n z_1|^n\}$ bounded and $|z| < |z_1|$, define $r := |z_1|$ and choose s such that $|z| < s < r$. Inserting this intermediate circle is the standard technique referred to above. By $\{|c_n z_1|^n\}$ bounded, there exists M such that $|c_n| r^n \leq M$ for all n.

Now,

$$\begin{aligned}\sum_{n=0}^{\infty} |c_n z^n| &= \sum_{n=0}^{\infty} |c_n| \frac{|z|^n}{s^n} \cdot \frac{s^n}{r^n} \cdot r^n \\ &\leq M \sum_{n=0}^{\infty} \frac{|z|^n}{s^n} \cdot \frac{s^n}{r^n} \\ &\leq M \sum_{n=0}^{\infty} \left(\frac{s}{r}\right)^n, \text{ by } \frac{|z|^n}{s^n} < 1 \text{ (look at the diagram)} \\ &= \frac{M}{1 - s/r}\end{aligned}$$



s/r is strictly less than 1 (look at the diagram again), so the above makes sense and we see that $\sum_{n=0}^{\infty} c_n z^n$ converges absolutely. Actually, ☐

6. 2003-09-11 (Fabrizio Polo)

6.1. **Lemma.** *$f(z) = z^n$ is differentiable everywhere.*

Proof. If z^n is differentiable at a, we should be able to say something about $|z^n - a^n - na^{n-1}(z-a)| \leq \epsilon|z-a|$ for $|z-a| < \delta$.

$$\begin{aligned} & z^n - a^n - na^{n-1}(z-a) = (z-a)[z^{n-1} + z^{n-2}a + \cdots + za^{n-2} + a^{n-1} - na^{n-1}] \\ = \; & (z-a)[(z^{n-1} - a^{n-1}) + (z^{n-2}a - a^{n-1}) + \cdots + (za^{n-2} - a^{n-1})] \\ = \; & (z-a)^2[(\overbrace{z^{n-2} + z^{n-3}a + \cdots + za^{n-3} + a^{n-2}}^{(n-1)\text{ terms}}) + \\ & a(\overbrace{z^{n-3} + z^{n-4}a + \cdots + za^{n-4} + a^{n-3}}^{(n-2)\text{ terms}}) + \cdots] \end{aligned}$$

So we have a total of $\frac{1}{2}(n-1)^n$ terms, each of the form $a^k z^j$ where $k+j = n-2$. If $|z| < r$ and $|a| < r$ then $|z^n - a^n - na^{n-1}(z-a)| \leq \frac{1}{2}(n-1)^n r^{n-2}|z-a|^2$ since $|a^k z^j| \leq r^{k+j} = r^{n-2}$. So

$$\begin{aligned} |z^n - a^n - na^{n-1}(z-a)| & \leq \frac{1}{2}(n-1)^n r^{n-2}|z-a|^2 \\ & = \left(\frac{1}{2}(n-1)^n r^{n-2}|z-a|\right)|z-a|. \end{aligned}$$

Which we may certainly make arbitrarily small. □

6.2. **Lemma.** *If $\sum c_n z^n$ has a radius of convergence R then so does $\sum nc_n z^n$.*

Proof. Suppose $|z| < R$ Pick r so that $|z| < r < R$. Then $|nc_n z^n| = n(|z|^n/r^n)|c_n r^n| \to 0$. Conversely, if $|z| > R$, then $|c_n z^n|$ is unbounded so $|nc_n z^n|$ is certainly unbounded. □

6.3. **Theorem.** *If $\sum c_n z^n$ has radius of convergence R and $f(z) = \sum c_n z^n$ for $|z| < R$ then f is analytic and $f'(z) = \sum nc_n z^{n-1}$.*

Proof. Fix $|a| < r < R$. Then

$$\begin{aligned} \left|f(z) - f(a) - \sum nc_n z^{n-1}(z-a)\right| & = \left|\sum c_n[z^n - a^n - na^{n-1}(z-a)]\right| \\ & \leq \sum |c_n| r^{n-2}|z-a|^2(1/2)n(n-1) \\ & = \left[\sum |c_n| r^{n-2}|z-a|(1/2)n(n-1)\right]|z-a|. \end{aligned}$$

The expression in brackets converges to $M < \infty$. So, the whole expression may be made arbitrarily small. (I believe an actual δ was found, but it is unclear in my notes.) □

6.4. **Corollary.** *If $R > 0$ then $c_k = \frac{f^{(k)}(0)}{k!}$.*

6.5. Theorem. *(Fubini)*

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|a_{j,k}|<\infty \implies \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{j,k}=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}a_{j,k}.$$

Recentering a Power Series

Suppose we have $f(z)=\sum c_n z^n$ for $|z|<R$. Then

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}c_n z^n=\sum_{n=0}^{\infty}c_n(z-a+a)^n \\ &= \sum_{n=0}^{\infty}c_n\sum_{k=0}^{n}\binom{n}{k}(z-a)^k a^{n-k}=\sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty}c_n\binom{n}{k}a^{n-k}\right)(z-a)^k. \end{aligned}$$

Now we have a power series in $z-a$. To make any use of it, we must prove that it converges absolutely.

$$\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}|c_n|\binom{n}{k}|z-a|^k|a|^{n-k}=\sum_{n=0}^{\infty}|c_n|\left(|z-a|+|a|\right)^n$$

which converges if $|z-a|+|a|<R \implies |z-a|<R-|a|$. In other words, the radius of convergence of our new series is at least as big as the largest disk centered at a that fits inside our old disk of convergence. Side note: comparing these calculations with Taylor's formula gives us a way to calculate the k-th derivative of f at a.

Example:

$f(z)=\frac{1}{1-z}=\sum z^n$ converges when $|z|<R=1$. Then $f'(z)=\frac{1}{(1-z)^2}, f''(z)=\frac{2}{(1-z)^3},\dots,f^{(k)}(z)=\frac{k!}{(1-z)^{k+1}}$. So $f^{(k)}(a)/k!=(\frac{1}{1-a})^{k+1}$. Applying Taylor's formula we get

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty}\left(\frac{1}{1-a}\right)^{k+1}(z-a)^k=\frac{1}{1-a}\sum_{k=0}^{\infty}\left(\frac{z-a}{1-a}\right)^{k+1} \\ &= \frac{1}{1-a}\left(\frac{1}{1-\frac{z-a}{1-a}}\right)=\frac{1}{(1-a)-(z-a)}=\frac{1}{1-z}. \end{aligned}$$

This converges if $|\frac{z-a}{1-a}|<1 \implies |z-a|<|1-a|$. But the theorem suggests $|z-a|<1-|a|$ as the radius of convergence. Moral: the theorem, being very general, may give a radius of convergence that is unnecessarily small.

7. 2003-09-12 (Nigar Tuncer)

7. **Problem.** The double summation theorem says that if $\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|a_{jk}| < \infty$ then $\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{jk}$ and $\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}a_{jk}$ converge and are equal. Show that, under the same assumption,

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{jk} = \sum_{n=0}^{\infty}\sum_{j+k=n}a_{jk}.$$

8. **Problem.** If f_1 and f_2 are given by power series centered at 0 with radii of convergence R_1 and R_2 respectively then $f_1 f_2$ has a power series expansion centered at 0 with radius of convergence at least as large as the minimum of R_1 and R_2.

Analytic Branches of Logarithm

Consider the power series of exponential function

$$\exp z = \sum_{n=0}^{\infty}\frac{z^2}{n!}$$

at $z = 0$.This series has a radius of convergence ∞.By proposition ? we can differantiate this power series and it has the same radius of convergence ∞.

$$f'(z) = \sum_{k=0}^{\infty}\frac{n}{n!}z^{(}n-1) = \sum_{k=0}^{i} nfty\frac{z^n}{n!} = \exp z$$

so $\frac{d}{dz}\exp z = \exp z$.

By using this property and recentering the power series of exponential function to a fixed point $z = a$ we get

$$\begin{aligned}\exp z &= \sum \frac{f^{(}k)(a)}{k!}(z-a)^k \\ &= \sum \frac{e^a}{k!}(z-a)^k \\ &= e^a \sum \frac{1}{k!}(z-a)^k = e^a e^{z-a}\end{aligned}$$

where $f(z) = e^z$ and $f^k(z) = e^z$ or by taking $f(z) = e^{z+a}e^{-z}$ where a is fixed. Then $f'(z) = e^{z+a}e{-}z + e^{z+a}(-e^{-z}) = 0$ So f is constant hence $f(z) = f(0)$ for all z.Then $e^{z+a}e^{-z} = e^a$.If we substitute $a = u + v$ and $z = -v$ we get $e^u e^v = e^{u+v}$.

Consequence of previous property of exponential function we have $e^{x+iy} = e^x(cosy+isiny)$ for $x, y \in \mathbb{R}$.

Some properties of exponential function

(1) exp is not onto, since $\exp z \neq 0.(e^z e^{-z} = 1)$

(2) $\exp(C) = C \setminus \{0\}$.

  Proof:If $w \in \mathbb{C} \setminus 0$ then $w = r(\cos\theta + i\sin\theta)$ in polar coordinates where $r \geq 0$ for $x = \ln r$, $y = \theta$, $e^{x+iy} = w$.

(3) exp is not one-to-one.Because $e^{z_1} = e^{z_2} \iff e^{z_1 - z_2} = 1 \iff z_1 - z_2 \in 2\pi i\mathbb{Z}$.

7.1. **Definition.** A "branch of the logarithm" is a function $f\colon V \to \mathbb{C}$ with V is open and connected so that $e^{f(z)} = z$ for all $z \in V$.

7.2. Lemma.

1. *If f is an analytic branch of logarithm then $f'(z) = \frac{1}{z}$ for all z.*
2. *Conversely, if $f\colon V \to \mathbb{C}$ satisfies $f'(z) = \frac{1}{z}$ for all z then $e^f(z) = Az$ for some constant $A \neq 0$.*
3. *$f(e^z) = B + z$*
4. *$f + C$ is a branch of the logarithm for suitable constant C.*

Proof.

1. $e^{f(z)} = z$ then $e^{f(z)}f'(z) = 1$.So $f'(z) = \frac{1}{z}$.
2. Let $g(z) = frace^{f(z)}z$.Then $g'(z) = \frac{ze^{f(z)}f'(z) - e^{f(z)}}{z^2} = 0$
3. $f(e^z) - z = g(z)$ then $g'(z) = f'(e^z)e^z - 1 = \frac{1}{e^z}e^z - 1 = 0$.
4. Pick $a \in V$, and choose $b \in \mathbb{C}$ so that $e^b = a$.Let $c = b - f(a)$.Then
$$e^{f(z)+c} = e^{f(z)}e^c = Az\frac{1}{A} = z$$
because $e^c = e^{b-f(a)} = e^be^{-f(a)} = \frac{e^b}{e^{f(a)}} = \frac{a}{Aa} = \frac{1}{A}$.

