The volume of the $10^{th}$ Birkhoff polytope

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The $n^{th}$ Birkhoff polytope is defined as

$$\mathcal{B}_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2} : x_{jk} \geq 0, \quad \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n, \quad \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \right\},$$

often described as the set of all $n \times n$ doubly stochastic matrices. $\mathcal{B}_n$ is a convex polytope with integer vertices. A long-standing open problem is the determination of the relative volume of $\mathcal{B}_n$. In [1] we introduced a method of calculating this volume and used it to compute $\text{vol} \mathcal{B}_9$. This note is an update on our progress: with the same program, we have now computed

$$\text{vol} \mathcal{B}_{10} = \frac{727291284016786420977508457990121862548823260052557333386607889}{82816096010676685512567631879807272934622463530389422671980721388055739956270293750883504892820848640000000},$$

We computed this using the spare time on most of the 50 Linux workstations in the Mathematics department at Binghamton. The total computation time, scaled to a 1GHz processor, was 6160 days, or almost 17 years.

As in [1], we computed part of the Ehrhart polynomial of $\mathcal{B}_{10}$, that is, the counting function

$$\# (t \mathcal{B}_{10} \cap \mathbb{Z}^{100}) ,$$

a polynomial in the integer variable $t$. The leading term of this polynomial is $\text{vol} \mathcal{B}_{10}/10^9$. For details, as well as the computational tricks which were again used in our computation, we refer to our paper [1] and the accompanying web site www.math.binghamton.edu/dennis/Birkhoff.

Some final remarks:

1. Unless the editors of Discrete & Computational Geometry will allow us to insert these new results in the final version of [1], this note will solely be published on the Mathematics ArXiv (front.math.ucdavis.edu).

2. We will not attempt to compute $\text{vol} \mathcal{B}_{11}$ with our current algorithm.

References


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