□

Note:There is no analytic branch of logarithm defined on $\mathbb{C} \setminus 0$.

Proof:Suppose f is such a branch.Define g by $g(t) = \frac{1}{i}f(e^{it})$.Then $g\colon \mathbb{R} \to \mathbb{R}$, $g'(t) = 1$, and so $g(0) \neq g(2\pi)$.

In fact, there is no branch of logarithm defined on V if V contains any circle around the origin.

If $a \neq 0$,
$$\begin{aligned}\frac{1}{z} = \frac{1}{a-(a-z)} &= \frac{1}{1-\frac{a-z}{a}}\frac{1}{a}\\ &= \frac{1}{a}\sum_{n=o}^{\infty}(\frac{a-z}{a})^n\\ &= \frac{1}{a}\sum_{n=0}^{\infty}\frac{1}{a^n}(a-z)^n\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{a^{n+1}}(z-a)^n\end{aligned}$$
where $|\frac{a-z}{a}| < 1$ so $|z-a| < |a|$.

Let $f(z) = c + \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)a^{n+1}}(z-a)^{n+1} = c + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{na^n}(z-a)^n$.Choose c so that $f(a) = c$ satisfies $e^c = a$.

Then f is an analytic branch of logarithm defined in the disk centered at a and have radius $|a|$.

8. 2003-09-15 (Apratim Roy)

9. **Problem.** Using the geometric series, evaluate $\displaystyle\int_{\gamma_r} \frac{dz}{z-a}$ where $\gamma_r(t) = re^{it}$ for $t \in [0, 2\pi]$ and $|a| > r$.

9. 2003-09-17 (Yishi Wang)

This is meant more-or-less as a joke:

10. **Problem.** Let A be the annulus $\{ z \in \mathbb{C} \colon r < |z| < R \}$ where $0 < r < R$. Find values of r and R so that the function $g : z \mapsto z^{-1}$ satisfies the TIPP in A. Does g have a primitive in A?

Let r be a curve and $f : r(I) \to \mathbb{C}$,where $I = [a, b]$, then

$$|\int_r f| = |\int_a^b f(r(t))r'(t)dt| \leq \int_a^b |f(r(t))||r'(t)|dt.$$

the above inequality hold because:

$$Re\int_a^b g(r(t))r'(t)dt = \int_a^b Re[g(r(t))r'(t)]dt \leq \int_a^b |Re[g(r(t))r'(t)]|dt \leq \int_a^b |[g(r(t))r'(t)]|dt$$

and in general

$$\int_r f = c\exp(i\alpha), so \int_r f\exp(-i\alpha) \in (0, +\infty)$$

where c is a real number.

9.1. **Corollary.** *By the above notation,*$|\int_r f| \leq ML(r)$ *,if* $|f(z)| \leq M$ *for* $\forall z \in r(I)$.

9.2. **Corollary.** *If* $f_n \to f$ *uniformly on* $r(I)$*, where* $r(I)$ *is smoothed and bounded, then* $\int_r f_n \to \int_r f$ *as* $n \to \infty$ *.*

Proof. $|\int_r f - \int_r f_n| = |\int_r f(r(t))r'(t)dt| \leq \sup_{z\in r(I)} |f(z) - f_n(z)|L(r)$ an it converges to 0 as $n \to \infty$, because $L(r)$ is finite.

□

From the last class, we know that : If $g\colon D \to \mathbb{C}$ has a primitive G, then $\int_{r(I)} g = G(\beta) - G(\alpha)$ where $r(a) = \alpha$ and $r(b) = \beta$, $I = [a, b]$. So g has the IPP properity.

9.3. **Definition.** *IPP(independent path properity)* If for $\forall$ paths r_1 and r_2 in D, with the same start and end point, $\int_{r_1} g = \int_{r_2} g$, we say that g has the IPP properity.

Remark : It's obvious that IPP $\Leftrightarrow$ for $\forall$ closed path r in D we have $\int_r g = 0$
To explain more clearly, let

(1) g has TIPP(trangular IPP) and the domain of g is starlike.
(2) g has a primitive
(3) g has IPP properity
(4) g has TIPP properity

then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4)

Proof. (1) $\Rightarrow$ (2) Choose $\alpha \in D$ such that D is starlike with respect to α. For $\forall z \in D$, define $G(z) = \int_{[\beta,Z]} g$. Let α,β,Z form a trangular and β and Z are close enough to be in a small neighborhood.

Notice that $(Z-\beta)g(\beta) = \int_{[\beta,Z]} g(\beta)$, so

$$|G(Z) - G(\beta) - (Z-\beta)g(\beta)| = |\int_{[\beta,Z]} (g - g(\beta))|$$

$$\leq \int_{[\beta,Z]} |g - g(\beta))| \leq L(\beta, Z) \sup_{D} |g - g(\beta)|$$

Then for any $\epsilon > 0$, $\exists \delta > 0$, such that $|g(Z) - g(\beta)| < \epsilon$ for $|Z - \beta| < \delta$

$$|G(Z) - G(\beta) - (Z-\beta)g(\beta)| \leq |\beta - Z|\epsilon$$

so, $\lim_{Z\to\beta} \frac{G(Z)-G(\beta)}{Z-\beta} = g(\beta)$

$\square$

An application of it is : Let $D = \mathbb{C}\backslash(-\infty, 0] \times \{0\}$, so D is starlike with respect to $(1, 0)$. Let $g(Z) = 1/Z$, if r is a trangular in D,then some open disk T in D contains r. Then $1/Z$ has a primitive in T.

Since $1/Z$ has a power series centered at some center c of T,with radius of converges equals to $|c|$, we have the following corollary:

9.4. **Corollary.** *There is a unique analytic branch of the logrithm defined in D with value 0 at (1,0), this is usually called the principal branch of the log.*

10. 2003-09-18 (Adam Perry)

11. 2003-09-19 (Mustafa Findik)

This is a slight weakening of Goursat's Lemma:

11**. Problem.** Suppose T is a triangular path and let K be the set of points on or in the interior of T. Suppose g is defined on an open set containing K and that g is differentiable on $K \setminus F$, where F is finite, and that g is continuous at each point of F. Then $\displaystyle\int_T g = 0$.

Suggestion: First reduce to the case that F consists of exactly one point a, which is a vertex of T. Then write the integral over T as the sum of an integral over a small triangle containing a and an integral over a quadrilateral. Show that the integral over the small triangle is small. What about the integral over the quadrilateral?

12**. Problem.** Let f be the function defined on $V = \mathbb{C} \setminus \{ -1, 1 \}$ by $f(z) = \dfrac{1}{1-z^2}$.

1. Describe an analytic function g defined on the complement of $(-\infty, -1] \cup [1, \infty)$ which satisfies $f = \exp g$.
2. If g is the function from part 1 and $h_0(z) = \exp(\frac{1}{2}g(z))$ then h_0 is analytic and $h_0(z)^2 = f(z)$. Describe an analytic function h defined on the compement of $[-1, 1]$ which also satisfies $h(z)^2 = f(z)$.

For $z \in \mathbb{C}$ let

$$f(z) = \sum c_n z^n,$$

R of C $= R > 0$,

$$c_n = \frac{f^{(n)}(0)}{n!} \text{ Taylor‘s Formula (*)}$$

For $0 < r < R$, γ_r

$$\int_{\gamma_r} \sum_{n=0}^{\infty} c_n z^n dz = \sum_{n=0}^{\infty} c_n \int_{\gamma_r} z^n dz = 0$$

Cauchy Thm for series (equaity holds because of being uniformly convergent for $|z| \le r$)

$$\int_{\gamma_r} \frac{f(z)}{z} dz = \int_{\gamma_r} (\frac{c_0}{z} + \sum_{n=1}^{\infty} c_n z^{n-1}) dz = \int_{\gamma} \frac{c_0}{z} dz = 2\pi i c_0$$

so

$$f(0) = c_0 = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z} dz$$

from (*) and

$$\int_{\gamma_r} \frac{1}{z^k} f(z) dz = \int_{\gamma_r} [\frac{c_0}{z^k} + \frac{c_1}{z_{k-1}} + ... + \frac{c_{k-1}}{z} + \sum_{n=k}^{\infty} c_n z^{n-k}] dz = 2\pi i c_{k-1}$$

so

$$c_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z^{n+1}} dz$$

Cauchy‘s Formula:

$$f^{(n)}(0) = \frac{n!}{2\pi i}\int_{\gamma_r}\frac{f(z)}{z^{n+1}}dz$$

Suppose f is differentiable on $B(0,R)$ where $R \geq r \geq 0$, $a \in B(0,r)$
Define
$g(z) = \frac{f(z)-f(a)}{z-a}$ if $z \neq a$ and $f'(a)$ if $z = a$
g is cts, g is diff except (possibly) at $a \Rightarrow$ TIPP for g by HW $\Rightarrow$ IPP for g
$\Rightarrow \int_{\gamma_r} g(z)dz = 0$
$\int_{\gamma_r}\frac{f(z)-f(a)}{z-a}dz = 0 = \int_{\gamma_r}\frac{f(z)}{z-a}dz - \int_{\gamma_r}\frac{f(a)}{z-a}dz \Rightarrow 2\pi i f(a) = \int_{\gamma_r}\frac{f(z)}{z-a}dz$
So CIF

$$f(a) = \frac{1}{2\pi i}\int_{\gamma_r}\frac{f(z)}{z-a}dz$$

After doing COV

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_r}\frac{f(w)}{w-z}dw$$

$$\frac{1}{w-z} = \frac{1}{1-\frac{z}{w}}\cdot\frac{1}{w} \text{ or } \frac{1}{w-z} = \frac{1}{\frac{1}{\frac{w}{z}-1}}\cdot\frac{1}{z}$$

for the first choise

$$\frac{1}{w}\sum_{n=0}^{\infty}(\frac{z}{w})^n = \sum_{n=0}^{\infty}\frac{z^n}{w^{n+1}}$$

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_n}(\sum_{n=0}^{\infty}\frac{z^n}{w^{n+1}}f(w))dw$$

"Wave"

$$= \sum_{n=0}^{\infty}(\frac{1}{2\pi i}\int_{\gamma_r}\frac{f(w)}{w^{n+1}})z^n$$

If we have CIF then $f =$ a power series expansion w / R of C $\leq R$ So f is diff.

11.1. **Theorem.** *If g is continuous and has IPP, then g is differentiable*

12. 2003-09-22 (Garry Bowlin)

12.1. **Theorem** (Cauchy's Theory)**.** *Let the domain in the all of the following be an open disk D and $f\colon V \to \mathbb{C}$. Then the following are equivelant:*

1. *f is differentiable*
2. *f has the triangle independance of path property (TIPP)*
3. *f has a primative*
4. *f has the independance of path property (IPP)*

Proof. We begin the proof by saying that f is differentiable. Then we define

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-a} & \text{if} z \neq a \\ f'(a) & \text{z=a} \end{cases}$$

Then g is differentiable except for finitley many exceptions, and is continuous everywhere. So we have that g has the TIPP, g has a primative, g has the IPP. Since g has the IPP, $\int_\gamma \frac{f(z)-f(a)dz}{z-a} = 0$ for any closed path γ in D. Consider w on γ we have $\int \frac{f(w)dw}{z-w} = \int \sum \frac{f(w)z^n dw}{w^{n+1}} = \sum \int \frac{f(w)z^n dw}{w^{n+1}}$ by uniform convergence. Since f has a power series expansion f is differentiable. □

12.2. **Definition.** The winding number around a is denoted by $n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$

Note: The CIF can be rewriten as $n(\gamma, a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)dw}{w-a}$

12.3. **Theorem.**

1. *$a \to n(\gamma, a)$ is continuous*
2. *$n(\gamma, a) \in \mathbb{Z}$*
3. *$n(\gamma, a)$ is constant on any connected set.*

Proof. Parameterize γ as $\gamma\colon [0,1] \to \mathbb{C}$, then $g(t) = \int_0^t \frac{\gamma'(s)ds}{\gamma'(s)-a}$ so $g\colon [0,1] \to \mathbb{C}$. Now $g'(t) = \frac{\gamma'(t)}{\gamma(t)-a}$ Let $h(t) = \frac{\exp g^t}{\gamma t - a}$ then we have $h'(t) = 0$ so $h(t)$ is constant. $h(0) = h(1)$, $h(0) = \frac{\exp(g(0)}{\gamma(0)-a} = h(1) = \frac{\exp(g(0)}{\gamma(0)-a}$. Since γ is a closed curve $\gamma(0) = \gamma(1)$ and it must be that $\exp g(1) = \exp(g(0)) = \exp \int_0^0 \mathit{something} = \exp 0 = 1$ so $g(1) \in 2\pi i\mathbb{Z}$ and $\frac{g(t)}{2\pi i} = n(\gamma, a) \in \mathbb{Z}$ □

13. **Problem.** Suppose $\gamma(t) = e^{it}$ for $t \in I = [0, 2\pi]$, so $\gamma(I) = C$ is the unit circle. Suppose $f\colon C \to \mathbb{R}$ is integrable, and define

$$F(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw.$$

Then F is analytic in the open unit disk $D = \{\, z\colon |z| < 1 \,\}$, so $F(z) = \sum c_n z^n$. Define $g(t) = f(e^{it})$ and determine the relation between the power series coefficients c_n and the Fourier coefficients of g:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(t)\,dt, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} g(t)\cos nt\,dt, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} g(t)\sin nt\,dt.$$

14**. Problem.** Referring to Problem 13, determine the coefficients c_n and the function F in elementary terms if

$$f(z) = \begin{cases} 1 & \text{if } \operatorname{Re} z > 0 \\ 0 & \text{otherwise} \end{cases}.$$

13. 2003-09-24 (Ken Ferry)

14. 2003-09-25 (Fabrizio Polo)

These are from Conway's book on complex variables:

15. Problem.

1. Find the series expansion of $\dfrac{e^z - 1}{z}$ about 0 and determine its radius of convergence.
2. Consider $f(z) = \dfrac{z}{e^z - 1}$ and let $f(z) = \sum_{k=0}^{\infty} \dfrac{a_k}{k!} z^k$ be its power series expansion about 0. What is its radius of convergence?
3. Show that
$$0 = a_0 + \binom{n+1}{1} a_1 + \cdots + \binom{n+1}{n} a_n.$$
4. Show that $f(z) + \frac{1}{2}z$ is an even function and use this to show that $a_k = 0$ for k odd and $k > 1$.
5. The numbers $B_{2n} = (-1)^{n-1} a_{2n}$ are called the *Bernoulli numbers.* Calculate B_{2n} for $n = 1, 2, 3, 4$.

16. Problem. Find the power series expansion of $\tan z$ about 0, expressing the coefficients in terms of the Bernoulli numbers. Hint: Prove the identity $\cot 2z = \frac{1}{2}\cot z - \frac{1}{2}\tan z$ and use this plus Problem 15.

15. 2003-09-26 (Nigar Tuncer)

16. 2003-09-29 (Apratim Roy)

17. 2003-09-29 PM (Yishi Wang)

17. **Problem.** Show that, for a function $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, the following are equivalent:

(1) f is continuous and is analytic except at a finite number of points.
(2) Either f is the constant function $z \mapsto \infty$ or there are polynomials P and Q, with Q not identically 0, so that $f(z) = P(z)/Q(z)$ for all $z \in \mathbb{C}$ with $Q(z) \neq 0$.

The functions described in Problem 17, except for the constant function $z \mapsto \infty$, are called *rational functions*.

18. **Problem.** Suppose $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational function and $f(z) = P(z)/Q(z)$ as in Problem 17. Suppose P and Q have no common factors of positive degree and let $p = \deg P$, $q = \deg Q$. What can you say about the cardinality of $f^{-1}\{b\}$ for $b \in \hat{\mathbb{C}}$?

Suppose $f : B(a,r)\backslash\{a\} \to \mathbb{C}$ is an analytic complex function ,we have 3 diffrent types of isolated singular points for$f$:

Case1: removable singualrity, if $\lim_{z\to a} f(z) = L \in \mathbb{C}$

Case2: pole singaularity,if $\lim_{z\to a} f(z) = \infty$, then $f(z) = \frac{g(z)}{(z-a)^d}$, both g and d are unique. d is a positive integer such that

$$\lim_{z\to a}(z-a)^d f(z) = L \neq 0, \infty.$$

Case3: essential singularity.

17.1. **Theorem.** (Casorati-Weierstrass Theorem)
Let $f : B(a,r)\backslash\{a\} \to \mathbb{C}$ is an analytic complex function. If f has an isolated singularity at $a \in \hat{\mathbb{C}}$, there exists some $r > 0$ and some nonempty open disk D in $\hat{\mathbb{C}}$ such that $f(B(a,r)\backslash\{a\}) \bigcap D = \phi$, then a is a removable or a pole singularity.

Proof.
(1) Let D be the open disk centered at ∞, then $D = \{z \in \mathbb{C} : |z| > M\}$ for some $M > 0$. Since the hypothesis $f(B(a,r)\backslash\{a\}) \bigcap D = \phi$ means that $|f(z)| \leq M$ for $z \in B(a,r)\backslash\{a\}$, $\lim_{z\to a} f(z) < \infty \Rightarrow a$ is removable singular.
(2) If $D = B(b,\epsilon)$ for some $b \in \mathbb{C}$, define $g(z) = \frac{1}{f(z)-b}$, then by (1) g has a removable singularity at a.

Since $f(z) = \frac{1}{g(z)} + b$, If $\lim_{z\to a} g(z) \neq 0$ then f has a removable singularity at a; If $\lim_{z\to a} g(z) = 0$ and g has 0 at a with multiplicity d, then f has a pole of order d at a □

17.2. **Corollary.** *Let $f : B(a,r)\backslash\{a\} \to \mathbb{C}$ is an analytic complex function. If a is an essential singular then $f(B(a,r)\backslash\{a\})$ is dense in $\hat{\mathbb{C}}$ for any $r > 0$.*

17.3. **Theorem.** (Big Picard theorem)
Let $f : B(a,r)\backslash\{a\} \to \mathbb{C}$ is an analytic complex function. If a is an essential singular, for any $r > 0$, $\hat{\mathbb{C}}\backslash f(B(a,r)\backslash\{a\})$ has less or equal to 2 points

17.4**. Theorem.** (Little Picard theorem)

Let $f : B(a,r)\backslash\{a\} \to \mathbb{C}$ *is an analytic complex function. If* f *is entire and not a pole, then for any* $r > 0$*,* $fZ : |Z| > r$ *omits at most 1 point.*

17.5**. Example.** *Let* $f(z) = \frac{z^3-2z+1}{z^2-3z+1}$*. Then* f *is continuous and analytic in* $\mathbb{C}\backslash\{1,2\}$*. We have removable singularity at 1, and pole singularity at 2.*

18. 2003-10-01 (Dennis Pixton)

Consider analytic functions $f\colon V \to \mathbb{C}$ where V is an open set and curves $\gamma\colon I \to V$ where $I = [0,1]$. The idea is to consider various properties of the mapping $(f,\gamma) \mapsto \int_\gamma f$.

We start by looking at continuity properties. If $f\colon X \to \mathbb{C}$ is any function and K is a non-empty subset of X we define

$$\|f\|_{0,K} = \sup_{x\in K} |f(x)|.$$

18.1. **Lemma.** *(1) $\|f\|_{0,K}$ is finite iff f is bounded on K.*
(2) $\|f\|_{0,K} \geq 0$, and $\|f\|_{0,K} = 0$ iff $f(x) = 0$ for all $x \in K$.
(3) $\|af\|_{0,K} = |a| \cdot \|f\|_{0,K}$ if $a \in \mathbb{C}$ (interpreting $a \cdot \infty$ as 0).
(4) $\|f+g\|_{0,K} \leq \|f\|_{0,K} + \|g\|_{0,K}$.

If $K = X$ we just write $\|f\|_0$ rather than $\|f\|_{0,X}$. According to the lemma, $\|\ \|_0$ is a *norm* on the vector space of bounded functions on X.

First we consider the functions f. We define $H(V)$ as the set of all analytic functions defined on V.

18.2. **Lemma.** *Suppose $K \subset V$ is compact and non-empty. Then, for any piecewise differentiable curve γ in K, and any $f_1, f_2 \in H(V)$,*

$$\left|\int_\gamma (f_1 - f_2)\right| \leq L(\gamma)\,\|f_1 - f_2\|_{0,K}\,.$$

where $L(\gamma)$ is the length of γ.

We say a sequence f_n of functions defined on V *converges uniformly on caompacta* to f iff, for any compact $K \subset V$,

$$\lim_{n\to\infty} \|f - f_n\|_{0,K} = 0.$$

18.3. **Theorem.** *Suppose $f_n \in H(V)$ converges uniformly on compacta to $h\colon V \to \mathbb{C}$. Then $f \in H(V)$.*

Proof. Pick $a \in V$ and let K be a closed disk of positive radius with center a so that $K \subset V$. First we show that f is continuous on K. This is the standard "$\frac{\epsilon}{3}$" argument: Given $\epsilon > 0$ choose N so that $\|f - f_n\|_{0,K} < \frac{\epsilon}{3}$ for all $n \geq N$. Now choose $\delta > 0$ so that $|f_N(z) - f_N(w)| < \frac{\epsilon}{3}$ for all $z, w \in K$ with $w - z| < \delta$; this is just uniform continuity of f_N on K. Now select such z and w and compute

$$\begin{aligned}
|f(z) - f(w)| &\leq |f(z) - f_N(z)| + |f_N(z) - f_N(w)| + |f_N(w) - f(w)| \\
&\leq \|f - f_N\|_{0,K} + |f_N(z) - f_N(w)| + \|f_N - f\|_{0,K} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Now if T is any triangular path in the interior of K then

$$\begin{aligned} \left|\int_T f\right| &\leq \left|\int_T f_n\right| + \left|\int_T (f-f_n)\right| \\ &= 0 + \left|\int_T (f-f_n)\right| && \text{by Cauchy's Theorem} \\ &\leq L(T)\,\|f-f_n\|_{0,K} \to 0 \end{aligned}$$

So $\int_T f = 0$. By Morera's Theorem, f is analytic in the interior of K. □

Next we look at the dependence of the integral on γ. We write $PC^1(I,V)$ for the set of piecewise differentiable curves $\gamma\colon I \to V$. Here is a preliminary inequality:

18.4. **Lemma.** *Suppose V is convex. If $f \in H(V)$ and $\gamma \in PC^1(I,V)$ then there is $\alpha > 0$ so that if $\gamma_1 \in PC^1(I,V)$ and $\|\gamma - \gamma_1\|_0 < \alpha$ then*

$$\left|\int_\gamma f - \int_{\gamma_1} f\right| \leq \|f\|_0 \cdot \left(|f(\gamma(0)) - f(\gamma_1(0))| + |f(\gamma(1)) - f(\gamma_1(1))|\right).$$

Proof. α is chosen so that the disks of radius α centered at $\gamma(0)$ and at $\gamma(1)$ lie in V. Then construct a closed path Γ consisting of γ, γ_1, and the straight segments σ_0 from $\gamma(0)$ to $\gamma_1(0)$ and σ_1 from $\gamma(1)$ to $\gamma_1(1)$ (with suitable orientations). Then $\int_\Gamma f = 0$ by Cauchy's Theorem, and this gives

$$\int_\gamma f - \int_{\gamma_1} f = \int_{\sigma_0} f - \int_{\sigma_1} f$$

Then the inequality follows since the lengths of σ_0 and σ_1 are $|f(\gamma(0)) - f(\gamma_1(0))|$ and $|f(\gamma(1)) - f(\gamma_1(1))|$. □

19. 2003-10-08 (Adam Perry)

20. 2003-10-09 (Garry Bowlin)

Recall, If $f\colon V \to \mathbb{C}$, and γ_0 and γ_1 are piecewise continuous, and that σ_0 and σ_1 are the paths between the endpoints of the $\gamma' s$ all oriented so that they complete a cycle. Then we have that $\int_{\gamma_1} f - \int_{\gamma_0} f = \int_{\sigma_1} f - \int_{\sigma_1} f$. Given γ_1 this works if $\|\gamma_1 - \gamma_0\|_0 < \frac{\alpha}{2}$ where α is chosen $N_\alpha(\gamma_1(I)) \subset V = \{z\colon \exists t, |z - \gamma_1(t)| < \alpha\}$ if $\gamma \in C^0(I, V)$ define $\int_\gamma f = \int_\Gamma f$ where Γ is PC^1 and has the same endpoints as γ and is close enough to γ Choose α so $N_\alpha(\gamma) \subset V$ and choose $\beta = \frac{\alpha}{2}$ then we have that $\|\gamma - \Gamma_1\|_0 < \frac{\beta}{4}$ and $\|\gamma - \Gamma_2\|_0 < \frac{\beta}{4}$ so now $\|\Gamma_1 - \Gamma_2\|_0 < \frac{\beta}{2}$ and we have $N_\beta(\Gamma_1) \subset N_\alpha(\gamma) \subset V$

Homology: Suppose γ_1, γ_2 are continuous paths in B. A homology between γ_1 and γ_2 is a function $H \in C^0(I \times I, V)$ so that $H(0, t) = \gamma_0(t)$ and $H(1, t) = \gamma_1(t)$

Define: $h\colon I \to C^0(I, V)$ and $h_s = h(s) = H(s, \dot{\ }) \in C^0(I, V)$ $h_s(t) = H(s, t)$ and $h(0) = \gamma_0)$ and $h(1) = \gamma_1)$

We have that h is continuous: $\|h_{s_1}(t) - h_{s_2}(t)\| = \sup_{t \in I} \|H(s_1, t) - H(s_2, t)\|$ by uniform continuity $\exists \delta$ so $|(s_1, t) - (s_2, t)| < \delta \Rightarrow \|H(s_1, t) - H(s_2, t)\| \leq \epsilon/2$ so $\sup_{t \in I} \|H(s_1, t) - H(s_2, t)\| < \epsilon$

20.1. **Lemma.** *If $H : I \times I \to V$ and $h\colon I \to C^0(I, V)$ are related by $H(s, t) = h_s(t$ then h is continuous $\Leftrightarrow$ H is continuous.*

20.2. **Corollary.** *If H is a homology in V then $s \to \int_{h_0} f$ is continuous.*

With this we now have $|\int_{h_{s_1}} f - \int_{h_{s_2}} f| = |\int_{h_{\sigma_1}} f - \int_{h_{\sigma_2}} f| \leq |\int_{h_{\gamma_1}} f - \int_{h_{\gamma_2}} f|$

20.3. **Definition.** V is simply connected $\Leftrightarrow$ V is connected and any closed path can be homotoped through closed paths in V to a constant curve.

20.4. **Corollary.** *If V is simply connected then $\int_\gamma f = 0$ for all closed curves γ in V and $f \in H(V)$*

20.5. **Theorem** (Riemann Mapping Theorem). *V is simply connected $\Leftrightarrow$ V is homeomorphic to a disk.*

20.6. **Corollary.** *If V is simply connected and $0 \notin V$ then there is a branch of the log defined on V.*

20.7. **Definition.** A cycle in V is a finite forma linear combination of closed curves in V.

We define $\int_c f = \sum \alpha_i \int_{\gamma_i} f$ where $c = \sum \alpha_i \gamma_i$ is a cycle and γ_i is a closed curve in V and $\alpha_i \in \mathbb{C}$

20.8. **Theorem.** *Suppose c is a cycle in V then $\int_c f = 0$ for $f \in H(V)$ $\Leftrightarrow$ for all $w \in \mathbb{C}$ we have that $n(c, w) = 0$*

20.9. **Definition.** for c_1, c_2 cycles in V we say c_1 c_2 ("is homologous to") if and only if $\int_{c_1} f = \int_{c_2} f$ for all $f \in H(V)$

21. 2003-10-10 (Ken Ferry)

22. 2003-10-13 (Mustafa Findik)

19**. Problem.** Let

$$V = \mathbb{C} \setminus \left\{ \frac{1}{n} \colon n \in \mathbb{Z},\ n > 0 \right\} \setminus \{ 0 \} .$$

Determine $H_1(V)$ as explicitly as you can. Determine a reasonable spanning set, and give a basis if you can.

If you prefer you can instead work with $V = \mathbb{C} \setminus K$ where K is the standard "middle thirds" Cantor set in $[0, 1]$.

20**. Problem.** Suppose f is entire and a and b are distinct points in $\mathbb{C}$. Let γ_1 and γ_2 be circles centered at a and b respectively which bound disjoint disks.

(1) Use the CIF to calculate

$$\int_{\gamma_1} \frac{f(z)}{(z-a)(z-b)} dz \text{ and } \int_{\gamma_2} \frac{f(z)}{(z-a)(z-b)} dz.$$

Your answers will be in terms of f, a and b.

(2) Let Γ be a circle centered at the origin which does not pass through a or b. Remembering that γ_1 and γ_2 form a basis for the homology of $\mathbb{C} \setminus \{ a, b \}$, calculate

$$\int_{\Gamma} \frac{f(z)}{(z-a)(z-b)} dz.$$

Your answer will depend on the relations between $|a|$, $|b|$, and the radius R of Γ. You may assume $|a| \leq |b|$.

21**. Problem.** Suppose f is entire and bounded. Use Problem 20 to give an alternate proof of Liouville's Theorem.

23. 2003-10-15 (Apratim Roy)

Suppose α is a positive real number and consider the mesh formed by all horizontal and vertical lines passing through the points $(j\alpha, k\alpha)$ for $j, k \in \mathbb{Z}$. This mesh divides $\mathbb{C}$ into closed squares of side length α. Designate a finite number of these squares as "K-squares" and the rest as "N squares". Say a directed segment σ from one of the vertices $(j\alpha, k\alpha)$ to one of its horizontal or vertical neighbors is a "KN segment" if the square on the left of σ is a K square and the one on the right is an N square.

This is purely combinatorial:

22**. Problem.** If σ is a KN segment then there is a closed path $\Gamma = \sigma_1\sigma_2\ldots\sigma_n$ where $\sigma_1 = \sigma$, each σ_k is a KN segment, and, for each j, k, if $j \neq k$ then σ_j is not equal to σ_k or to the reversal of σ_k.

And this is easy:

23**. Problem.** If there is at least one K square then there is a KN path Γ as in Problem 22 and some K square Q so that $n(\Gamma, w) = 1$ for each point of Q not on Γ.

Now apply Problems 22 and 23:

24**. Problem.** Suppose K is compact and not empty, F is closed, and $F \cap K = \emptyset$. Let $V = \mathbb{C} \setminus (K \cup F)$. Show that there is a closed path γ in V so that, for some $w \in K$, $n(\gamma, w) = 1$.

24. 2003-10-16 (Dennis Pixton)

There are three miscellaneous topics for today.

24.1. **Calculating winding numbers.** Suppose $\gamma\colon [0,1] \to \mathbb{C}$ is a closed curve, and $w \in \mathbb{C} \setminus |\gamma|$. Suppose $\alpha \in \mathbb{R}$ and define

$T = \{\, w + re^{i\alpha}\colon r \geq 0 \,\}$	the ray from w with angle α
$S = \{\, w + re^{i\alpha}\colon r \in \mathbb{R} \,\}$	the line containing T
$L = \{\, w + re^{i\theta}\colon r > 0,\ \alpha < \theta < \alpha + \pi \,\}$	the halfplane on the left of T
$R = \{\, w + re^{i\theta}\colon r > 0,\ \alpha - \pi < \theta < \alpha \,\}$	the halfplane on the right of T.

We suppose $\gamma(0) \notin T$ and $\gamma^{-1}(T)$ is finite, and we enumerate $\gamma^{-1}(T)$ as $\{\, t_1, t_2, \ldots, t_n \,\}$. Let T' be the closed ray from w with angle $-\alpha$. By continuity $\gamma^{-1}(T)$ and $\gamma^{-1}(T')$ are closed in $[0,1]$, and so are compact, and they are disjoint since γ does not go through w. Hence there is $\epsilon > 0$ so that, for each j, both $(t_j - \epsilon, t_j)$ and $(t_j, t_j + \epsilon)$ lie in $(0,1)$ and are disjoint from $\gamma^{-1}(T) \cup \gamma^{-1}(T') = \gamma^{-1}(S)$. Then the connected sets $\gamma(t_j - \epsilon, t_j)$ and $\gamma(t_j, t_j + \epsilon)$ lie in $\mathbb{C} \setminus S = L \cup R$, so each lies in either L or in R. So there are four possible configurations. We define the intersection number of γ and T at t_j as follows:

$$\operatorname{int}(\gamma, T, t_j) = \begin{cases} 1 & \text{RL crossing: } \gamma(t_j - \epsilon, t_j) \subset R,\ \gamma(t_j, t_j + \epsilon) \subset L \\ -1 & \text{LR crossing: } \gamma(t_j - \epsilon, t_j) \subset L,\ \gamma(t_j, t_j + \epsilon) \subset R \\ 0 & \text{bounce: both on same side.} \end{cases}$$

24.1. **Theorem.**

$$n(\gamma, w) = \sum_{j=1}^{n} \operatorname{int}(\gamma, T, t_j).$$

Proof. We first eliminate the “bounces”: If $\operatorname{int}(\gamma, T, t_j) = 0$ then we can homotope the curve γ near t_j to remove t_j from the intersection set. This homotopy does not change the winding number, and it does not change the sum of intersection numbers.

So we assume all the intersection numbers are ± 1. For each j let γ_j be a circle with center at w which intersects T at $\gamma(t_j)$, oriented to have the same intersection number with T as γ does. Form a closed curve Γ by piecing together γ and the circles γ_j: Follow $\gamma(t)$ for $t \in [0, t_1]$, then follow γ_1 *in the opposite direction*; continue with $\gamma(t)$ for $t \in [t_1, t_2]$, then follow γ_2 in the opposite direction; etc. This eventually ends with $\gamma(t)$ for $t \in [t_n, 1]$. Now Γ meets T for $2n$ different values of the parameter and all its intersection numbers with T are 0. So by the previous paragraph, there is a small homotopy which moves Γ into $\mathbb{C} \setminus T$. This is starlike, so Cauchy's Theorem applies to show that $n(\Gamma, w) = 0$.

Finally, reassembling the integrals over the $2n + 1$ subpaths that define Γ gives

$$0 = n(\Gamma, w) = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z - w} = \frac{1}{2\pi i} \int_{\gamma - c} \frac{dz}{z - w} = n(\gamma - c, w)$$

where c is the cycle $\sum_{j=1}^{n} \gamma_j$, so $n(\gamma, w) = \sum_{j=1}^{n} n(\gamma_j, w)$. This finishes the proof since $n(\gamma_j, w) = \operatorname{int}(\gamma, T, t_j)$. □

24.2. Complements of null-homologous sets.

24.2. **Theorem.** *If V is a connected open subset of $\mathbb{C}$ then $H_1(V) = \{ 0 \}$ if and only if $\hat{\mathbb{C}} \setminus V$ is connected.*

Proof. If $\hat{\mathbb{C}} \setminus V$ is connected then any component of $\mathbb{C} \setminus V$ is unbounded. Let c be a cycle in V and consider $w \in \mathbb{C} \setminus V$. Then w lies in such a component F of $\mathbb{C} \setminus V$. But $n(c, w)$ is constant as w varies over the connected set F and limits to 0 as $w \to \infty$, so $n(c, w) = 0$ for all $w \in F$. Hence $n(c, w) = 0$ for all $w \in \mathbb{C} \setminus V$ so c is homologous to 0. Thus $H_1(V) = \{ 0 \}$.

Conversely, if $\hat{\mathbb{C}} \setminus V$ is not connected then we can write $\hat{\mathbb{C}} \setminus V = K \cup L$ where K and L are disjoint, compact and non-empty, and $\infty \in L$. Then $K \subset \mathbb{C}$ and $F = L \cap \mathbb{C}$ is closed, and Problem 24 provides a curve Γ in V so that $n(\Gamma, w) = 1$ for some $w \in K$. Hence Γ is not homologous to 0 in V, so $H_1(V) \neq \{ 0 \}$. $\square$

24.3. Preparation for the Argument Principle.

Suppose V is a connected open set and $f \colon V \to \mathbb{C}$ is analytic and non-constant. For $b \in \mathbb{C}$ consider $f^{-1} \{ b \}$.

24.3. **Lemma.** *No limit point of $f^{-1} \{ b \}$ lies in V.*

Proof. It follows from the classification of singularities theorem that if $f(a) - b = 0$ then there is $r > 0$ so that $f(z) - b \neq 0$ for $0 < |z - a| < r$. $\square$

24.4. **Lemma.** *Let $F = \mathbb{C} \setminus V$. For a positive integer n define*

$$U_n = \left\{ z \in \mathbb{C} \colon |z| < n \text{ and } d_F(z) > \frac{1}{n} \right\}$$

$$K_n = \left\{ z \in \mathbb{C} \colon |z| \leq n \text{ and } d_F(z) \geq \frac{1}{n} \right\}$$

Then U_n is open, K_n is compact, $U_n \subset K_n \subset U_{n+1}$, and $V = \bigcup_n U_n = \bigcup_n K_n$.

24.5. **Lemma.** *Suppose c is a cycle in V which is homologous to 0, and suppose $|c| \cap f^{-1} \{ b \} = \emptyset$. Then $\{ a \in f^{-1} \{ b \} : n(c, a) \neq 0 \}$ is finite.*

Proof. By compactness there is n so that $|c| \subset U_n \subset K_n \subset V$. Since K_n is compact, Lemma 24.3 implies that $K_n \cap f^{-1} \{ b \}$ is finite. Suppose $w \notin K_n$; then either $|w| > n$ or $d_F(w) < \frac{1}{n}$.

Since $|c| \subset B(0, n)$ the function $n(c, \cdot)$ is zero on the connected unbounded set $\mathbb{C} \setminus B(0, n)$. So in the first case $n(c, w) = 0$.

In the second case there is some $w_0 \in F$ so that $w \in B(w_0, \frac{1}{n})$. Then $B(w_0, \frac{1}{n}) \cap |c| \subset B(w_0, \frac{1}{n}) \cap U_n = \emptyset$ so $B(w_0, \frac{1}{n})$ is a connected set in the complement of $|c|$. Hence $n(c, w) = n(c, w_0)$. But $n(c, w_0) = 0$ since c is homologous to 0 in V. $\square$

25. 2003-10-17 (Nigar Tuncer)

25.1. **Theorem** (The Inverse Function Theorem, IFT). *If V is an open set and $f\colon V \to \mathbb{C}$ is an analytic injection then $f(V)$ is open and $f^{-1}\colon f(V) \to \mathbb{C}$ is analytic.*

The following in needed in the proof of the IFT:

25. **Problem.** Use the argument principle to show that if $f\colon V \to \mathbb{C}$ is an analytic injection then $f'(a) \neq 0$ for all $a \in V$.

26. 2003-10-20 (Fabrizio Polo)

27. 2003-10-30 (Garry Bowlin)

28. 2003-10-31 (Yishi Wang)

26. **Problem.** Using the metric ρ, show that $C(V, X)$ is complete if X is complete.

Here's an outline for Problem 26:

1. Start with a ρ-Cauchy sequence f_k in $C(V, X)$.
2. Show that, for fixed n, f_k is ρ_n-Cauchy, and then that it is d_n-Cauchy.
3. Show that, for any $z \in K_n$, $f_k(z)$ is a d-Cauchy sequence in X, so it converges to some $x_z \in X$.
4. Notice that you can define $f \colon V \to V$ by $f(z) = x_z$. You need to argue that this makes sense; that is, that for any $z \in V$ you can find K_n so that $z \in K_n$ and that x_z is independent of the choice of such n. This should be trivial.
5. Fix n, and suppose $\epsilon > 0$. Choose N so that $d_n(f_k, f_j) < \frac{\epsilon}{3}$ for all $j, k > N$, and argue that $d(f_k(z), f(z) < \frac{2\epsilon}{3}$ for all $k > N$ and all $z \in K_n$. To do this, note that you can choose $j > N$, *depending on* z, so that $d(f_j(z), f(z)) < \frac{\epsilon}{3}$.

   Conclude that, for fixed n, $f_k|K_n$ converges uniformly to $f|K_n$.
6. Step 5 implies (by a well known theorem about uniform convergence) that $f|K_n$ is continuous for each n. Show that f is continuous, so $f \in C(V, X)$.
7. Finally, show that $\rho(f_k, f) \to 0$ as $k \to \infty$.

29. 2003-11-03 (Dennis Pixton)

We are looking at subsets $\mathcal{F} \subset C(V, X)$. We shall need the following assumptions:

1. (X, d) is a complete metric space.
2. V has an exhaustion: $U_n \subset K_n \subset U_{n+1}$ with K_n compact, U_n open, and $\bigcup_n U_n = V$.
3. V is a metric space.
4. For any $z \in V$ there is some positive r so that $\bar{B}(z, r)$ is compact.

These are satisfied if V is an open subset of $\mathbb{C}$ (or of $\mathbb{R}^k$ or $\mathbb{C}^k$) and $X = \mathbb{C}$. Part (4) is equivalent to the topological requirement that V is *locally compact*; that is, that each point of V has a compact neighborhood in V.

Under these assumptions the topology of uniform convergence on compacta on $C(V, X)$ is defined by the metric ρ, and, according to Problem 26, the metric space $(C(V, X), \rho)$ is complete.

29.1. **Definition.** $\mathcal{F}$ is *normal* iff every sequence in $\mathcal{F}$ has a subsequence which converges uniformly on compacta.

29.2. **Theorem.** *The following are equivalent:*

1. $\mathcal{F}$ *is normal.*
2. $\operatorname{Cl} \mathcal{F}$ *is sequentially compact.*
3. $\operatorname{Cl} \mathcal{F}$ *is compact.*

Proof. (2) and (3) are equivalent for any metric space, and (2) trivially implies (1).

Suppose (1) is true and let f_n be any sequence in $\operatorname{Cl} \mathcal{F}$. For each n choose $h_n \in \mathcal{F}$ so that $\rho(f_n, h_n) < \frac{1}{n}$. Then some subsequence h_{n_k} of h_n converges to some h (since $\mathcal{F}$ is normal), and it follows that f_{n_k} also converges to h. $\square$

Suppose (A, d_A) and (B, d_B) are metric spaces. Then the product topology on $A \times B$ is the same as the topology determined by the metric μ, defined by $(\mu((a_1, b_1), (a_2, b_2)) = \max\{\, d_A(a_1, a_2), d_B(b_1, b_2) \,\}$.

Define the *evaluation function* $e\colon C(V, X) \times V \to X$ by $e(f, z) = f(z)$.

29.3. **Lemma.** *e is continuous.*

Proof. Given $(f, z) \in C(V, X) \times V$ and $\epsilon > 0$, first determine $\delta > 0$ so that $d(f(z), f(w)) < \frac{\epsilon}{2}$ whenever $d(w, z) < \delta$. Choose $r > 0$ so that $\bar{B}(z, r)$ is compact and let $\alpha = \min\left\{ r, \frac{\delta}{2} \right\}$. Then $K = \bar{B}(z, \alpha)$ is compact. Choose $\beta > 0$ so that $B(\rho, f, \beta) \subset B(d_K, f, \frac{\epsilon}{2})$.

Now if $\mu((g, w)(f, z)) < \min\{\, \alpha, \beta \,\}$ we have

$$\begin{aligned} d(e(g, w), e(f, z)) = d(g(w), f(z)) &\le d(g(w), f(w)) + d(f(w), f(z)) \\ &\le d_K(g, f) + d(f(w), f(z)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

$\square$

29.4. **Definition.** $\mathcal{F}$ is *equicontinuous* on the set $K \subset V$ iff

$$\forall \epsilon > 0\, \exists \delta > 0 \text{ s.t. } \forall f \in \mathcal{F}\, \forall z, w \in K,\ d(x, y) < \delta \implies d(f(z), g(z)) < \epsilon.$$

$\mathcal{F}$ is *equicontinuous on compacta* iff it is equicontinuous on all compact subsets of V.

29.5**. Definition.** For $z \in V$, $\mathcal{F}(z) = \{\, f(z) \colon f \in \mathcal{F}\,\}$. More generally, if $K \subset V$, $\mathcal{F}(K) = \{\, f(z) \colon f \in \mathcal{F},\, z \in K\,\}$.

29.6**. Theorem.** *Suppose $\mathcal{F}$ is compact and $K \subset V$ is compact. Then*

(1) *$\mathcal{F}(K)$ is compact.*
(2) *$\mathcal{F}$ is equicontinuous on K.*

Proof. $\mathcal{F}(K) = e(\mathcal{F} \times K)$ is compact since it is the continuous image of a compact set. Also, e is conntinuous on the compact set $\mathcal{F} \times K$, so it is uniformly continuous on $\mathcal{F} \times K$. Given $\epsilon > 0$ fine $\delta > 0$ so that $d(e(g, w), e(f, z)) < \epsilon$ whenever (g, w), $(f, z) \in \mathcal{F} \times K$ with $\mu((g, w), (f, z)) < \delta$. Then for any $f \in \mathcal{F}$ and any $w, z \in K$ with $d(w, z) < \delta$ we have $\mu((f, w), (f, z)) = d(w, z) < \delta$ so $d(f(w), f(z)) = d(e(f, w), e(f, z)) < \epsilon$. $\square$

30. 2003-11-05 (Adam Perry)

27. Problem. If V is a metric space with an exhaustion then V has a countable dense subset.

31. 2003-11-06 (Mustafa Findik)

32. 2003-11-07 (Yishi Wang)

The *open annulus* with center a and radii r and R is $\operatorname{Ann}(a,r,R) = \{\, z \in \mathbb{C} \colon r < |z| < R \,\}$. This is only interesting, of course, if $0 \le r < R$. If $r = 0$ then $\operatorname{Ann}(a,0,R)$ is the *punctured disk* with center a and radius R; that is, $\operatorname{Ann}(a,0,R) = B(a,R) \setminus \{\, a \,\}$.

28**. Problem.** Suppose $0 < r < 1$. Exhibit a homeomorphism between $\operatorname{Ann}(0,0,1)$ and $\operatorname{Ann}(0,r,1)$ but show that there is no analytic bijection f of $\operatorname{Ann}(0,0,1)$ onto $\operatorname{Ann}(0,r,1)$. [Suggestion: First show that f has a removable singularity at 0.]

Note: For $0 \le s < r < 1$ there is no analytic bijection of $\operatorname{Ann}(0,s,1)$ onto $\operatorname{Ann}(0,r,1)$. If you find a simple proof of this I would like to see it.

33. 2003-11-10 (Nigar Tuncer)

Notation: $D = B(0,1)$ is the open unit disk and $S^1 = \{\, z \in \mathbb{C} \colon |z| = 1 \,\}$ is the unit circle. If V is an open subset of $\mathbb{C}$ then $\operatorname{Aut}(V)$ denotes the group of analytic bijections of V onto itself.

33.1. **Theorem.** $\operatorname{Aut}(D)$ *is the group of fractional linear transformations of the form*

$$T_{\lambda,a} : z \mapsto \lambda \frac{z-a}{1-\overline{a}z}$$

with λ in the unit circle S^1 and $a \in D$, and the parameters λ and a are uniquely determined by the transformation.

29. **Problem.** The identification of $\operatorname{Aut}(D)$ as the set of transformations $T_{\lambda,a}$ induces a group structure on the open torus $S^1 \times D$. Determine this group structure; that is, give the multiplication and inversion rules for pairs of the form (λ, a).

Let H be the *upper half plane*, so $H = \{\, z \in \mathbb{C} \colon \operatorname{Im} z > 0 \,\}$.

30. **Problem.** Since H is simply connected there is an analytic bijection f of H onto D. Find such a map. [Suggestion: There is a fractional linear transformation which will do this. Determine this by picking three points on $\mathbb{R} \cup \{\, \infty \,\}$ and three points on S^1 and require that f take the first triple onto the second triple. Your map may send D to the lower half plane instead of the upper; in this case try again, reversing the order of one of the triples.]

If A is an invertible 2×2 matrix over $\mathbb{C}$ then T_A is the corresponding fractional linear transformation:

$$\text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } T_A(z) = \frac{az+b}{cz+d}.$$

Using conjugation by the map f of Problem 30 it is easy to define an isomorphism between $\operatorname{Aut}(D)$ and $\operatorname{Aut}(H)$, and it follows from the form of this isomorphism that $\operatorname{Aut}(H)$ is a group of fractional linear transformations. More precisely:

31. **Problem.** The group $\operatorname{Aut}(H)$ is the group of fractional linear transformations T_A, where the entries of A are real and A has positive determinant.

$\mathrm{PSL}(2,\mathbb{R})$ is the *real projective special linear group of rank 2*, defined as the group of 2×2 real matrices A with $\det A = 1$, under the equivalence relation $A \sim -A$. Then Problem 31 can be adapted to give an isomorphism between $\operatorname{Aut}(H)$ and $\mathrm{PSL}(\mathbb{R}, 2)$.

32. **Problem.** The groups $\operatorname{Aut}(D)$, $S^1 \times D$, $\operatorname{Aut}(H)$ and $\mathrm{PSL}(2,\mathbb{R})$ are all isomorphic. Give an explicit isomorphism between $\mathrm{PSL}(2,\mathbb{R})$ and $S^1 \times D$.

34. 2003-11-12 (Apratim Roy)

33**. Problem.** Find explicitly an analytic bijection of the semi-infinite strip $S = (0, \infty) \times (-\frac{\pi}{2}i, \frac{\pi}{2}i)$ onto D. If you can, write your answer in terms of the trig functions of the complex variable z.

35. 2003-11-13 (Garry Bowlin)

36. 2003-11-14 (Ken Ferry)

36.1**. Definition.** A function $f\colon \mathbb{C} \to \mathbb{C}$ is *doubly periodic* iff there are $a, b \in \mathbb{C}$, linearly independent over $\mathbb{R}$, so that $f(z + ma + nb) = f(z)$ for all $m, n \in \mathbb{Z}$ and all $z \in \mathbb{C}$. This is equivalent to the requirement that $f(z + a) = f(z + b) = f(z)$ for all $z \in \mathbb{C}$.

34**. Problem.** Any analytic doubly periodic function is constant.

37. 2003-11-17 (Fabrizio Polo)

38. 2003-11-17 SUPPLEMENT (DENNIS PIXTON)

Here are a few facts about hyperbolic geometry.

Remember $H = \{ z \in \mathbb{C} \colon \operatorname{Im} z > 0 \}$ is the upper half plane. A *hyperbolic line* is the intersection of H with a circle in $\hat{\mathbb{C}}$ which is perpendicular to the real axis. Given two distinct points of H there is a unique hyperbolic line which passes through both.

If two hyperbolic lines intersect at the point a then the angle between them is just the Euclidean angle between their tangent vectors at a. Given a point a on a hyperbolic line L and an angle $\theta \in [0, \pi]$ there is a unique hyperbolic line L' passing through a making the angle θ with L.

If $\gamma \colon [a, b] \to H$ is a piecewise C^1 curve in H then its *hyperbolic arc length* is

$$\int_a^b \frac{1}{y} ds = \int_a^b \frac{1}{y} \sqrt{dx^2 + dy^2} = \int_a^b \frac{1}{y(t)} |\gamma'(t)| \, dt = \int_a^b \frac{1}{y(t)} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$$

35. Problem. If $M \in \operatorname{Aut}(H)$ (see Problem 31) and γ is a piecewise C^1 curve in H then $M \circ \gamma$ and γ have the same arc length.

It now follows that the hyperbolic line between points a and b of H has the least hyperbolic arc length among curves in H between a and b. This is easily seen if a and b lie on a vertical line, and any other pair of points of H can be mapped to be on a vertical line by an element of H.

Define the *hyperbolic distance* between a and b to be the hyperbolic length of the hyperbolic line connecting them. Then Problem 35 says that the elements of $\operatorname{Aut}(H)$ are *isometries*; that is, they preserve hyperbolic distances. Also, of course, since they are analytic maps they preserve angles.

A *triangle* in H is defined as in Euclidean geometry, but using hyperbolic lines. As a limiting case we also consider triangles in which one or more of the vertices may be on $\hat{\mathbb{R}} = \mathbb{R} \cup \{ \infty \}$. A vertex of a triangle is on $\hat{\mathbb{R}}$ iff the corresponding angle is 0.

38.1. Theorem. *If T and $\tilde{T}$ are triangles then there is an isometry $M \in \operatorname{Aut}(H)$ taking T onto $\tilde{T}$ if and only if their corresponding angles are equal.*

Proof. One direction is obvious.

For the reverse we first assume none of the angles are 0. Let the vertices of T be A, B and C (in counterclockwise order) and let the corresponding angles be α, β, γ. We shall first assume that $\alpha = \frac{\pi}{2}$. There is an element M of $\operatorname{Aut}(H)$ which maps the segment AC to a segment in the Y axis, so that A maps to i and C maps to some ti with $t > 1$. Thus M maps the rest of T into the right half plane.

Since $\alpha = \frac{\pi}{2}$ the segment AB is mapped to an arc of the unit circle from i to a point $w = M(B)$ in the first quadrant. The segment BC is mapped onto an arc of some circle C with center on the X-axis, say with center $c \in \mathbb{R}$ and radius r. Since this arc connects ti, with $t > 1$, to the point w on the unit circle in the first quadrant, it is clear that $c < 0$ and $r > 1$. The Euclidean triangle with vertices at c, 0 and w has angle β at the vertex w, and it follows from the law of cosines that $2r \cos \beta = 1 + r^2 - c^2$. The Euclidean right triangle with vertices c, 0 and ti has angle γ at c, and we conclude $r^2 = c^2 + t^2$ and $r \sin \gamma = t$. Putting these together and doing some algebra gives

$$\cos\beta = \frac{1+r^2-c^2}{2r} = \frac{1+t^2}{2r} = \frac{1+t^2}{2t}\sin\gamma,$$

and so

$$\frac{2t}{1+t^2} = \frac{\sin\gamma}{\cos\beta}.$$

For $t > 1$ the left hand side is a strictly decreasing function of t, so t is uniquely determined by γ and β.

Now if $\tilde{T}$ is another triangle with corresponding $\tilde{\alpha} = \frac{\pi}{2}$, $\tilde{\beta} = \beta$ and $\tilde{\gamma} = \gamma$ then there will be some $\tilde{M} \in \operatorname{Aut}(H)$ mapping $\tilde{T}$ onto the same triangle with vertices i, w and ti, so $\tilde{M}^{-1}M$ maps T onto $\tilde{T}$.

The general case is reduced to the right triangle case by dropping a perpendicular from one vertex to the opposite side (perhaps extended), thus representing T as the "sum" or "difference" of two right triangles; details are left to the reader.

Finally, the cases when one or more vertex is on $\hat{\mathbb{R}}$ is left to the reader. ☐

Notice that, as a special case, any two triangles with all three vertices on $\hat{\mathbb{R}}$ are equivalent by an element of $\operatorname{Aut}(H)$.

To define the hyperbolic area of a region $R \subset H$ we use a double integral. Since the lengths of small hyperbolic segments in the X and Y directions are, respectively, approximately $\frac{1}{y}\Delta x$ and $\frac{1}{y}\Delta y$, we define the area of R as

$$\iint_R \frac{1}{y^2}\,dx\,dy.$$

In general, one must assume that R is Lebesgue measurable in order for the integral to make sense. It is easy to check that H has infinite area.

38.2. **Theorem.** (1) *If $R \subset H$ and $M \in \operatorname{Aut}(H)$ then $M(R)$ has the same area as R.*

(2) *If T is a triangle with angles α, β, γ then the area of T is*

$$\pi - (\alpha+\beta+\gamma).$$

38.3. **Corollary.** *The sum of the angles in a hyperbolic triangle is less than π.*

38.4. **Corollary.** *All vertices of a hyperbolic triangle T lie on $\hat{\mathbb{R}}$ if and only if T has area π.*

39. 2003-11-19 (Mustafa Findik)

40. 2003-11-20 (Apratim Roy)

36**. Problem.** Let C be the set of images of vertices of the fundamental domain X under the action of $\mathcal{G}$. That is,

$$C = \{\, M(v)\colon M \in \mathcal{G},\, v \in \{\, -1, 0, 1, \infty \,\}\,\}\,.$$

Show that C is dense in $\hat{\mathbb{R}}$.

41. 2003-11-21 (Yishi Wang)

42. 2003-11-05 (Adam Perry)

Remember that S and T are the fractional linear transformations

$$S(z) = \frac{z}{2z+1}, \quad T(z) = z + 2,$$

and r_A, r_B and r_C are the reflections of H through the hyperbolic lines

$$\begin{aligned} A &= \{ z \in H \colon \operatorname{Re} z = 0 \} \\ B &= \{ z \in H \colon |2z - 1| = 1 \} \\ C &= \{ z \in H \colon \operatorname{Re} z = 1 \} . \end{aligned}$$

37. **Problem.** It was shown in class that $T = r_C r_A$. Show that S is the composition of two of the reflections in $\{ r_A, r_B, r_C \}$. (I may have given the wrong product in class.) Now express each composition of two of these reflections in terms of S and T.

Suppose L and L' are circles in $\hat{\mathbb{C}}$ with corresponding reflections r and r'.

42.1. **Lemma** (Schwartz reflection principle). *If $f \colon V \to \mathbb{C}$ is analytic then $r' \circ f \circ r$ is analytic on $r(V)$.*

42.2. **Lemma.** *Suppose $f \colon V \to \mathbb{C}$ is continuous and analytic except on $V \cap L$. Then f is analytic on V.*

43. 2003-11-24 PM (Dennis Pixton)

Before proving the Picard Theorem we review some details of the construction of λ. The fundamental domain for the group $\mathcal{G}$ is

$$X = \{\, z \in H \colon -1 < \operatorname{Re} z \leq 1,\ |2z+1| > 1,\ |2z-1 \geq 1 \,\}.$$

This is actually the "mirror image" of the fundamental domain I used in class; that is, X contains the boundary curves on the right rather than the ones on the left. There is no substantive difference, and this version makes the rest somewhat neater.

The construction of λ starts with λ_0 on the triangle

$$Y_0 = \{\, z \in H \colon 0 \leq \operatorname{Re} z \leq 1,\ |2z-1| \geq 1 \,\}$$

courtesy of the RMT. This is extended by the Schwartz reflection principle, using the reflections r_1, r_2 and r_3 in the three sides of Y_0. The interiors of the sets $Y_0 \cup r_j(Y_0)$ are

$$\begin{aligned} Y_1 &= \{\, z \in H \colon -1 < \operatorname{Re} z < 1,\ |2z+1| > 1,\ |2z-1 > 1 \,\} \\ Y_2 &= \{\, z \in H \colon 0 < \operatorname{Re} z < 2,\ |2z-1 > 1,\ |2z-3| > 1 \,\} \\ Y_3 &= \{\, z \in H \colon 0 < \operatorname{Re} z < 1,\ |4z-1| > 1,\ |4z-3| > 1 \,\} \end{aligned}$$

43.1. **Lemma.** *(1) $\lambda_j = \lambda|Y_j$ is an analytic bijection onto the open set $\lambda(Y_j)$.*
(2) $\lambda(Y_1 \cup Y_2 \cup Y_3) = \mathbb{C} \setminus \{0, 1\}$.
(3) $X \subset Y_1 \cup Y_2 \cup Y_3$ and Y_1 is the interior of X.
(4) If M_1 and M_2 are in $\mathcal{G}$ and $M_1(Y_j) \cap M_2(Y_j) \neq \emptyset$ then $M_1 = M_2$.

Proof. This should all be clear from the construction in class except, perhaps, part 4. This is clear for Y_1 since $Y_1 \subset X$. For Y_2 notice that $Y_2 \subset X \cup T(X)$, so $M(Y_2) \cap Y_2 \neq \emptyset$ implies that one of $M(X) \cap X$, $MT(X) \cap X$, $M(X) \cap T(X)$ or $MT(X) \cap T(X)$ is not empty. Hence M must be id or T or T^{-1}. But $T(Y_2)$ and $T^{-1}(Y_2)$ are disjoint from Y_2, so $M = \text{id}$. A similar argument works for Y_3, using S instead of T. □

43.2. **Lemma.** *$\lambda(z_1) = \lambda(z_2)$ if and only if $z_2 = M(z_1)$ for some $M \in \mathcal{G}$. Moreover M is uniquely determined by z_1 and z_2.*

Proof. Find M_1 and M_2 in $\mathcal{G}$ so that $z_j \in M_j(X)$ and let $w_j = M_j^{-1}(z_j)$. Then w_1 and w_2 are in X and $\lambda(w_1) = \lambda(w_2)$. If w_1 and w_2 are both in Y_0 then $w_1 = w_2$ since λ_0 is injective. If w_1 and w_2 are both in the reflection $r_1(Y_0)$ through the imaginary axis then $r_1(w_1)$ and $r_1(w_2)$ are in Y_0 and $\lambda(r_1(w_1)) = \overline{\lambda(w_1)} = \overline{\lambda(w_2)} = \lambda(r_1(w_2))$. Hence $r_1(w_1) = r_1(w_2)$, so $w_1 = w_2$. In the remaining case one point, say w_1, is in Y_0 and w_2 is in $X \setminus Y_0$. In this case $\operatorname{Im} \lambda(w_1) \geq 0$ and $\operatorname{Im} \lambda(w_2) < 0$ so we can't have $\lambda(w_1) = \lambda(w_2)$.

Hence $w_1 = w_2$, so $M(z_1) = z_2$ where $M = M_2 M_1^{-1}$. M is unique since if M' also maps z_1 to z_2 then $MM_1(w_1) = z_2 = M'M_1(w_1)$ so $MM_1(X) \cap M'M_1(X) \neq \emptyset$ so $MM_1 = M'M_1$. □

43.3. **Theorem** (Little Picard Theorem). *If $f \colon \mathbb{C} \to \mathbb{C}$ is analytic and $\mathbb{C} \setminus f(\mathbb{C})$ contains more than 1 point then f is constant.*

Proof. Suppose $f(\mathbb{C}) \subset \mathbb{C} \setminus \{a, b\}$ with $a \neq b$. Replacing f with

$$z \mapsto \frac{f(z) - a}{b - a}$$

we may assume that $f(\mathbb{C}) \subset \mathbb{C} \setminus \{0, 1\}$. The theorem now follows from the following:

43.4. **Theorem.** *If V is simply connected and $f\colon V \to \mathbb{C} \setminus \{0, 1\}$ is analytic then there is an analytic $g\colon V \to H$ so that $f = \lambda \circ g$.*

Assuming Theorem 43.4, find such $g\colon \mathbb{C} \to H$ so that $f = \lambda \circ g$. By the Casorati-Weierstrass Theorem g is constant, and so f is constant. ☐

Proof of Theorem 43.4. The Riemann Mapping Theorem provides an analytic bijection ϕ of V onto either $\mathbb{C}$ or D. Replacing f by $f \circ \phi^{-1}$, we may assume $V = \mathbb{C}$ or $V = D$.

Now suppose $a \in V$. Select $j \in \{1, 2, 3\}$ so that $f(a) \in \lambda(Y_j)$ and then (by continuity of f) choose $r > 0$ so that $f(B(a, r)) \subset \lambda(Y_j)$. Now choose any $c \in H$ so that $\lambda(c) = f(a)$. There is a unique $M \in \mathcal{G}$ so that $c \in M(Y_j)$. Define $g_a = M \circ \lambda_j^{-1} \circ f|B(a, r)$. Then g_a is analytic on $B(a, r)$ and satisfies $g_a(a) = c$ and $f(z) = \lambda \circ g_a(z)$ for $z \in B(a, r)$. Thus we have proved "local existence" of the function g.

We can also prove "local uniqueness" of the function g. Specifically, suppose g_1 and g_2 are defined in a neighborhood of some $b \in D$ and satisfy

$$g_1(b) = g_2(b) \text{ and } f(z) = \lambda \circ g_1(z) = \lambda \circ g_2(z).$$

Then, for some $r > 0$, $g_1(z) = g_2(z)$ for all $z \in B(b, r)$. To prove this it is only necessary to see that, for some $j \in \{1, 2, 3\}$ and some $M \in \mathcal{G}$, $M \circ g_1(b)$ and $M \circ g_2(b)$ are in Y_j; for then $M \circ g_1(z)$ and $M \circ g_2(z)$ are in Y_j for z close enough to b and satisfy $\lambda_j(M \circ g_1(z)) = f(z) = \lambda_j(M \circ g_2(z))$. Since λ_j is injective, $M \circ g_1(z) = M \circ g_2(z)$ so $g_1(z) = g_2(z)$. Note that this argument only requires that g_1 and g_2 be continuous.

Next, "local uniqueness" implies uniqueness on connected sets. That is, if

$$g_1(b) = g_2(b) \text{ and } f(z) = \lambda \circ g_1(z) = \lambda \circ g_2(z).$$

for all z in some connected set C containing b then $g_1(z) = g_2(z)$ for all $z \in C$. To prove this consider $E = \{ z \in C\colon g_1(z) = g_2(z) \}$. Then E contains b; E is closed in C since $E = g^{-1}\{0\} \cap C$ where g is the continuous function $g_1 - g_2$; and E is open in C by local uniqueness. By connectedness, $E = C$.

Now we are ready to define g. Choose $c_0 \in H$ so that $\lambda(c_0) = f(0)$. Pick $z \in V$ and define $\gamma(t) = tz$ for $t \in I = [0, 1]$ (remember that V is either D or $\mathbb{C}$). Cover the compact set $\gamma(I)$ by open disks $B(\gamma(t), r_t)$ so that each $f(B(\gamma(t), 4r_t))$ lies in one of $\lambda(Y_j)$, with $j \in \{1, 2, 3\}$. Let $\alpha > 0$ be a Lebesgue number for this cover, and partition I into subintervals $I_k = [t_{k-1}, t_k]$ with $0 = t_0 < t_1 < \cdots < t_n = 1$, so that, for each k, $\gamma(I_k)$ has length less than $\frac{\alpha}{2}$. Let B_k be the open disk with center $\gamma(t_{k-1})$ and radius α. It follows that $f(B_k)$ lies in some $\lambda(Y_j)$. Now define values c_k and functions h_k recursively as follows: c_0 was specified above. Define h_k on B_k by "local existence", so that $\lambda \circ h_k = f|B_k$ and $h_k(\gamma(t_{k-1})) = c_{k-1}$, and then define $c_k = h_k(\gamma(t_k))$. Note that h_k and h_{k-1} agree on their common domain (which is the intersection of two disks containing $\gamma(t_{k-1})$, so is connected). It follows, by induction on $m - k$ and some geometry, that h_m and h_k agree on $B_m \cap B_k$ whenever $m > k$. Hence we can define an analytic function h on the neighborhood $B = \bigcup_k B_k$ of $\gamma(I)$ by requiring that $h|B_k = h_k$. Finally, define $g(z) = h(\gamma(1)) = h(z)$.

For another point z_1 we must repeat this construction to define a function h_1 on a neighborhood of the segment $\gamma_1(I)$. However, if z_1 is close enough to z then the segment $\gamma_1(I)$

will lie in B. Then, by uniqueness, h_1 and h must agree when restricted to $\gamma_1(I)$. In other words, $g(z)$ agrees with $h(z)$ in a neighborhood of z, so g is analytic at z.

This finishes the proof of Theorem 43.4. $\square$

44. 2003-11-26 (DENNIS PIXTON)

38**. Problem.** Define f by

$$f(z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(z-m-ni)^3}.$$

(You should interpret a series of the form $\sum_{n=-\infty}^{\infty} c_n$ as $\sum_{j=1}^{\infty} c_{-j} + \sum_{k=0}^{\infty} c_k$.) Show that

1. f is defined and analytic except at points of $\mathbb{Z} + i\mathbb{Z}$ (use uniform convergence on a neighborhood of z.)
2. f is doubly periodic, with periods 1 and i.
3. f has poles of order 3 at each point of $\mathbb{Z} + i\mathbb{Z}$. Suggestion: It is enough to show that f has a pole at 0 of order 3. Multiply the series by z^3 (to cancel the denominator when $m = n = 0$) and then show that the result is uniformly convergent for z near 0.

39**. Problem.** Let Q be the closed square with vertices at 0, 1, $1+i$ and i. Using the RMT (and its "black box" extension) find a homeomorphism f of Q onto $\operatorname{Cl} H \cup \{\infty\}$ which is analytic in the interior. Using Problem 42, arrange

$$f\colon \quad 0 \mapsto 0, \quad 1 \mapsto 1, \quad 1+i \mapsto \infty, \quad i \mapsto -1.$$

Now restrict f to $Q_0 = Q \setminus \{0, 1, 1+i, i\}$ and call the restriction ϕ_0. Finally, extend ϕ_0 to ϕ by reflections through the sides of Q_0. Show:

1. ϕ is defined and analytic at all points of $\mathbb{C}$ except points of $\mathbb{Z} + i\mathbb{Z}$.
2. ϕ is doubly periodic, with periods 2 and $2i$. Moreover, $\phi(-z) = \phi(z)$.
3. ϕ has removable singularities at each point $m + ni$ with m, n in $\mathbb{Z}$, unless both are odd.
4. ϕ has a pole at each point of the form $m + ni$ with m, n in $\mathbb{Z}$, if both are odd.
5. The zeros of ϕ have multiplicity 2, and the poles have order 2.

40**. Problem.** A function which is analytic on a domain V except for poles at isolated points of V is called *meromorphic*. Show that if f is a nonconstant meromorphic doubly periodic function then $f(\mathbb{C}) = \hat{\mathbb{C}}$.

41**. Problem.** This is actually a precursor to Problem 38.

1. Show that
$$\cot z = \frac{\cos x - i \sin x \tanh y}{\sin x + i \cos x \tanh y}$$
where $z = x + iy$. Use this to show
$$|\cot \pi z| \le \begin{cases} \dfrac{1}{|\tanh \pi y|} & \text{if } y \neq 0 \\ |\tanh \pi y| & \text{if } x \in \mathbb{Z} + \frac{1}{2} \end{cases}$$
2. For a positive integer n let Γ_n be the counterclockwise rectangular path with vertices at $n + \frac{1}{2} + ni$, $-n - \frac{1}{2} + ni$, $-n - \frac{1}{2} - ni$, and $n + \frac{1}{2} - ni$. Show that $|\cot \pi z| \le 2$ for z on Γ_n. Remember that tanh is increasing and bounded by 1, and check that $\tanh \frac{\pi}{2}$ is greater than $\frac{1}{2}$.

(3) Show that

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Gamma_n}\frac{\cot\pi z}{(z-w)^2}dz=0$$

where w is fixed.

(4) Suppose $w\in\mathbb{C}\setminus\mathbb{Z}$. If γ_w is a small circle around w then evaluate

$$\frac{1}{2\pi i}\int_{\gamma_w}\frac{\cot\pi z}{(z-w)^2}dz$$

using Cauchy's formula for the derivative.

(5) If γ_k is a small circle around $k\in\mathbb{Z}$ then evaluate

$$\frac{1}{2\pi i}\int_{\gamma_k}\frac{\cot\pi z}{(z-w)^2}=\frac{1}{2\pi i}\int_{\gamma_k}\frac{f(z)}{z-k}dz$$

by Cauchy's formula, where

$$f(z)=\frac{\cos\pi z}{(z-w)^2}\frac{z-k}{\sin\pi z}$$

is analytic inside and on γ_k. (Technically, it has a removable singularity at k, so you need to evaluate $\lim_{z\to k}f(z)$).

(6) By the homological version of Cauchy's Theorem the integral of $\cot\pi z/(z-w)^2$ around Γ_n is a sum of integrals around γ_k $(-n\le k\le n)$ and γ_w (if w is inside Γ_n). Combine parts 3, 4 and 5 to obtain

$$\sum_{k=-\infty}^{\infty}\frac{1}{(w-k)^2}=\pi^2\csc^2\pi w$$

(This is the singly periodic analog of the function defined in Problem 38.)

(7) Deduce

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}.$$

By the way, this implies

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$$

if you use the identity

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}+\sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}+\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}.$$

45. 2003-12-01 (Garry Bowlin)

42**. Problem.** This is an addendum to Problem 39. Using the RMT plus its "black box" extension there is a homeomorphism f of Q onto $\operatorname{Cl} H \cup \{\infty\}$ which is analytic in the interior. There is an element T of $\operatorname{Aut}(H)$ which maps $f(0)$ to 0, $f(1)$ to 1, and $f(1+i)$ to ∞. Replacing f with $T \circ f$ we have

$$f\colon \quad 0 \mapsto 0, \quad 1 \mapsto 1, \quad 1+i \mapsto \infty.$$

The point of this problem is to see that f must map i to -1.

Define $S(z) = iz + 1$. Notice that this is the rotation through angle $\pi/2$ with center $c = \frac{1}{2} + \frac{1}{2}i$, so it maps the square Q onto itself. Hence $h = f \circ S \circ f^{-1}$ is defined.

(1) Show that $h|H$ is in $\operatorname{Aut}(H)$ and that

$$h\colon \quad 0 \mapsto 1, \quad 1 \mapsto \infty, \quad \infty \mapsto k, \quad k \mapsto 0$$

where $k = f(i)$.

(2) Using Problem 31, show that $h(z) = \dfrac{1+z}{1-z}$ and that $k = -1$. Hence $f(i) = -1$.

(3) Show that f satisfies the functional equation

$$f(iz+1) = \frac{1+f(z)}{1-f(z)}.$$

Now find a formula for $f(1+i-z)$ in terms of $f(z)$. [Hint: $S(S(z)) = 1 + i - z$.]

(4) Remember that c is the center of the rotation S. Find $f(c)$.

46. 2003-12-03 (Ken Ferry)

47. 2003-12-04 (Fabrizio Polo)

48. 2003-12-05 (Mustafa Findik)

49. 2003-12-08 (Adam Perry)